TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 5, pp. 1563-1581, October 2014
DOI: 10.11650/tjm.18.2014.3169
This paper is available online at http://journal.taiwanmathsoc.org.tw

DECOMPOSITION OF COMPLETE GRAPHS INTO TRIANGLES AND CLAWS

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Abstract. Let K_n be a complete graph with n vertices, C_k denote a cycle of length k, and S_k denote a star with k edges. If $k = 3$, then we call C_3 a triangle and S_3 a claw. In this paper, we show that for any nonnegative integers p and q and any positive integer n, there exists a decomposition of K_n into p copies of C_3 and q copies of S_3 if and only if $3(p + q) = \binom{n}{2}$, $q \neq 1, 2$ if n is odd, $q = 1$ if $n = 4$, and $q \geq \max\{3, \lceil \frac{n}{4} \rceil\}$ if n is even and $n \geq 6$.

1. INTRODUCTION

All graphs considered here are finite and undirected, unless otherwise noted. For the standard graph-theoretic terminology the reader is referred to [3]. Let K_n be the complete graph with n vertices and $K_{m,n}$ be the complete bipartite graph with parts of sizes m and n. The cycle with k vertices is denoted by C_k. The k-star, denoted by S_k, consists of a vertex x of degree k, and k edges joining x to its neighbor. S_k is isomorphic to $K_{1,k}$. When $k = 3$, we call C_3 a triangle and S_3 a claw. Let G be a simple graph and $\Gamma = \{G_1, G_2, \ldots, G_t\}$ be a family of subgraphs of G. A Γ-decomposition of G is an edge-disjoint decomposition of G into positive integer α_i copies of G_i, where $i \in \{1, 2, \ldots, t\}$, denoted by $G = \alpha_1 G_1 \oplus \alpha_2 G_2 \oplus \ldots \oplus \alpha_t G_t$. Furthermore, if $\Gamma = \{H\}$, we say that G has an H-decomposition. It is easy to see that $\sum_{i=1}^{t} \alpha_i e(G_i) = e(G)$ is one of the necessary conditions for the existence of a $\{G_1, G_2, \ldots, G_t\}$-decomposition of G. In [7] Shyu obtained four necessary conditions for a decomposition of K_n into C_l and S_k and gave the necessary and sufficient conditions for $l = k = 4$.

In this paper, we will prove the following result.

Main Theorem. *For any nonnegative integers p and q and any positive integer n, $K_n = pC_3 \oplus qS_3$ if and only if $3(p+q) = \binom{n}{2}$, $q \neq 1, 2$ if n is odd, $q = 1$ if $n = 4$, and $q \geq \max\{3, \lceil \frac{n}{4} \rceil\}$ if $n \geq 6$ and n is even.*

Received April 9, 2013, accepted February 20, 2014.
Communicated by Gerard Jennhwa Chang.
2010 *Mathematics Subject Classification*: 05C51.
Key words and phrases: Graph decomposition, Complete graph, Cycle, Star.
*This research is supported by NSC 101-2115-M-032-002.

2. Notation and Preliminaries

A Steiner triple system is an ordered pair (S, T), where S is a finite set of symbols, and T is a set of 3-element subsets of S called triples, such that each pair of distinct elements of S occurs together in exactly one triple of T. The order of a Steiner triple system (S, T) is the size of the set S, denoted by $|S|$. A Steiner triple system (S, T) is equivalent to a complete graph $K_{|S|}$ in which the edges have been partitioned into triangles (corresponding to the triples in T). For convenience, we let $\mathrm{STS}(v)$ denote a Steiner triple system of order v. In 1847 Kirkman [5] proved the following result.

Theorem 2.1. *A* $\mathrm{STS}(v)$ *exists if and only if* $v \equiv 1, 3 \pmod 6$.

Therefore, K_v has a C_3-decomposition if and only if $v \equiv 1, 3 \pmod 6$.

A pairwise balanced design (or simply, PBD) is an ordered pair (S, B), where S is a finite set of symbols, and B is a collection of subsets of S called blocks, such that each pair of distinct elements of S occurs together in exactly one block of B. If $|S| = v$ and $K = \{|b| | b \in B\}$, then (S, B) is a PBD of order v with block sizes in K, denoted by $\mathrm{PBD}(v, K)$. A group divisible design (GDD) is an ordered triple (S, G, B) where S is a finite set, G is a collection of sets called groups which partition S, and B is a set of subsets of S called blocks, such that $(S, G \cup B)$ is a PBD. If $|S| = v$, $|G| > 1$ and $|b| = 3$, for each $b \in B$, then we call (S, G, B) is a 3-GDD of order v. If $v = a_1 g_1 + a_2 g_2 + \ldots + a_s g_s$ and there are a_i groups of size g_i, $i = 1, 2, \ldots, s$, then we call the 3-GDD is of type $g_1^{a_1} g_2^{a_2} \ldots g_s^{a_s}$.

Theorem 2.2. ([4]). *Let* g, t, *and* u *be nonnegative integers. There exists a* 3*-GDD of type* $g^t u^1$ *if and only if the following conditions are all satisfied:*

1. *If* $g > 0$, *then* $t \geq 3$, *or* $t = 2$ *and* $u = g$, *or* $t = 1$ *and* $u = 0$, *or* $t = 0$;
2. $u \leq g(t-1)$ *or* $gt = 0$;
3. $g(t-1) + u \equiv 0 \pmod 2$ *or* $gt = 0$;
4. $gt \equiv 0 \pmod 2$ *or* $u = 0$;
5. $g^2 t(t-1)/2 + gtu \equiv 0 \pmod 3$.

Let $Q = \{1, 2, \ldots, 2n\}$ and let $H = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$. In what follows, the two-element subsets $\{2i-1, 2i\} \in H$ are called *holes*. *A quasigroup with holes* H is a quasigroup $(Q, \circ)$ of order $2n$ in which for each $h \in H$, $(h, \circ)$ is a subquasigroup of $(Q, \circ)$. For clearness, we give the construction of a quasigroup with holes, which is shown in [6], as follows.

Theorem 2.3. ([6]). *For all* $n \geq 3$ *there exists a commutative quasigroup of order* $2n$ *with holes* $H = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$.

