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WEIGHTED HARDY SPACES ON SPACE OF HOMOGENEOUS TYPE WITH APPLICATIONS

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Abstract. In this paper, we develop a theory of weighted Hardy spaces H^p_ω on spaces of homogeneous type and prove that certain class of singular integral operators are bounded from H^p_ω to itself and from H^p_ω to L^p_ω. As an application, we give weighted endpoint estimates for Nagel-Stein's NIS operators studied in [26].

1. INTRODUCTION

In 2004, Nagel and Stein [26] introduced a new class of singular integral operators on smooth manifolds and proved the L^p boundedness of them. The geometry on the manifolds is given by a Carnot-Carathéodory metric induced by a collection of vector fields of finite type and the operators includes the so-called non-isotropic smoothing (NIS) operators of order zero arising in several complex varieties, see [26, 27]. Later on, Ding and the first author of this paper studied the mapping properties of a class of fractional integral operators on smooth manifolds in [6]. Recently, Han, Li and Lu [15] developed a theory of multiparameter Hardy spaces on a more general setting, namely, spaces of homogeneous type and proved the $H^p - H^p$ and $H^p - L^p$ boundedness of certain class of singular integral operators.

On the other hand, weighted Hardy spaces have been studied extensively in Euclidean setting (see for example Garcia-Guerva [8] and Strömberg-Torchinsky [29] and many other references therein), where the weighted Hardy space was defined using the non-tangential maximal functions and atomic decompositions were derived. The wavelet characterization of weighted Hardy spaces were established by Wu [32] and by Garcia-Cuerva and Martell [9]. Strömberg and Wheeden [30] studied the relationship between weighted Hardy spaces H^p_ω and weighted Lebesgue spaces L^p_ω. The molecular

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characterization of weighted Hardy spaces were established by Lee and Lin [23] and the H^p_ω boundedness of Riesz transforms were obtained in [24] by using the atomic and molecular decompositions. Recently, Ding, Han, Lu and the first author of this paper [5] proved the $H^p_\omega - H^p_\omega$ and $H^p_\omega - L^p_\omega$ boundedness of singular integral operators on weighted Hardy spaces, under a rather weak assumption $w \in A_\infty$.

Motivated by these results and the recent development of discrete Littlewood-Paley analysis on spaces of homogeneous type, in this paper we study the boundedness of the singular integral operators on weighted Hardy spaces H^p_ω over space of homogeneous type. To achieve our goal, we develop the weighted discrete Littlewood-Paley-Stein theory in the current setting and this allows us to avoid the use of complicated atomic and molecular decompositions of H^p_ω. Our result naturally extent the recent result in [5] and can be applied to variant different settings such as Ahlfors n-regular metric measure spaces, Lie groups of polynomial growth and Carnot-Carathéodory spaces (see, for instance, [20, 31, 26, 27, 28]).

Before stating the main results, let us first recall some definitions and notions. Throughout this paper, we use C to denote a positive constant independent of main parameters involved, which may vary at different occurrences. Let $A \lesssim B$ denote $A \le CB$ and let $A \approx B$ mean $A \lesssim B$ and $B \lesssim A$.

The following notion of spaces of homogeneous type was introduced by Coifman and Weiss in [4].

Definition 1.1. $(\mathcal{X}, d, \mu)$ is called a space of homogeneous type if d is a quasi-metric, that is, (1) $d(x, y) = 0$ iff $x = y$; (ii) $d(x, y) = d(y, x)$; (iii) $d(x, z) \le A[d(x, y) + d(y, z)]$ for some $A \ge 1$, and μ is a nonnegative measure satisfying the doubling property

$$\mu(B(x, 2r)) \le C_1 \mu(B(x, r)). \tag{1.1}$$

In [25], Macias and Segovia have proved that one can replace the quasi-metric d by another quasi-metric $\tilde{d}$ such that $\tilde{d}$ yields the same topology on $\mathcal{X}$ as d and, moreover,

$$\mu(\widetilde{B}(x, r)) \approx r \tag{1.2}$$

where $\widetilde{B}(x, r) = \{y \in \mathcal{X}, \tilde{d}(y, x) < r\}$ and $\tilde{d}$ has the following regularity property

$$|\tilde{d}(x, y) - \tilde{d}(x', y)| \le C_0 \tilde{d}(x, x')^\vartheta [\tilde{d}(x, y) + \tilde{d}(x', y)]^{1-\vartheta}, \tag{1.3}$$

for some regularity exponent ϑ: $0 < \vartheta < 1, 0 < r < \infty$ and all $x, x', y \in \mathcal{X}$. Throughout this paper, we only assume that (1.3) holds for d and a condition like (1.2) is *not* required.

To simplify notation, throughout this paper, we use dx and $|B(x, r)|$ to denote $d\mu(x)$ and $\mu(B(x, r))$, respectively. Denote $V(x, y) = |B(x, d(x, y))|$ and $V_t(x) = \mu(B(x, t)), t > 0$. It is easy to see $V(x, y) \approx V(y, x)$. Note that the doubling condition

(1.1) implies that there exist positive constants C and Q such that for all $x \in \mathcal{X}$ and $\lambda \geq 1$,

$$|B(x, \lambda r)| \leq C\lambda^Q |B(x, r)|. \tag{1.4}$$

Here Q, if chosen minimal, measures the "dimension" of the space $\mathcal{X}$ in some sense.

We now recall some notions on space of homogeneous type in [15].

Definition 1.2. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of operators is said to be an approximation to the identity if there exists constant $C > 0$ such that for all $k \in \mathbb{Z}$ and all $x, x', y, y' \in \mathcal{X}$, $S_k(x, y)$, the kernel of S_k satisfy the following conditions:

(i) $S_k(x, y) = 0$ if $d(x, y) \geq C2^{-k}$ and $|S_k(x, y)| \leq C\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;

(ii) $|S_k(x, y) - S_k(x', y)| \leq C2^{k\vartheta} d(x, x')^\vartheta \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;

(iii) property (ii) holds with x and y interchanged;

(iv) $|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]| \leq C2^{2k\vartheta} d(x, x')^\vartheta d(y, y')^\vartheta \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;

(v) $\int_{\mathcal{X}} S_k(x, y)d\mu(y) = \int_{\mathcal{X}} S_k(x, y)d\mu(x) = 1$.