Proof. Let $S = \{1, 2, \ldots, 2n+1\}$. If $2n+1 \equiv 1$ or $3 \pmod 6$ then let (S, B) be a Steiner triple system of order $2n+1$, and if $2n+1 \equiv 5 \pmod 6$ then let (S, B)

be a PBD of order $2n+1$ with exactly one block, say b, of size 5, and the rest of size 3. By renaming the symbols in the triples (blocks) if necessary, we can assume that the only triples containing symbol $2n+1$ are:

$$\{1, 2, 2n+1\}, \{3, 4, 2n+1\}, \ldots, \{2n-1, 2n, 2n+1\}.$$

(In forming the quasigroup, these triples are ignored.) Define a quasigroup $(Q, \circ) = (\{1, 2, \ldots, 2n\}, \circ)$ as follows:

(a) for each $h \in H = \{\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}\}$ let $(h, \circ)$ be a subquasigroup of $(Q, \circ)$;

(b) for $1 \le i \ne j \le 2n$, $\{i, j\} \notin H$ and $\{i, j\} \not\subset b$, let $\{i, j, k\}$ be the triple in B containing symbols i and j and define $i \circ j = k = j \circ i$; and

(c) if $2n+1 \equiv 5 \pmod 6$ then let $(b, \otimes)$ be an idempotent commutative quasigroup of order 5 and for each $\{i, j\} \subseteq b$ define $i \circ j = i \otimes j = j \circ i$. ■

By using commutative quasigroups with holes, Lindner et al. give a constructon for STS and PBD in [6], stated as follows. **L-Construction.** Let $(\{1, 2, \ldots, 2n\}, \circ)$ be a commutative quasigroup of order $2n$ with holes H. Then

(a) $(\{\infty\} \cup (\{1, 2, \ldots, 2n\} \times \{1, 2, 3\}), B')$ is a $\mathrm{STS}(6n+1)$, where B' is defined by:

  (1) for $1 \le i \le n$ let B'_i contain the triples in a $\mathrm{STS}(7)$ on the symbols $\{\infty\} \cup (\{2i-1, 2i\} \times \{1, 2, 3\})$ and let $B'_i \subseteq B'$, and

  (2) for $1 \le i \ne j \le 2n$, $\{i, j\} \notin H$, place the triples $\{(i,1), (j,1), (i \circ j, 2)\}$, $\{(i,2), (j,2), (i \circ j, 3)\}$, and $\{(i,3), (j,3), (i \circ j, 1)\}$ in B'.

(b) $(\{\infty_1, \infty_2, \infty_3\} \cup (\{1, 2, \ldots, 2n\} \times \{1, 2, 3\}), B'')$ is a $\mathrm{STS}(6n+3)$, where B'' is defined by replacing (1) in (a) with:

  (1′) for $1 \le i \le n$ let B''_i contain the triples in a $\mathrm{STS}(9)$ on the symbols $\{\infty_1, \infty_2, \infty_3\} \cup (\{2i-1, 2i\} \times \{1, 2, 3\})$ in which $\{\infty_1, \infty_2, \infty_3\}$ is a triple, and let $B''_i \subseteq B''$, and

(c) $(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (\{1, 2, \ldots, 2n\} \times \{1, 2, 3\}), B''')$ is a $\mathrm{PBD}(6n+5)$ with one block of size 5, the rest of size 3, where B''' is defined by replacing (1) in (a) with:

  (1″) for $1 \le i \le n$ let B'''_i contain the blocks in a $\mathrm{PBD}(11)$ on the symbols $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (\{2i-1, 2i\} \times \{1, 2, 3\})$ in which $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ is a block, and let $B'''_i \subseteq B'''$.

In 1975 Yamamoto et al. [10], and independently in 1979 Tarsi [9] got the following result.

Theorem 2.4. ([9]). *Let n and k be positive integers. There is an S_k-decomposition of K_n if and only if $n \geq 2k$ and $n(n-1) \equiv 0 \pmod{2k}$.*

In [7], Shyu showed the necessary condition for decomposing K_n into p copies of C_3 and q copies of S_3 as follows.

Theorem 2.5. ([7]). *Let n be an integer. If $K_n = pC_3 \oplus qS_3$ for any nonnegative integers p and q, then $3(p+q) = \binom{n}{2}$, $q \neq 1, 2$ if n is odd, $q = 1$ if $n = 4$, and $q \geq \max\{3, \lceil \frac{n}{4} \rceil\}$ if $n \geq 6$ and n is even.*

Next we will show that given any nonnegative integers p and q if they satisfy the necessary condition in Theorem 2.5, then there is a $\{C_3, S_3\}$-decomposition of K_n.

By counting the edges of K_n, we can get the necessary condition for the existence of a $\{C_3, S_3\}$-decomposition of K_n as follows.

Theorem 2.6. *Let n be a positive integer. If there is a $\{C_3, S_3\}$-decomposition of K_n, then $n \equiv 0, 1 \pmod 3$.*

For convenience, we define $I(G) = \{q | G = pC_3 \oplus qS_3$, for any nonnegative integers p and $q\}$,

$$J_n = \left\{ q \,\middle|\, p+q = \frac{n(n-1)}{6},\ p, q \geq 0 \text{ and } q \neq 1, 2 \right\} \text{ if } n \text{ is odd, and}$$
$$J_n = \left\{ q \,\middle|\, p+q = \frac{n(n-1)}{6},\ p, q \geq 0 \text{ and } q \geq \max\left\{3, \left\lceil \frac{n}{4} \right\rceil \right\} \right\} \text{ if } n \text{ is even and } n \geq 6.$$

Then $I(K_n) \subseteq J_n$. Let $A + B = \{a + b | a \in A, b \in B\}$, and $k \cdot A = A + A + \ldots + A$ (the addition of k A's).

Example 2.7. $n = 4$. It is easy to see that K_4 can be decomposed into one C_3 and one S_3, there is neither C_3-decomposition nor S_3-decomposition of K_4. Thus $I(K_4) = \{1\}$.

It is easy to see that if $K_n = G_1 \oplus G_2$, then $I(G_1) + I(G_2) \subseteq I(K_n)$. Next we just only need to prove $I(K_n) \supseteq J_n$, for $n \equiv 0, 1 \pmod 3$ and $n \geq 6$.

3. Some Small Cases

In this section, we will show that $I(K_n) = J_n$, for $n \equiv 0, 1 \pmod 3$ and $6 \leq n \leq 15$. For convenience, we let $V(K_n) = Z_n = \{1, 2, \ldots, n\}$, (a,b,c) means a 3-cycle with vertices a,b,c and S(a;b,c,d) means a star(or claw) with center vertex a and end vertices b,c,d.

Example 3.1. $J_6 = \{3, 4, 5\}$ and there are following decompositions of K_6.

(1) (1, 2, 3), (4, 5, 6), S(1; 4, 5, 6), S(2; 4, 5, 6), S(3; 4, 5, 6). Then $3 \in I(K_6)$.

(2) (1, 2, 3), S(3; 4, 5, 6), S(4; 1, 2, 5), S(5; 1, 2, 6), S(6; 1, 2, 4). Then $4 \in I(K_6)$.

(3) By Theorem 2.4, $5 \in I(K_6)$.

Therefore, $I(K_6) \supseteq J_6$.

Example 3.2. $J_7 = \{0, 3, 4, 5, 6, 7\}$ and there are following decompositions of K_7.

(1) By Theorem 2.1 and 2.4, we have $0, 7 \in I(K_7)$.

(2) (1, 2, 3), (1, 4, 7), (2, 5, 7), (3, 6, 7), S(4; 2, 3, 6), S(5; 1, 3, 4), S(6; 1, 2, 5). Then $3 \in I(K_7)$.

(3) (1, 2, 3), (3, 4, 5), (5, 6, 7), S(1; 4, 5, 7), S(2; 4, 5, 6), S(6; 1, 3, 4), S(7; 2, 3, 4). Then $4 \in I(K_7)$.

(4) (1, 2, 3), (4, 5, 6), S(1; 4, 5, 6), S(2; 4, 5, 6), S(3; 4, 5, 6), S(7; 1, 2, 3), S(7; 4, 5, 6). Then $5 \in I(K_7)$.

(5) (1, 2, 3), S(1; 4, 6, 7), S(2; 4, 6, 7), S(3; 4, 5, 7), S(5; 1, 2, 4), S(6; 3, 4, 5), S(7; 4, 5, 6). Then $6 \in I(K_7)$.

Therefore, $I(K_7) \supseteq J_7$.

Example 3.3. $J_9 = \{i | i = 0 \text{ or } 3 \leq i \leq 12\}$ and there are following decompositions of K_9.

(1) (1, 2, 3), (4, 5, 6), (7, 8, 9), (1, 4, 7), (2, 5, 8), (3, 6, 9), (1, 5, 9), (2, 6, 7), (3, 4, 8), (1, 6, 8), (2, 4, 9), (3, 5, 7). Then $0 \in I(K_9)$.

(2) (1, 5, 9), (1, 6, 8), (2, 4, 9), (2, 5, 8), (3, 4, 8), (3, 5, 7), (3, 6, 9), (4, 5, 6), (7, 8, 9), S(1; 2, 3, 4), S(2; 3, 6, 7), S(7; 1, 4, 6). Then $3 \in I(K_9)$.

(3) (1, 2, 3), (3, 4, 5), (5, 6, 7), (7, 8, 9), (1, 6, 9), (1, 5, 8), (1, 4, 7), (2, 6, 8), S(2; 5, 7, 9), S(3; 6, 7, 8), S(4; 2, 6, 8), S(9; 3, 4, 5). Then $4 \in I(K_9)$.

(4) (1, 2, 3), (3, 4, 5), (5, 6, 7), (7, 8, 9), (1, 6, 9), (1, 5, 8), (1, 4, 7), S(2; 4, 5, 7), S(3; 7, 8, 9), S(6; 2, 3, 4), S(8; 2, 4, 6), S(9; 2, 4, 5). Then $5 \in I(K_9)$.

(5) (1, 2, 3), (3, 4, 5), (5, 6, 7), (7, 8, 9), (1, 6, 9), (1, 5, 8), S(2; 5, 8, 9), S(4; 1, 2, 7), S(6; 2, 3, 4), S(7; 1, 2, 3), S(8; 3, 4, 6), S(9; 3, 4, 5). Then $6 \in I(K_9)$.

(6) (1, 2, 3), (3, 4, 5), (5, 6, 7), (7, 8, 9), (1, 6, 9), S(1; 4, 5, 7), S(2; 4, 5, 6), S(2; 7, 8, 9), S(3; 6, 7, 8), S(4; 6, 7, 8), S(8; 1, 5, 6), S(9; 3, 4, 5). Then $7 \in I(K_9)$.

(7) (1, 2, 3), (3, 4, 5), (5, 6, 7), (7, 8, 9), S(1; 4, 5, 6), S(2; 4, 5, 7), S(6; 2, 3, 4), S(7; 1, 3, 4), S(8; 1, 2, 3), S(8; 4, 5, 6), S(9; 1, 2, 3), S(9; 4, 5, 6). Then $8 \in I(K_9)$.

(8) (1, 2, 3), (3, 4, 5), (5, 6, 7), S(1; 4, 5, 6), S(1; 7, 8, 9), S(2; 4, 5, 6), S(2; 7, 8, 9), S(3; 6, 7, 8), S(4; 6, 7, 8), S(8; 5, 6, 7), S(9; 3, 4, 5), S(9; 6, 7, 8). Then $9 \in I(K_9)$.

(9) (1, 2, 3), (3, 4, 5), S(1; 4, 5, 6), S(1; 7, 8, 9), S(2; 4, 5, 7), S(5; 6, 7, 9), S(6; 2, 3, 4), S(7; 3, 4, 6), S(8; 2, 3, 4), S(8; 5, 6, 7), S(9; 2, 3, 4), S(9; 6, 7, 8). Then $10 \in I(K_9)$.

(10) (1, 2, 3), S(1; 4, 5, 6), S(2; 4, 5, 9), S(3; 4, 5, 9), S(4; 5, 6, 8), S(6; 2, 3, 5), S(7; 1, 2, 3), S(7; 4, 5, 6), S(8; 1, 2, 3), S(8; 5, 6, 7), S(9; 1, 4, 5), S(9; 6, 7, 8). Then $11 \in I(K_9)$.

(11) By Theorem 2.4, $12 \in I(K_9)$.