Definition 1.3. Let $0 < \beta, \gamma \leq \vartheta$ where ϑ is the regularity exponent on $\mathcal{X}$ given in and $r > 0$. A function φ on $\mathcal{X}$ is said to be a test function of type (x_0, r, β, γ) if f satisfies the following conditions:

(i) $|\varphi(x)| \leq C\frac{1}{V_r(x_0)+V(x,x_0)}\left(\frac{r}{r+d(x,x_0)}\right)^\gamma$;

(ii) $|\varphi(x)-\varphi(y)| \leq C\left(\frac{d(x,y)}{r+d(x,x_0)}\right)^\beta \frac{1}{V_r(x_0)+V(x,x_0)}\left(\frac{r}{r+d(x,x_0)}\right)^\gamma$ for all $x, y \in \mathcal{X}$ with $d(x, y) \leq (r + d(x, x_0))/(2A)$.

We denote by $\mathcal{G}(x_1, r, \beta, \gamma)$ the set of all test functions of type (x_1, r, β, γ). If $\varphi \in \mathcal{G}(x_1, r, \beta, \gamma)$ we define its norm by $\|\varphi\|_{\mathcal{G}(x_1,r,\beta,\gamma)} \equiv \inf\{C : (i)\ and\ (ii)\ hold\}$. Now fix $x_0 \in \mathcal{X}$ we denote $\mathcal{G}(\beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$ and by $\mathcal{G}_0(\beta, \gamma)$ the collection of all test functions in $\mathcal{G}(\beta, \gamma)$ with $\int_{\mathcal{X}} f(x)dx = 0$. It is easy to check that $\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$ with equivalent norms for all $x_1 \in \mathcal{X}$ and $r > 0$. Furthermore, it is also easy to see that $\mathcal{G}(\beta, \gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta, \gamma)$. Let $\overset{\circ}{\mathcal{G}}_\vartheta(\beta, \gamma)$ be the completion of the space $\mathcal{G}_0(\vartheta, \vartheta)$ in the norm of $\mathcal{G}(\beta, \gamma)$ when $0 < \beta, \gamma < \vartheta$. If $f \in \overset{\circ}{\mathcal{G}}_\vartheta(\beta, \gamma)$, we then define $\|f\|_{\overset{\circ}{\mathcal{G}}_\vartheta(\beta,\gamma)} = \|f\|_{\mathcal{G}(\beta,\gamma)}$. $(\overset{\circ}{\mathcal{G}}_\vartheta(\beta, \gamma))'$, the distribution space, is defined to be the set of all linear functionals L from $\overset{\circ}{\mathcal{G}}_\vartheta(\beta, \gamma)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in \overset{\circ}{\mathcal{G}}_\vartheta(\beta, \gamma)$, $|L(f)| \leq C\|f\|_{\overset{\circ}{\mathcal{G}}_\vartheta(\beta,\gamma)}$.

Christ [3] provides an analogue of the grid of Euclidean dyadic cubes on space of homogeneous type.

Lemma 1.1. *Let $\mathcal{X}$ be a space of homogeneous type, then, there exists a collection $\{I_\alpha^k \subset \mathcal{X} : k \in \mathbb{Z}, \alpha \in \mathcal{I}^k\}$ of open subsets, where $\mathcal{I}^k$ is some index set, and $C_1, C_2 > 0$, such that*

(i) $\mu(\mathcal{X} \backslash \cup_\alpha I_\alpha^k) = 0$ *for each fixed k and $I_\alpha^k \cap I_\beta^k = \emptyset$, if $\alpha = \beta$;*

(ii) *for any α, β, k, l with $l \geq k$, either $I_\beta^l \subset I_\alpha^k$ or $I_\beta^l \cap I_\alpha^k = \emptyset$;*

(iii) *for each (k, α) and each $l \leq k$ there is a unique β such that $I_\alpha^k \subset I_\beta^l$;*

(iv) *diam$(I_\alpha^k) \leq C_1 2^{-k}$;*

(v) *each I_α^k contains some ball $B(z_\alpha^k, C_2 2^{-k})$, where $z_\alpha^k \in \mathcal{X}$.*

We can think of I_α^k as being a dyadic cube with side-length $\ell(I_\alpha^k) = 2^{-k}$ centered at z_α^k.

Based on Lemma 1.1, Han, Li and Lu [15] established the following discrete Calderón's reproducing formula.

Lemma 1.2. *Let $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity with regularity exponent ϑ. Set $D_k = S_k - S_{k-1}$, $k \in \mathbb{Z}$. Then there exist families of linear operators $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$ and $\{\widetilde{\widetilde{D}}_k\}_{k\in\mathbb{Z}}$ such that for any fixed $x_I \in I$, where N is a fixed constant, and all $(\overset{\circ}{\mathcal{G}}_\vartheta(\beta, \gamma))'$ with $0 < \beta, \gamma < \vartheta$*

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{Q}_k} |I|\, \widetilde{D}_k(x, x_I) D_k(f)(x_I) \\ &= \sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{Q}_k} |I|\, D_k(x, x_I) \widetilde{\widetilde{D}}_k(f)(x_I), \end{aligned}$$

where $\mathcal{Q}_k$ denotes the set of all dyadic cubes I with sidelength $\ell(I) = 2^{-(k+N)}$ for some fixed large constant N and the series converges in $(\overset{\circ}{\mathcal{G}}_\vartheta(\beta, \gamma))'$ with $0 < \beta, \gamma < \vartheta$, and in $L^p(\mathcal{X}), 1 < p < \infty$. Moreover, for $0 < \epsilon < \vartheta$, $\widetilde{D}_k(x, y)$, the kernel of $\widetilde{D}_k$ satisfies

$$\text{(i) } |\widetilde{D}_k(x, y)| \leq C \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon};$$

$$\text{(ii) } |\widetilde{D}_k(x, y) - \widetilde{D}_k(x', y)| \leq C\Big(\frac{d(x, x')}{2^{-k} + d(x, y)}\Big)^\epsilon \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)},$$

$$\times \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon} \qquad \textit{for } d(x, x') \leq (2^{-k} + d(x, y))/2A;$$

$$\text{(iii) } \int_{\mathcal{X}} \widetilde{D}_k(x, y) d\mu(y) = \int_{\mathcal{X}} \widetilde{D}_k(x, y) d\mu(x) = 0,$$

and $\widetilde{\widetilde{D}}_k(x, y)$ the kernel of $\widetilde{\widetilde{D}}_k$ satisfies the similar estimates but with x and y interchanged in (ii).

We remark that the continuous and discrete version of Calderón's reproducing formula on spaces of homogeneous type with the conditions (1.2) and (1.3) were developed in [19] and [13]. Such kind of formula is also a key tool in establishing the $T(b)$ theorem in the Euclidean setting (see [11]).

Let $D_k = S_k - S_{k-1}$, where S_k is an approximation to the identity on $\mathcal{X}$ with the regularity exponent ϑ. For each $f \in (\mathring{\mathcal{G}}_\vartheta(\beta,\gamma))'$ with $0 < \beta, \gamma < \vartheta$, $S(f)$, the Littlewood-Paley square function of f, is defined by

$$\mathcal{G}(f)(x) = \Big\{ \sum_{k\in\mathbb{Z}} |D_k(f)(x)|^2 \Big\}^{1/2}.$$

Definition 1.4. Let $\omega \in L^1_{loc}(\mathcal{X})$ be a nonnegative function in $\mathcal{X}$. We say that ω is an $A_p(\mathcal{X})$ weight, if there exists a constant $C > 0$ such that for every dyadic cube $I \subset \mathcal{X}$,

$$\Big(\frac{1}{|I|}\int_I \omega(x)dx\Big)\Big(\frac{1}{|I|}\int_I \omega(x)^{-\frac{1}{p-1}}dx\Big)^{p-1} \le C, \quad \text{if } 1 < p < \infty,$$
$$\mathcal{M}(\omega)(x) \le C\omega(x), \quad \text{if } p = 1,$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function on $\mathcal{X}$. In this case, we write $\omega \in A_p(\mathcal{X})$. Define $A_\infty(\mathcal{X}) \equiv \bigcup_{1\le p<\infty} A_p(\mathcal{X})$. Let $q_\omega \equiv \inf\{q : \ \omega \in A_q(\mathcal{X})\}$ denote the critical index of ω. We use $\omega(A)$ to denote $\int_A \omega(x)dx$.

For more details about the A_p weight, we refer the reader to [10]. We now give the definition of weighed Hardy spaces $H^p_\omega(\mathcal{X})$.

Definition 1.5. Let $\omega \in A_\infty(\mathcal{X})$ with $q_\omega < 1+\frac{\vartheta}{Q}$, $p \in (\frac{Qq_\omega}{Q+\vartheta}, \infty)$ and $\beta, \gamma \in (0, \vartheta)$. The *weighed Hardy space* $H^p_\omega(\mathcal{X})$ is defined by

$$H^p_\omega(\mathcal{X}) = \{f \in (\mathring{\mathcal{G}}_\vartheta(\beta,\gamma))' : \mathcal{G}(f) \in L^p_\omega(\mathcal{X})\}$$

with H^p_ω quasi-norm $\|f\|_{H^p_\omega(\mathcal{X})} \equiv \|\mathcal{G}(f)\|_{L^p_\omega(\mathcal{X})}$.

To show that $H^p_\omega(\mathcal{X})$ is well defined, we prove the following Plancherel-Pôlya inequalities.

Theorem 1.1. *Suppose* $\omega \in A_\infty(\mathcal{X})$ *with* $q_\omega < 1 + \frac{\vartheta}{Q}$. *Let* $\{S_k\}_{k\in\mathbb{Z}}$ *and* $\{P_k\}_{k\in\mathbb{Z}}$ *be two approximations to the identity with regularity exponent* ϑ. *For* $k \in \mathbb{Z}$, *set* $D_k = S_k - S_{k-1}$ *and* $E_k = P_k - P_{k-1}$. *For a fixed large integer* N *as in Lemma 1.2 and all* $f \in (\mathring{\mathcal{G}}_\vartheta(\beta,\gamma))'$ *with* $0 < \beta, \gamma < \vartheta$, $p \in (\frac{Qq_\omega}{Q+\vartheta}, \infty)$, *where* Q *is the dimension of* $\mathcal{X}$ *given in* (1.4),

$$\Big\|\Big\{\sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{Q}_k}\sup_{z\in I}|D_k(f)(z)|^2\chi_I\Big\}^{1/2}\Big\|_{L^p_w} \approx \Big\|\Big\{\sum_{k'\in\mathbb{Z}}\sum_{I'\in\mathcal{Q}_{k'}}\inf_{z'\in I'}|E_{k'}(f)(z')|^2\chi_{I'}\Big\}^{1/2}\Big\|_{L^p_w}.$$

Remark 1.1. In the unweighted case, such kind of inequalities were first proved in [12] on space of homogeneous type with the conditions (1.2) and (1.3). In this paper, we establish new Plancherel-Pôlya inequality for H^p_ω over space of homogeneous type, which implies that the weighted Hardy spaces H^p_ω are well introduced.