Therefore, $I(K_9) \supseteq J_9$.

Example 3.4. $J_{10} = \{i|3 \leq i \leq 15\}$. Let $V(K_{10}) = \{\infty\} \cup Z_9$.

Since $K_{10} = K_{1,9} \oplus K_9$, we have $I(K_{10}) \supseteq I(K_{1,9}) + I(K_9) = \{3\} + \{i|i = 0$ or $3 \leq i \leq 12\} = \{i|i = 3$ or $6 \leq i \leq 15\} = J_{10} - \{4, 5\}$.

From Example 3.3 (1), there are 4 triangles (1, 2, 3), (4, 5, 6), (7, 8, 9), and (3, 6, 9) in the decomposition of K_9, see Figure 3.1. Consider the union of these 4 triangles and 3 stars $S(\infty$; 1, 2, 3), $S(\infty$; 4, 5, 6), $S(\infty$; 7, 8, 9), it can be viewed as $3C_3 \oplus 4S_3$ or $2C_3 \oplus 5S_3$ as follows:

(∞, 1, 2), (∞, 4, 5), (∞, 7, 8), $S(\infty$; 3, 6, 9), S(3; 1, 2, 6), S(6; 4, 5, 9), S(9; 3, 7, 8) or (4, 5, 6), (7, 8, 9), $S(\infty$; 4, 5, 6), $S(\infty$; 2, 7, 8), S(1; ∞, 2, 3), S(3; ∞, 2, 6), S(9; ∞, 3, 6). Therefore $I(K_{10}) \supseteq J_{10}$.

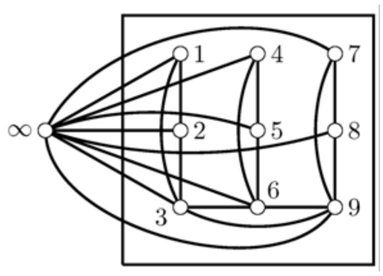

Figure 3.1. $4C_3 \oplus 3S_3$.

Example 3.5. $J_{12} = \{i|3 \leq i \leq 22\}$ and if $K_{12} = K_3 \oplus K_{3,9} \oplus K_9$, then $I(K_{12}) \supseteq I(K_3) + I(K_{3,9}) + I(K_9)$ =$\{9\}$+$\{i|i = 0$ or $3 \leq i \leq 12\}$ = $\{i|i = 9$ or $12 \leq i \leq 21\}$.

Do the same process as in Example 3.4, we can get $10, 11 \in I(K_{12})$. For $q = 3, 4, \ldots, 8$, we discuss them as follows.

(1) Take $K_{12} = 3K_4 \oplus K_{4,4,4}$.

Let $V(K_{4,4,4}) = \{1, 2, 3, 4\} \cup \{5, 6, 7, 8\} \cup \{9, 10, 11, 12\}$.

(i) There is a C_3-decomposition of $K_{4,4,4}$ as follows: (1, 5, 9), (1, 6, 10), (1, 7, 11), (1, 8, 12), (2, 5, 10), (2, 6, 11), (2, 7, 12), (2, 8, 9), (3, 5, 11), (3, 6, 12), (3, 7, 9), (3, 8, 10), (4, 5, 12), (4, 6, 9), (4, 7, 10), (4, 8, 11). We can get 3 copies of S_3 from $3K_4$. Thus $3 \in I(K_{12})$.

(ii) Take $5C_3$: (3, 5, 11), (3, 6, 12), (4, 5, 12), (4, 6, 9), (4, 8, 11) from (i) and $3S_3$: S(4; 1, 2, 3), S(8; 5, 6, 7), S(12; 9, 10, 11) from $3K_4$, we can get the following results:

(a) (3, 4, 6), (3, 11, 12), (4, 5, 11) and S(4; 1, 2, 8), S(5; 3, 8, 12), S(8; 6, 7, 11), S(9; 4, 6, 12), S(12; 4, 6, 10). Thus $5 \in I(K_{12})$.

(b) (3, 4, 6), (3, 11, 12) and S(4; 1, 2, 8), S(5; 3, 4, 12), S(8; 5, 6, 7), S(9; 4, 6, 12), S(11; 4, 5, 8), S(12; 4, 6, 10). Thus $6 \in I(K_{12})$.

(c) (3, 11, 12) and S(3; 4, 5, 6), S(4; 1, 2, 12), S(4; 5, 6, 8), S(8; 5, 6, 7), S(9; 4, 6, 12), S(11; 4, 5, 8), S(12; 5, 6, 10). Thus $7 \in I(K_{12})$.

(d) S(3; 4, 6, 11), S(4; 1, 2, 8), S(4; 5, 9, 11), S(5; 3, 8, 11), S(6; 4, 9, 12), S(8; 6, 7, 11), S(12; 3, 4, 5), S(12; 9, 10, 11). Thus $8 \in I(K_{12})$.

(2) Take $K_{12} = K_8 \oplus K_{4,8} \oplus K_4$. Let $V(K_8) = \{1, 2, 3, \ldots, 8\}$ and $V(K_4) = \{9, 10, 11, 12\}$. We can decompose K_8 into S(1; 4, 7, 8), S(2; 5, 7, 8), S(3; 6, 7, 8), (4, 5, 6), and 4 1-factors: {12, 34, 57, 68}, {13, 26, 47, 58}, {15, 23, 48, 67} and {16, 24, 35, 78}. The union of these four 1-factors and $K_{4,8}$ has a C_3-decomposition. In K_4, we have one copy of S_3, thus $4 \in I(K_{12})$.

(3) By Theorem 2.4, $22 \in I(K_{12})$.

Therefore $I(K_{12}) \supseteq J_{12}$.

Lemma 3.6. *Let the graph* M_1 *be the union of seven cycles,* $(1, 2, 7), (1, 3, 5)$, $(1, 4, 6)$, $(2, 3, 4), (5, 6, 8), (5, 7, 10)$, *and* $(6, 7, 9)$, *see Figure* 3.2. *Then* $I(M_1) \supseteq \{0, 3, 4, 5, 6, 7\}$.