We consider a class of singular integral operators T which are initially defined from $C^\eta_0(\mathcal{X})$, C^η functions with compact supports, $0 < \eta \le \vartheta$ to $C^\eta(\mathcal{X})$ with a distribution kernel $K(x, y)$ and satisfy the following properties:

(I-1) If $\varphi, \psi \in C^\eta_0(\mathcal{X})$ have disjoint supports, then

$$\langle T\varphi, \psi\rangle = \int_{\mathcal{X}\times\mathcal{X}} K(x, y)\varphi(y)\psi(x)dydx.$$

(I-2) If φ is a normalized bump function associated to a ball of radius r, then $\|T\varphi\|_\infty \lesssim 1$ and $\|T\varphi\|_\epsilon \lesssim r^{-\epsilon}$, $\epsilon \le \eta$.

(I-3) If $x \ne y$, then $|K(x, y)| \lesssim V(x, y)^{-1}$ and $|K(x, y) - K(x, y')| \lesssim (\frac{d(y,y')}{d(x,y)})^\epsilon V(x, y)^{-1}$ for $d(y, y') \le \frac{1}{2A} d(x, y)$.

(I-4) Properties (I-1) through (I-3) also hold with x and y interchanged. That is, these properties also hold for the adjoint operator T^t defined by $\langle T^t\varphi, \psi\rangle = \langle T\psi, \varphi\rangle$.

We now give our main result as follows.

Theorem 1.2. *Let $\omega \in A_\infty(\mathcal{X})$ with $q_w < 1 + \epsilon/Q$. Then each singular integral operator T satisfying (I-1) through (I-4) is bounded from $H^p_\omega(\mathcal{X})$ to $H^p_\omega(\mathcal{X})$ for $\frac{q_w Q}{Q+\epsilon} < p < \infty$, and bounded from $H^p_\omega(\mathcal{X})$ to $L^p_\omega(\mathcal{X})$ for $\frac{q_w Q}{Q+\epsilon} < p \le 1$.*

We end the introduction with the following remarks.

First, the singular integral operators considered in this paper are similar to NIS operator considered in [26]. Theorem 1.2 thus provides the weighted endpoint estimate for the NIS operators studied in [26]. Moreover, our results naturally generalize the results of Bownik-Li-Yang-Zhou [2] and Ding-Han-Lu-Wu [5].

Second, there is only one moment condition on spaces of homogeneous type, namely, the moment condition of order 0. Consequently, singular integral operators are bounded on Hardy spaces only for $p > Q/(Q+\epsilon)$ in the unweighted case (see [15]). The ranges of p in Theorem 1.2 are best possible in the sense that when $w \equiv 1 \in A_1(\mathcal{X})$ they become the same as in the unweighted case in [15].

Third, if the double measure μ satisfies certain reverse doubling condition, then the space of homogeneous type $(\mathcal{X}, d, \mu)$ is called RD-space. Han-Müller-Yang [17, 18] developed the Littlewood-Paley theory of Hardy, Triebel-Lizorkin and Besov spaces on RD-spaces. Maximal function characterizations of Hardy spaces on RD-spaces were established by Grafakos-Liu-Yang in [21, 22]. The theory of weak Hardy spaces in this setting was set up in [7, 33].

Fourth, the main tools used in establishing our whole theory are the discrete Littlewood-Paley theory with weights and discrete Calderón-type identity in the current setting. These ideas have been used before in other one-parameter or multiparameter settings, see [5, 14, 15, 16] etc.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following two lemmas (see [15]).

Lemma 2.1. *Let* $\{S_k\}_{k\in\mathbb{Z}}$ *and* $\{P_k\}_{k\in\mathbb{Z}}$ *be two approximations to the identity with regularity exponent* ϑ *and* $D_k = S_k - S_{k-1}$, $E_k = P_k - P_{k-1}$. *Then for any* $\epsilon \in (0, \vartheta)$, *there exists a positive constant* C *depending only on* ϵ *such that* $D_l E_k(x, y)$, *the kernel of* $D_l E_k$, *satisfy the following estimate,*

$$
\begin{aligned}
&|D_l E_k(x, y)| \\
&\leq C 2^{-\epsilon|k-l|} \frac{1}{V_{2^{-(k\wedge l)}}(x) + V_{2^{-(k\wedge l)}}(y) + V(x, y)} \frac{2^{-(k\wedge l)\epsilon}}{(2^{-(k\wedge l)} + d(x, y))^{\epsilon}}.
\end{aligned} \tag{2.1}
$$

Lemma 2.2. *Let* $\epsilon > 0$, $k, k' \in \mathbb{Z}$ *and* y_τ^k *be any point in* I_τ^k *for* $\tau \in \mathcal{I}_k$. *If* $\frac{Q}{Q+\epsilon} < r < p \leq 1$, *then there exists a constant* $C > 0$ *depending only on* r *such that for all* $a_\tau^k \in \mathbb{C}$ *and all* $x \in \mathcal{X}$,

$$
\begin{aligned}
&\sum_{\tau\in\mathcal{I}_k} |I_\tau^k| \frac{1}{V_{2^{-(k\wedge k')}}(x) + V(x, y_\tau^k)} \frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')} + d(x, y_\tau^k))^{\epsilon}} |a_\tau^k| \\
&\lesssim 2^{|k'-k|Q(1/r-1)} \Big\{ \mathcal{M}\Big(\sum_{\tau\in\mathcal{I}_k} |a_\tau^k|^2 \chi_{I_\tau^k} \Big)^{r/2}(x) \Big\}^{1/r},
\end{aligned}
$$

where $[a]_+ = \max(a, 0)$.