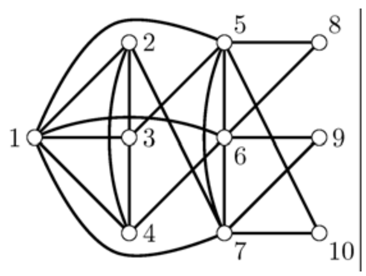

Figure 3.2. M_1.

Proof. We can decompose M_1 as follows:

(1) (1, 3, 5), (5, 6, 8), (5, 7, 10), (6, 7, 9), S(1; 2, 6, 7), S(2; 3, 4, 7), S(4; 1, 3, 6). Then $3 \in I(M_1)$.

(2) (5, 6, 8), (5, 7, 10), (6, 7, 9), S(1; 5, 6, 7), S(2; 1, 4, 7), S(3; 1, 2, 5), S(4; 1, 3, 6). Then $4 \in I(M_1)$.

(3) (5, 6, 8), (6, 7, 9), S(1; 3, 6, 7), S(2; 1, 3, 4), S(4; 1, 3, 6), S(5; 1, 3, 10), S(7; 2, 5, 10). Then $5 \in I(M_1)$.

(4) (5, 6, 8), S(1; 4, 6, 7), S(2; 1, 4, 7), S(3; 1, 2, 4), S(5; 1, 3, 10), S(6; 4, 7, 9), S(7; 5, 9, 10). Then $6 \in I(M_1)$.

(5) S(1; 5, 6, 7), S(2; 1, 4, 7), S(3; 1, 2, 5), S(4; 1, 3, 6), S(5; 7, 8, 10), S(6; 5, 8, 9), S(7; 6, 9, 10). Then $7 \in I(M_1)$.

Therefore, $I(M_1) \supseteq \{i|i = 0 \text{ or } 3 \leq i \leq 7\}$. ■

Lemma 3.7. $I(K_{13}) = J_{13}$.

Proof. Let (S, T) be a $\mathrm{STS}(13)$, where $S = (\{1, 2, 3, 4\} \times \{1, 2, 3\}) \cup \{\infty\}$ and the elements of T is defined as follows.

$$\begin{aligned}
&\text{Type 1}: \{(1,1),(1,2),(1,3)\},\ \{(2,1),(2,2),(2,3)\};\\
&\text{Type 2}: \{\infty,(3,i),(1,i+1)\},\ \{\infty,(4,i),(2,i+1)\},\ 1 \leq i \leq 3;\\
&\text{Type 3}: \{(1,i),(2,i),(3,i+1)\},\ \{(1,i),(3,i),(2,i+1)\},\\
&\qquad \{(1,i),(4,i),(4,i+1)\},\ \{(2,i),(3,i),(4,i+1)\},\\
&\qquad \{(2,i),(4,i),(1,i+1)\},\ \{(3,i),(4,i),(3,i+1)\},\ \text{for } 1 \leq i \leq 3.
\end{aligned}$$

(1) Pick two 7 copies of C_3 from T:

$\{(1,1),(1,2),(1,3)\}$, $\{\infty,(3,i),(1,i+1)\}$, $\{(3,i),(4,i),(3,i+1)\}$, for $1 \leq i \leq 3$, and $\{(2,1),(2,2),(2,3)\}$, $\{\infty,(4,i),(2,i+1)\}$, $\{(1,i),(4,i),(4,i+1)\}$, for $1 \leq i \leq 3$. The union of each 7 copies of C_3 forms a graph isomorphic to M_1 as in Figure 3.2, respectively.

By Lemma 3.6, $I(M_1) \supseteq \{0, 3, 4, 5, 6, 7\}$.

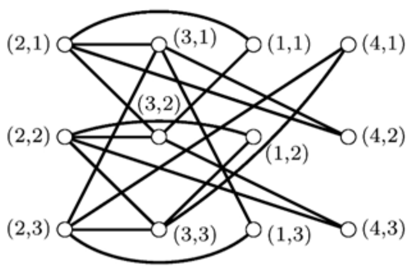

Figure 3.3. W.

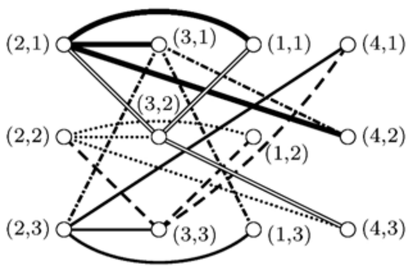

Figure 3.4. $W = 6S_3$.

(2) Pick two 6 copies of C_3 from T:

$\{(1,i),(2,i),(3,i{+}1)\}$, $\{(2,i),(3,i),(4,i{+}1)\}$, for $1 \le i \le 3$, and $\{(1,i),(3,i),(2,i+1)\}$, $\{(2,i),(4,i),(1,i+1)\}$, for $1 \le i \le 3$. The union of these 6 copies of C_3 forms a graph isomorphic to W as in Figure 3.3.

From Figure 3.4, there is a S_3-decomposition of W. Thus $I(W) \supseteq \{0,6\}$. Since K_{13}= $2M_1 \oplus 2W$, we conclude that $I(K_{13}) \supseteq 2{\cdot}I(M_1)+2{\cdot}I(W) \supseteq \{0,3,4,\ldots,26\} = \{i|i=0$ or $3 \le i \le 26\} = J_{13}$. ■

Lemma 3.8. *Let the graph* M_2 *be the union of 11 cycles,* $(1,2,3),(1,4,14)$, $(1,5,7)$, $(1,8,10)$, $(2,5,15)$, $(2,6,8)$, $(2,9,11)$, $(3,4,9)$, $(3,6,13)$, $(3,7,12)$, *and* $(4, 5,6)$, *as in Figure* 3.5. *Then* $I(M_2) \supseteq \{0,3,4,\ldots,11\}$.

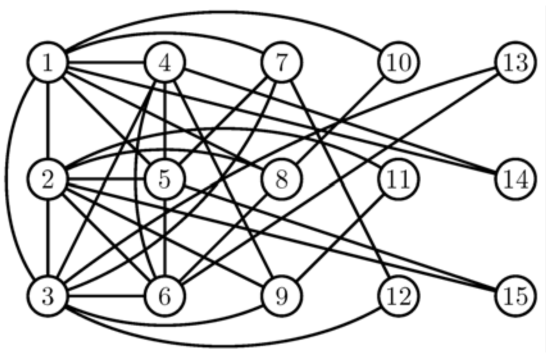

Figure 3.5. M_2.