We now give

Proof of Theorem 1.1. For $f \in (\overset{\circ}{\mathcal{G}}_\vartheta(\beta, \gamma))'$, we use the discrete Calderón's reproducing formula in Lemma 1.2 to write

$$
f = \sum_{k'} \sum_{I'\in\mathcal{Q}_{k'}} |I'| \widetilde{E}_{k'}(\cdot, x_{I'}) E_{k'}(f)(x_{I'}),
$$

where the series converges in $(\overset{\circ}{\mathcal{G}}_\vartheta(\beta, \gamma))'$ and $x_{I'}$ is any fixed point in the dyadic cube I. Note that by Lemma 1.2, $\widetilde{E}_{k'}(x, y)$ satisfies the same cancellation and smoothness conditions as $E_{k'}(x, y)$. Therefore $D_k \widetilde{E}_{k'}(x, x_{I'})$ satisfy the same almost orthogonality estimate in (2.1) as $D_k E_{k'}(x, x_{I'})$. Applying the Calderón's identity in Lemma 1.2 and the almost orthogonality estimate for $D_k \widetilde{E}_{k'}(x, x_{I'})$, we get that for any $k \in \mathbb{Z}$, any $x, x_I \in I$ and any $x_{I'} \in I'$,

$$\begin{aligned}|D_k(f)(x)| =& \Big|\sum_k \sum_{I'\in\mathcal{Q}_{k'}} |I'| D_k \widetilde{E}_{k'}(x, x_{I'}) E_{k'}(f)(x_{I'})\Big| \\ \lesssim & \sum_{k'} 2^{-\epsilon|k-k'|} \sum_{I'\in\mathcal{Q}_{k'}} |I'||E_{k'}(f)(x_{I'})| \\ & \times \frac{1}{V(x, x_{I'}) + V_{2^{-(k\wedge k')}}(x) + V_{2^{-(k\wedge k')}}(x_{I'})} \Big(\frac{2^{-(k\wedge k')}}{2^{-(k\wedge k')} + d(x, x_{I'})}\Big)^{\epsilon} \\ \sim & \sum_{k'} 2^{-\epsilon|k-k'|} \sum_{I'\in\mathcal{Q}_{k'}} |I'||E_{k'}(f)(x_{I'})| \\ & \times \frac{1}{V(x_I, x_{I'}) + V_{2^{-(k\wedge k')}}(x_I) + V_{2^{-(k\wedge k')}}(x_{I'})} \Big(\frac{2^{-(k\wedge k')}}{2^{-(k\wedge k')} + d(x_I, x_{I'})}\Big)^{\epsilon},\end{aligned}$$

for any $\epsilon \in (0, \vartheta)$, where in the last equivalence we have used

$$V(x, x_{I'}) + V_{2^{-(k\wedge k')}}(x) + V_{2^{-(k\wedge k')}}(x_{I'}) \sim V(x_I, x_{I'}) + V_{2^{-(k\wedge k')}}(x_I) + V_{2^{-(k\wedge k')}}$$

and

$$2^{-(k\wedge k')} + d(x, x_{I'}) \sim 2^{-(k\wedge k')} + d(x_I, x_{I'}).$$

Given any r satisfying $Q/(Q+\vartheta) < r < \min(p/q_w, 1)$, we choose ϵ sufficiently close to ϑ in the last inequality so that

$$\frac{Q}{Q+\epsilon} < r < \min\Big(\frac{p}{q_w}, 1\Big). \tag{2.2}$$

For the above ϵ and r, applying Lemma 2.2 yields

$$\begin{aligned}|D_k(f)(x_I)| \lesssim & \sum_{k'} 2^{-\epsilon|k-k'|} \sum_{I'\in\mathcal{Q}_{k'}} |I'||E_{k'}(f)(x_{I'})| \\ & \times \frac{1}{V(x_I, x_{I'}) + V_{2^{-(k\wedge k')}}(x_I) + V_{2^{-(k\wedge k')}}(x_{I'})} \Big(\frac{2^{-(k\wedge k')}}{2^{-(k\wedge k')} + d(x_I, x_{I'})}\Big)^{\epsilon} \\ \lesssim & \sum_{k'} 2^{-|k-k'|\epsilon'} \Big[\mathcal{M}\Big(\sum_{I'\in\mathcal{Q}_{k'}} |E_{k'}(f)(x_{I'})|^2 \chi_{I'}\Big)^{r/2}(x)\Big]^{1/r},\end{aligned}$$

where $\epsilon' = \epsilon - Q(1/r - 1) > 0$ by (2.2).

Using the fact that x_I and $x_{I'}$ are arbitrary points in I and I' respectively and applying Cauchy-Schwarz's inequality, we get

$$
\begin{aligned}
&\sup_{u\in I}|D_k(f)(u)|^2\\
&\lesssim \Big\{\sum_{k'}2^{-|k-k'|\epsilon'}\Big[\mathcal{M}\Big(\sum_{I'\in\mathcal{Q}_{k'}}\inf_{v\in I'}|E_{k'}(f)(v)|^2\chi_{I'}\Big)^{r/2}(x)\Big]^{1/r}\Big\}^2\\
&\le \Big\{\sum_{k'}2^{-|k-k'|\epsilon'}\Big\}\Big\{\sum_{k'}2^{-|k-k'|\epsilon'}\Big[\mathcal{M}\Big(\sum_{I'\in\mathcal{Q}_{k'}}\inf_{v\in I'}|E_{k'}(f)(v)|^2\chi_{I'}\Big)^{r/2}(x)\Big]^{2/r}\Big\}\\
&\lesssim \sum_{k'}2^{-|k-k'|\epsilon'}\Big[\mathcal{M}\Big(\sum_{I'\in\mathcal{Q}_{k'}}\inf_{v\in I'}|E_{k'}(f)(v)|^2\chi_{I'}\Big)^{r/2}(x)\Big]^{2/r},
\end{aligned}
$$

where x is arbitrary point in I. Then it is easy to see that for any $x\in\mathcal{X}$,

$$
\begin{aligned}
&\sum_{I\in\mathcal{Q}_k}\sup_{u\in I}|D_k(f)(u)|^2\chi_I(x)\\
&\lesssim \sum_{k'}2^{-|k-k'|\epsilon'}\Big[\mathcal{M}\Big(\sum_{I'\in\mathcal{Q}_{k'}}\inf_{v\in I'}|E_{k'}(f)(v)|^2\chi_{I'}\Big)^{r/2}(x)\Big]^{2/r}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_k\sum_{I\in\mathcal{Q}_k}\sup_{u\in I}|D_k(f)(u)|^2\chi_I(x)\\
&\lesssim \sum_k\sum_{k'}2^{-|k-k'|\epsilon'}\Big[\mathcal{M}\Big(\sum_{I'\in\mathcal{Q}_{k'}}\inf_{v\in I'}|E_{k'}(f)(v)|^2\chi_{I'}\Big)^{r/2}(x)\Big]^{2/r}\\
&\le \sum_{k'}\Big[\sum_k 2^{-|k-k'|\epsilon'}\Big]\Big[\mathcal{M}\Big(\sum_{I'\in\mathcal{Q}_{k'}}\inf_{v\in I'}|E_{k'}(f)(v)|^2\chi_{I'}\Big)^{r/2}(x)\Big]^{2/r}\\
&\lesssim \sum_{k'}\Big[\mathcal{M}\Big(\sum_{I'\in\mathcal{Q}_{k'}}\inf_{v\in I'}|E_{k'}(f)(v)|^2\chi_{I'}\Big)^{r/2}(x)\Big]^{2/r},
\end{aligned}
\tag{2.3}
$$

where in the last inequality we have used the inequality $\sum_k 2^{-|k-k'|\epsilon'}<C$.

Since $p/r>q_w$ by (2.2), we see that $w\in A_{p/r}(\mathcal{X})$. Taking the square root first and then the $L^p_\omega(\mathcal{X})$ norm on both sides of (2.3) and using $L^{p/r}_\omega(\ell^{2/r})$ boundedness of $\mathcal{M}$ (by the weighted Fefferman-Stein's vector-valued inequality in [1]) yield

$$
\begin{aligned}
&\Big\|\Big\{\sum_k\sum_{I\in\mathcal{Q}_k}\sup_{u\in I}|D_k(f)(u)|^2\chi_I\Big\}^{1/2}\Big\|_{L^p_w(\mathcal{X})}\\
&\lesssim \Big\|\Big\{\sum_{k'}\Big[\mathcal{M}\Big(\sum_{I'\in\mathcal{Q}_{k'}}\inf_{v\in I'}|E_{k'}(f)(v)|^2\chi_{I'}\Big)^{r/2}\Big]^{2/r}\Big\}^{1/2}\Big\|_{L^p_w(\mathcal{X})}\\
&\lesssim \Big\|\Big\{\sum_{k'}\sum_{I'\in\mathcal{Q}_{k'}}\inf_{v\in I'}|E_{k'}(f)(v)|^2\chi_{I'}\Big\}^{1/2}\Big\|_{L^p_w(\mathcal{X})}.
\end{aligned}
$$

The converse inequality can be proved in the same way This concludes the proof of Theorem 1.1. ■

We also prove the following proposition, which will be used in proof of Theorem 1.2.

Proposition 2.1. *For* $\omega \in A_\infty(\mathcal{X})$ *with* $q_\omega < 1 + \frac{\vartheta}{Q}$ *and* $\frac{q_\omega Q}{Q+\vartheta} < p \le 1$, $\mathcal{G}_0(\vartheta, \vartheta)$ *is dense in* $H^p_\omega(\mathcal{X})$.

Proof. Suppose that notations are the same as in the proof of Theorem 1.1. Fix $x_0 \in \mathcal{X}$ and let

$$\mathcal{R}_L = \{(k', I') : I' \in \mathcal{Q}_{k'}, |k'| \le L, I' \subseteq B(x_0, L)\} \quad \text{for } L \in \mathbb{Z}^+.$$

Repeating the same proof as in Theorem 1.1, we can get

$$\begin{aligned}
&\Big\| f - \sum_{(k',I') \in \mathcal{R}_L} |I'| \, \widetilde{E}_{k'}(\cdot, x_{I'}) \, E_{k'}(f)(x_{I'}) \Big\|_{H^p_\omega(\mathcal{X})} \\
&= \Big\| \Big\{ \sum_k \sum_{I \in \mathcal{Q}_k} \Big| \sum_{(k',I') \in \mathcal{R}_L^c} |I'| \, D_k \widetilde{E}_{k'}(x, x_{I'}) \, E_{k'}(f)(x_{I'}) \Big|^2 \chi_I \Big\}^{1/2} \Big\|_{L^p_w(\mathcal{X})} \\
&\lesssim \Big\| \Big\{ \sum_{(k',I') \in \mathcal{R}_L^c} |E_{k'}(f)|^2 \chi_{I'} \Big\}^{1/2} \Big\|_{L^p_w(\mathcal{X})} \to 0,
\end{aligned}$$

as $N \to \infty$ whenever $f \in H^p_\omega(\mathcal{X})$. Note that for $(k, I) \in \mathcal{R}_L$, $\widetilde{D}_k(x, x_I)$ belongs to $\mathcal{G}_0(\vartheta, \vartheta)$. Therefore, the finite linear combination

$$\sum_{(k,I) \in \mathcal{R}_L} |I| \, \widetilde{D}_k(x, x_I) \, D_k(f)(x_I)$$

also belongs to $\mathcal{G}_0(\vartheta, \vartheta)$. This concludes the proof of Proposition 2.1. ■