Proof. We can decompose M_2 as follows:

(1) (1, 4, 14), (1, 5, 7), (1, 8, 10), (2, 5, 15), (2, 9, 11), (3, 4, 9), (3, 7, 12), (4, 5, 6), S(2; 1, 6, 8), S(3; 1, 2, 13), S(6; 3, 8, 13). Then $3 \in I(M_2)$.

(2) (1, 4, 14), (1, 8, 10), (2, 5, 15), (2, 9, 11), (3, 4, 9), (3, 6, 13), (3, 7, 12), S(1; 3, 5, 7), S(2; 1, 3, 8), S(5; 4, 6, 7), S(6; 2, 4, 8). Then $4 \in I(M_2)$.

(3) (1, 4, 14), (1, 8, 10), (2, 5, 15), (2, 9, 11), (3, 7, 12), (3, 4, 9), S(1; 3, 5, 7), S(2; 1, 6, 8), S(3; 2, 6, 13), S(5; 4, 6, 7), S(6; 4, 8, 13). Then $5 \in I(M_2)$.

(4) (1, 8, 10), (2, 5, 15), (2, 9, 11), (3, 6, 13), (3, 7, 12), S(1; 4, 7, 14), S(2; 1, 3, 8), S(3; 1, 4, 9), S(4; 5, 9, 14), S(5; 1, 6, 7), S(6; 2, 4, 8). Then $6 \in I(M_2)$.

(5) (1, 8, 10), (2, 9, 11), (3, 6, 13), (3, 7, 12), S(1; 2, 3, 4), S(1; 5, 7, 14), S(2; 5, 8, 15), S(3; 2, 4, 9), S(4; 5, 9, 14), S(5; 6, 7, 15), S(6; 2, 4, 8). Then $7 \in I(M_2)$.

(6) (1, 8, 10), (2, 9, 11), (3, 7, 12),S(1; 2, 7, 14), S(2; 6, 8, 15), S(3; 1, 2, 4), S(3; 6, 9, 13), S(4; 1, 9, 14), S(5; 1, 2, 4), S(5; 6, 7, 15), S(6; 4, 8, 13). Then $8 \in I(M_2)$.

(7) (1, 8, 10), (2, 9, 11), S(1; 2, 3, 4),S(1; 5, 7, 14), S(2; 6, 8, 15), S(3; 2, 4, 6), S(3; 9, 12, 13), S(4; 6, 9, 14), S(5; 2, 4, 15), S(6; 5, 8, 13), S(7; 3, 5, 12). Then $9 \in I(M_2)$.

(8) (2, 9, 11), S(1; 4, 5, 7), S(1; 8, 10, 14), S(2; 1, 6, 15), S(3; 1, 2, 6), S(3; 9, 12, 13), S(4; 3, 9, 14), S(5; 2, 4, 15), S(6; 4, 5, 13), S(7; 3, 5, 12), S(8; 2, 6, 10). Then $10 \in I(M_2)$.

(9) S(1; 3, 5, 7), S(1; 8, 10, 14), S(2; 1, 3, 6), S(2; 9, 11, 15), S(3; 6, 12, 13), S(4; 1, 3, 14), S(5; 2, 4, 15), S(6; 4, 5, 13), S(7; 3, 5, 12), S(8; 2, 6, 10), S(9; 3, 4, 11). Then $11 \in I(M_2)$.

Therefore, $I(M_2) \supseteq \{0, 3, 4, \ldots, 11\}$. ∎

Lemma 3.9. $I(K_{15}) = J_{15}$.

Proof. Let (S, T) be a STS(15), where $S = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\}$ and the elements of T is defined as follows.

Type 1 : $\{(i, 1), (i, 2), (i, 3)\}$, for $1 \le i \le 5$;

Type 2 : $\{\{(1, i), (2, i), (4, i+1)\}, \{(1, i), (3, i), (2, i+1)\}, \{(1, i), (4, i), (5, i+1)\},$
$\{(1, i), (5, i), (3, i+1)\}, \{(2, i), (3, i), (5, i+1)\}, \{(2, i), (4, i), (3, i+1)\},$
$\{(2, i), (5, i), (1, i+1)\}, \{(3, i), (4, i), (1, i+1)\}, \{(3, i), (5, i), (4, i+1)\},$
$\{(4, i), (5, i), (2, i+1)\}\}$, for $1 \le i \le 3$.

(1) We pick 11 copies of C_3: $\{(1, 1), (1, 2), (1, 3)\}, \{(2, 1), (2, 2), (2, 3)\}, \{(1, i), (2, i), (4, i+1)\}, \{(1, i), (3, i), (2, i+1)\}, \{(1, i), (4, i), (5, i+1)\}$, for $1 \le i \le 3$. The union of these 11 C_3 forms a copy of M_2 as in Figure 3.5. By Lemma 3.8, we have $I(M_2) \supseteq \{0, 3, 4, \ldots, 11\}$.

(2) Since $K_{15} = K_2 \oplus K_{2,13} \oplus K_{13}$ and $K_2 \oplus K_{2,13}$ can be decomposed into one C_3 and 8 copies of S_3, we have $8 \in I(K_2 \oplus K_{2,13})$. By Example 3.4, we have $I(K_{13}) = \{0, 3, 4, \ldots, 26\}$. Thus $I(K_{15}) \supseteq I(K_2 \oplus K_{2,13}) + I(K_{13}) \supseteq \{8, 11, 12, \ldots, 34)\}$.

(3) By Theorem 2.4, $35 \in I(K_{15})$.

From (1), (2), and (3), we conclude that $I(K_{15}) \supseteq \{0, 3, 4, \ldots, 35\} = \{i | i = 0$ or $3 \le i \le 35\} = J_{15}$. ∎

4. Main Theorem for Odd n

In this section, we will consider $n \ge 19$ and $n \equiv 0, 1 \pmod 3$. From Theorem 2.3, we get a commutative quasigroup of order $2n$ with holes H by using $\mathrm{STS}(2n+1)$ or $\mathrm{PBD}(2n+1, \{3, 5\})$. In the L-Construction, we use a commutative quasigroup of order $2n$ with holes H to get a $\mathrm{STS}(6n+1)$, a $\mathrm{STS}(6n+3)$ and a $\mathrm{PBD}(6n+5, \{3, 5\})$.