3. Proof of Theorem 1.2

To prove Theorem 1.2, we first establish the following

Theorem 3.1. *Let* $\omega \in A_\infty(\mathcal{X})$ *with* $q_\omega < 1 + \frac{\vartheta}{Q}$ *and* $\frac{q_\omega Q}{Q+\vartheta} < p \le 1$. *If* $f \in L^2(\mathcal{X}) \cap H^p_\omega(\mathcal{X})$, *then* $f \in L^p_\omega(\mathcal{X})$ *and there exists a constant* $C > 0$ *which is independent of the* L^2 *norm of* f *such that*

$$\|f\|_{L^p_\omega(\mathcal{X})} \le C \|f\|_{H^p_\omega(\mathcal{X})}.$$

Proof. Assume $f \in L^2(\mathcal{X}) \cap H^p_w(\mathcal{X})$. By Lemma 1.2,

$$f = \sum_k \sum_{I \in \mathcal{Q}_k} |I|\, D_k(\cdot, x_I)\, \widetilde{\widetilde{D}}_k(f)(x_I), \tag{3.1}$$

where the series converges in $L^2(\mathcal{X})$ and hence converges almost everywhere. Since $S_k(x,y)$ are supported where $d(x,y) < C2^{-k}$ by Definition 1.2,

$$D_k(x, x_I) = S_k(x, x_I) - S_{k-1}(x, x_I)$$

also has compact support. Moreover, by the same proof as the proof of Theorem 1.1, we get

$$\|f\|_{H^p_\omega} \approx \Big\| \Big\{ \sum_k \sum_{I \in \mathcal{Q}_k} |\widetilde{\widetilde{D}}_k(f)|^2 \chi_I \Big\}^{1/2} \Big\|_{L^p_\omega}.$$

Set

$$\Omega_i = \Big\{ x \in \mathcal{X} : \Big\{ \sum_k \sum_{I \in \mathcal{Q}_k} |\widetilde{\widetilde{D}}_k(f)(x)|^2 \chi_I(x) \Big\}^{1/2} > 2^i \Big\}$$

and

$$B_i = \{(k, I) : I \in \mathcal{Q}_k, |I \cap \Omega_i| > (1/2A)|I|, |I \cap \Omega_{i+1}| \le (1/2A)|I|\}.$$

We claim

$$\Big\| \sum_{(k,I) \in B_i} |I| D_k(\cdot, x_I) \widetilde{D}_k(f)(x_I) \Big\|^p_{L^p_\omega(\mathcal{X})} \le C 2^{ip} \omega(\Omega_i). \tag{3.2}$$

Assume the claim for the moment. This together with the fact $(\sum_i |a_i|)^p \le \sum_i |a_i|^p, 0 < p \le 1$ would yield

$$\begin{aligned}
\|f\|^p_{L^p_\omega(\mathcal{X})} &= \Big\| \sum_{i \in \mathbb{Z}} \sum_{(k,I) \in B_i} |I| D_k(\cdot, x_I) \widetilde{D}_k(f)(x_I) \Big\|^p_{L^p_\omega(\mathcal{X})} \\
&\le \sum_i \Big\| \sum_{(k,I) \in B_i} |I| D_k(\cdot, x_I) \widetilde{D}_k(f)(x_I) \Big\|^p_{L^p_\omega(\mathcal{X})} \\
&\lesssim \sum_i 2^{ip} \omega(\Omega_i) \lesssim \|f\|^p_{H^p_\omega(\mathcal{X})}.
\end{aligned}$$

To finish the proof of Theorem 3.1, it thus suffices to verify claim (3.2). Note that if $(k, I) \in B_i$, then the support of $D_k(x, x_I)$ is contained in

$$\widetilde{\Omega}_i = \Big\{ x : \mathcal{M}(\chi_{\Omega_i})(x) > \frac{1}{(2A)^{10}} \Big\}.$$

Therefore, by Hölder's inequality,

$$\begin{aligned}&\Big\|\sum_{(k,I)\in B_i}|I|D_k(\cdot,x_I)\widetilde{\widetilde{D}}_k(f)(x_I)\Big\|^p_{L^p_\omega(\mathcal{X})}\\&\le[w(\widetilde{\Omega}_i)]^{1-\frac{p}{q}}\Big\|\sum_{(k,I)\in B_i}|I|D_k(\cdot,x_I)\widetilde{\widetilde{D}}_k(f)(x_I)\Big\|^p_{L^q_\omega(\mathcal{X})}.\end{aligned}$$

We now estimate the last $L^q_\omega(\mathcal{X})$ norm by the duality argument. For all $g\in L^{q'}_{\omega^{1-q'}}(\mathcal{X})$ with $\|g\|_{L^{q'}_{\omega^{1-q'}}}\le 1$,

$$\begin{aligned}&\Big|\Big\langle\sum_{(k,I)\in B_i}|I|D_k(\cdot,x_I)\widetilde{\widetilde{D}}_k(f)(x_I),g\Big\rangle\Big|\\&=\Big|\int_{\mathcal{X}}\sum_{(k,I)\in B_i}|I|D_k^*(g)(x_I)\widetilde{\widetilde{D}}_k(f)(x_I)\chi_I(x)dx\Big|\\&\le\Big\|\Big(\sum_{(k,I)\in B_i}|D_k^*(g)(x_I)|^2\chi_I(\cdot)\Big)^{1/2}\Big\|_{L^{q'}_{\omega^{1-q'}}(\mathcal{X})}\Big\|\Big(\sum_{(k,I)\in B_i}|\widetilde{\widetilde{D}}_k(f)(x_I)|^2\chi_I(\cdot)\Big)^{1/2}\Big\|_{L^q_\omega(\mathcal{X})},\end{aligned}$$

where D_k^* is an operator defined by

$$D_k^*(g)(x)=\int_{\mathcal{X}}D_k(y,x_I)\overline{g(y)}dy.$$

By Definition 1.2, we can see that $S_k(x,y)$ satisfies the same properties as $S_k(y,x)$. Thus $D_k(y,x_I)$ satisfies the same properties as $D_k(x_I,y)$. Note that $\omega\in A_q(\mathcal{X})$ implies $\omega^{1-q'}\in A_{q'}(\mathcal{X})$. Thus by the weighted Fefferman-Stein vector-valued inequality, we have