In the proof of Theorem 2.3 (b), we have that for $1 \leq i \neq j \leq 2n$, $\{i,j\} \notin H$ and $\{i,j\} \not\subset b$, let $\{i,j,k\}$ be the triple in B containing symbols i and j and define $i \circ j = k = j \circ i$, and in L-Construction (a)(2), for $1 \leq i \neq j \leq 2n$, $\{i,j\} \notin H$, place the triples $\{(i,1),(j,1),(i\circ j,2)\}$, $\{(i,2),(j,2),(i\circ j,3)\}$, and $\{(i,3),(j,3),(i\circ j,1)\}$ in B'. Thus for each $\{i,j\} \notin H$ if $\{i,j,k\}$ is a triple in B, then we obtain three triangles: $((i,1),(j,1),(k,2))$, $((k,1),(j,1),(i,2))$, and $((i,1),(k,1),(j,2))$. The graph G corresponding to this three triangles is shown in Figure 3.6.

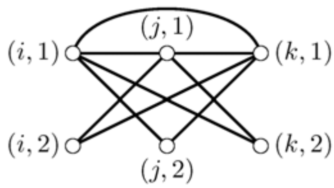

Figure 3.6. G.

Lemma 4.1. *Let the graph G be the union of three cycles, $((i,1),(j,1),(k,2))$, $((k,1),(j,1),(i,2))$, and $((i,1),(k,1),(j,2))$, shown in Figure 3.6. Then there is an S_3-decomposition of G, i.e., $I(G) \supseteq \{0,3\}$.*

Proof. There is an S_3-decomposition of G: $S((i,1);(j,1),(j,2),(k,2))$, $S((j,1);(i,2),(k,1),(k,2))$, $S((k,1);(i,1),(i,2),(j,2))$. ■

Lemma 4.2. *If $n \equiv 1 \pmod 6$ and $n \geq 19$, then $I(K_n) = J_n = \{i | i = 0$ or $3 \leq i \leq \frac{n(n-1)}{6}\}$.*

Proof. Let $n = 6k+1$ and $k \geq 3$.

(a) If $2k \equiv 0,2 \pmod 6$, then $2k+1 \equiv 1,3 \pmod 6$. There exists a $\mathrm{STS}(2k+1)$. Using this $\mathrm{STS}(2k+1)$, we can get a commutative quasigroup $(Q,\circ)$ with holes of order $2k$. By L-Construction, there are k edge-disjoint copies of $\mathrm{STS}(7)$ and there are $\frac{2k(k-1)}{3}$ triples not containing $2k+1$ in $\mathrm{STS}(2k+1)$. For each triple $\{r,s,t\}$ not containing $2k+1$, there are three copies of G which is shown in Figure 3.6. By Example 3.2 and Lemma 4.1, we have $I(K_7) = \{0,3,4,5,6,7\}$ and $I(G) \supseteq \{0,3\}$. Thus, we obtain that

$$\begin{aligned} I(K_n) &\supseteq k \cdot I(K_7) + \frac{2k(k-1)}{3} \cdot (3 \cdot I(G)) \\ &\supseteq k \cdot \{0,3,4,5,6,7\} + 2k(k-1) \cdot \{0,3\} \\ &= \{i | i = 0 \text{ or } 3 \leq i \leq k(6k+1)\} = \left\{ i | i = 0 \text{ or } 3 \leq i \leq \frac{n(n-1)}{6} \right\} = J_n. \end{aligned}$$

(b) If $2k \equiv 4 \pmod 6$, then $2k+1 \equiv 5 \pmod 6$. There exists a $\mathrm{PBD}(2k+1)$ with only one block size 5 and the rest of 3. By L-Construction, there are k edge-disjoint copies of $\mathrm{STS}(7)$, and there are $\frac{2}{3}(k^2-k-5)$ triples in $\mathrm{PBD}(2k+1)$ not containing $2k+1$. Let $\{1,2,3,4,5\}$ be the block of size 5 in $\mathrm{PBD}(2k+1)$. Let $(\{1,2,3,4,5\}, \otimes)$ be defined as follows.

$\otimes$	1	2	3	4	5
1	1	4	2	5	3
2	4	2	5	3	1
3	2	5	3	1	4
4	5	3	1	4	2
5	3	1	4	2	5

The triples corresponding to this idempotent commutative quasigroup of order 5 can be grouped into 5 pairs: $\{\{1,2,4\},\{1,3,2\}\}$, $\{\{1,4,5\},\{1,5,3\}\}$, $\{\{2,3,5\},\{2,4,3\}\}$, $\{\{3,4,1\},\{3,5,4\}\}$, and $\{\{2,5,1\},\{4,5,2\}\}$ where the pair $\{\{a,b,c\},\{d,e,f\}\}$ means $\{\{(a,i),(b,i),(c,i+1)\}$,$\{(d,i),(e,i),(f,i+1)\}\}$ for
$1 \le i \le 3$. The graph corresponding to each pair is isomorphic to the graph W, (see Figure 3.3) and $I(W) \supseteq \{0,6\}$.

Thus, we obtain that

$$\begin{aligned} I(K_n) &\supseteq k \cdot I(K_7) + \frac{2}{3}(k^2-k-5)\cdot(3\cdot I(G)) + 5\cdot I(W) \\ &\supseteq k\cdot\{0,3,4,5,6,7\} + 2(k^2-k-5)\cdot\{0,3\} + 5\cdot\{0,6\} \\ &= \{i \mid i=0 \ or \ 3\le i\le k(6k+1)\} = \left\{i \mid i=0 \ or \ 3\le i\le \frac{n(n-1)}{6}\right\} = J_n. \ \blacksquare \end{aligned}$$

Lemma 4.3. *If* $n \equiv 3 \pmod 6$ *and* $n \ge 21$, *then* $I(K_n) = J_n = \{i \mid i = 0$ *or* $3 \le i \le \frac{n(n-1)}{6}\}$.

Proof. Let $n = 6k+3$ and $k \ge 3$.