$$\begin{aligned}&\Big\|\Big(\sum_{(k,I)\in B_i}|D_k^*(g)(x_I)|^2\chi_I\Big)^{1/2}\Big\|_{L^{q'}_{w^{1-q'}}(\mathcal{X})}\\&\Big\|\Big(\sum_{(k,I)\in B_i}|\inf_{u\in I}\mathcal{M}(D_k^*(g))(u)|^2\chi_I\Big)^{1/2}\Big\|_{L^{q'}_{\omega^{1-q'}}(\mathcal{X})}\\&\le\Big\|\Big(\sum_k|\mathcal{M}(D_k^*(g))(\cdot)|^2\chi_I(\cdot)\Big)^{1/2}\Big\|_{L^{q'}_{\omega^{1-q'}}(\mathcal{X})}\\&\lesssim\|g\|_{L^{q'}_{\omega^{1-q'}}(\mathcal{X})}\le 1,\end{aligned}$$

where in the next to the last inequality we have used weighted Littlewood-Paley inequality in [1]. Altogether yields

$$\begin{aligned}&\Big\|\sum_{(k,I)\in B_i}|I|D_k(\cdot,x_I)\widetilde{\widetilde{D}}_k(f)(x_I)\Big\|_{L^q_\omega(\mathcal{X})}\\&\lesssim\Big\|\Big\{\sum_{(k,I)\in B_i}|\widetilde{\widetilde{D}}_k(f)(x_I)|^2\chi_I\Big\}^{1/2}\Big\|_{L^q_\omega(\mathcal{X})}.\end{aligned}\tag{3.3}$$

Note also that

$$
\begin{aligned}
2^{qi}\omega(\Omega_i) &\gtrsim \int_{\widetilde{\Omega}_i\backslash\Omega_{i+1}} \Big\{\sum_k \sum_{I\in\mathcal{Q}_k} |\widetilde{\widetilde{D}}_k(f)(x)|^2\chi_I(x)\Big\}^{q/2}\omega(x)dx \\
&\gtrsim \int_{\mathcal{X}} \Big\{\sum_{(k,I)\in B_i} |\widetilde{\widetilde{D}}_k(f)(x_I)\mathcal{M}(I\cap(\widetilde{\Omega}_i\backslash\Omega_{i+1}))(x)|^2\Big\}^{q/2}\omega(x)dx \\
&\gtrsim \int_{\mathcal{X}} \Big\{\sum_{(k,I)\in B_i} |\widetilde{\widetilde{D}}_k(f)(x_I)|^2\chi_I(x)\Big\}^{q/2}\omega(x)dx,
\end{aligned}
$$

where in the last inequality we have used the fact that $\omega(I\cap(\widetilde{\Omega}_i\backslash\Omega_{i+1}))\geq\frac{1}{2A}\omega(I)$ whenever $(k,I)\in B_i$. This finishes the proof of claim (3.2) and hence Theorem 3.1. ∎

Now, we are ready to give

Proof of Theorem 1.2. We assume $f\in L^2\cap H^p_\omega(\mathcal{X})$. Let x_I and $x_{I'}$ be arbitrary points in I and I', respectively. Repeating the same argument as in the proof of Theorem 1.1, we get

$$
\begin{aligned}
\|T(f)\|_{H^p_\omega(\mathcal{X})} &\sim \Big\|\Big\{\sum_k\sum_{I\in\mathcal{Q}_k} D_k(Tf)(x_I)\chi_I\Big\}^{1/2}\Big\|_{L^p_\omega(\mathcal{X})} \\
&\lesssim \Big\|\Big\{\sum_k\sum_{I\in\mathcal{Q}_k}\sum_{k'}\sum_{I'\in\mathcal{Q}_{k'}} |I'|D_kT\widetilde{D}_{k'}(x_I,x_{I'})D_{k'}(f)(x_{I'})\chi_I\Big\}^{1/2}\Big\|_{L^p_\omega(\mathcal{X})} \\
&\lesssim \Big\|\Big\{\sum_{k'}\Big[\mathcal{M}\Big(\sum_{I'\in\mathcal{Q}_{k'}}|D_{k'}(f)(x_{I'})|^2\chi_{I'}\Big)^{r/2}\Big]^{2/r}\Big\}^{1/2}\Big\|_{L^p_w(\mathcal{X})} \\
&\lesssim \Big\|\Big\{\sum_{k'}\sum_{I\in\mathcal{Q}_{k'}}|D_{k'}(f)(x_{I'})|^2\chi_{I'}\Big\}^{1/2}\Big\|_{L^p_\omega(\mathcal{X})} \lesssim \|f\|_{H^p_\omega(\mathcal{X})},
\end{aligned}
$$

where we have used the following estimate (see [15])

$$
|D_kT\widetilde{D}_{k'}(x,y)|\lesssim 2^{-|k-k'|\epsilon'}\frac{1}{V(x,y)+V_{2^{-(k\wedge k')}}(x)+V_{2^{-(k\wedge k')}}(y)}\Big(\frac{2^{-(k\wedge k')}}{2^{-(k\wedge k')}+d(x,y)}\Big)^{\epsilon'}
$$

for any $\epsilon'<\epsilon$. By Proposition 2.1, a limiting argument yields the $H^p_\omega(\mathcal{X})$ boundedness of T.

To prove $H^p_\omega(\mathcal{X})-L^p_\omega(\mathcal{X})$ boundedness of T, we assume $f\in L^2\cap H^p_w(\mathcal{X})$. Then from the $H^p_\omega(\mathcal{X})$ boundedness and Theorem 3.1, it follows that

$$
\|T(f)\|_{L^p_\omega(\mathcal{X})}\lesssim\|T(f)\|_{H^p_\omega(\mathcal{X})}\lesssim\|f\|_{H^p_\omega(\mathcal{X})}.
$$

Use Proposition 2.1 again to get the desired conclusion. Hence the proof of Theorem 1.2 is complete. ∎

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