(a) If $2k \equiv 0,2 \pmod 6$, then $2k+1 \equiv 1,3 \pmod 6$. There exists an $\mathrm{STS}(2k+1)$. By L-Construction, there are k copies of $\mathrm{STS}(9)$ in which $\{\infty_1,\infty_2,\infty_3\}$ is a common triple and there are $\frac{2k(k-1)}{3}$ copies of triple not containing $2k+1$ in $\mathrm{STS}(2k+1)$. By Example 3.3, $I(K_9) = \{0,3,4,\ldots,12\}$ and $I(K_9 \backslash C_3) = \{0,3,4,\ldots,11\}$. Thus, we have

$$
\begin{aligned}
I(K_n) &\supseteq 1 \cdot I(K_9) + (k-1) \cdot I(K_9 \backslash C_3) + \frac{2}{3}k(k-1) \cdot (3 \cdot I(G)) \\
&\supseteq \{0,3,4,\ldots,12\} + (k-1) \cdot \{0,3,4,\ldots,11\} + 2k(k-1) \cdot \{0,3\} \\
&\supseteq \{i | i=0 \ or \ 3 \leq i \leq (2k+1)(3k+1)\} = \left\{ i | i=0 \ or \ 3 \leq i \leq \frac{n(n-1)}{6} \right\} \\
&= J_n.
\end{aligned}
$$

(b) If $2k \equiv 4 \pmod 6$, then $2k+1 \equiv 5 \pmod 6$. There exists a $\mathrm{PBD}(2k+1)$ with only one block size 5 and the rest of size 3. By L-Construction, there are k edge-disjoint copies of $\mathrm{STS}(9)$ containing a common triple and there are $\frac{2}{3}(k^2-k-5)$ triples in $\mathrm{PBD}(2k+1)$ not containing $2k+1$. As in the proof of Lemma 4.2 (b), we have

$$
\begin{aligned}
&I(K_n) \\
&\supseteq 1 \cdot I(K_9) + (k-1) \cdot I(K_9 \backslash C_3) + \frac{2}{3}(k^2-k-5) \cdot (3 \cdot I(G)) + 5 \cdot I(W) \\
&\supseteq \{0,3,4,\ldots,12\} + (k-1) \cdot \{0,3,4,\ldots,11\} + 2(k^2-k-5) \cdot \{0,3\} \\
&\quad + 5 \cdot \{0,6\} \\
&= \{i | i=0 \ or \ 3 \leq i \leq (2k+1)(3k+1)\} \\
&= \left\{ i \,\middle|\, i=0 \ or \ 3 \leq i \leq \frac{n(n-1)}{6} \right\} = J_n.
\end{aligned}
$$

■

By Examples 3.2, 3.3, and Lemmas 3.7, 3.9, 4.2 and 4.3, we have the following result.

Theorem 4.4. *If* $n \equiv 1,3 \pmod 6$ *and* $n \geq 7$, *then* $I(K_n) = J_n = \{i | i = 0$ *or* $3 \leq i \leq \frac{n(n-1)}{6}\}$.

5. Main Theorem for Even n

In this section we will concern that n is an even integer. A Skolem triple system of order t is a partition of the set $\{1,2,\ldots,3t\}$ into triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for each $i = 1,2,\ldots,t$.

Theorem 5.1. ([8]). *A Skolem triple system of order* t *exists if and only if* $t \equiv 0$ *or* $1 \pmod 4$.

Let $n \geq 2$ be an integer and let $D \subseteq \{1,2,\ldots,\lfloor n/2 \rfloor\}$. The circulant graph $\langle D \rangle_n$ is the graph with vertices $V = Z_n$ and edges $E = \{\{i,j\} || i-j| \in D$ or $n-|i-j| \in D\}$. For all $t \equiv 0,1 \pmod 4$, A Skolem triple system provides a partition of $\{1,2,\ldots,3t\}$ into t triples giving a cyclic 3-cycle system of $K_{6t+1} = \langle \{1,2,\ldots,3t\} \rangle_{6t+1}$.

Theorem 5.2. ([2]). *Let s, t and n be integers with $s < t < n/2$. If $\gcd(s,t,n) = 1$, then the graph $\langle\{s,t\}\rangle_n$ can be decomposed into two Hamilton cycles. If n is even, then the graph $\langle\{s,t\}\rangle_n$ can be decomposed into four 1-factors.*

Lemma 5.3. *Let k be a positive integer and $k \geq 1$.*

(i) *If $n = 6k+4$, then $I(K_n) \supseteq \{2k+1, 2k+2, 2k+3, 2k+4, \ldots, 6k^2+7k+2\}$.*

(ii) *If $n = 6k+6$, then $I(K_n) \supseteq \{6k+3, 6k+4, 6k+5, 6k+6, 6k+7, 6k+8, \ldots, 6k^2+11k+4, 6k^2+11k+5\}$.*

Proof.

(i) It is easy to see that $I(K_{1,6k+3}) = \{2k+1\}$. By Theorem 4.4, we have $I(K_{6k+3}) = \{0, 3, 4, 5, \ldots, 6k^2+5k+1\}$. If we take $K_{6k+4} = K_{1,6k+3} \oplus K_{6k+3}$, then $I(K_{6k+4}) \supseteq I(K_{1,6k+3}) + I(K_{6k+3}) = \{2k+1, 2k+4, 2k+5, 2k+6, \ldots, 6k^2+7k+2\}$.

(ii) It is easy to see that $I(K_{3,6k+3}) = \{6k+3\}$. If we take $K_{6k+6} = K_3 \oplus K_{3,6k+3} \oplus K_{6k+3}$, then $I(K_{6k+6}) \supseteq I(K_3) + I(K_{3,6k+3}) + I(K_{6k+3})$. By Theorem 2.4, there is an S_3-decomposition of K_{6k+6}, thus $6k^2+11k+5 \in I(K_{6k+6})$. Therefore, $I(K_{6k+6}) \supseteq \{6k+3, 6k+6, 6k+7, 6k+8, \ldots, 6k^2+11k+4, 6k^2+11k+5\}$.

By L-Construction, there is a subsystem $\mathrm{STS}(9)$ in $\mathrm{STS}(6k+3)$. As in Example 3.4, we can get a subgraph which is the union of 4 triangles and three S_3 in K_{6k+4}. Thus $2k+2, 2k+3 \in I(K_{6k+4})$ and $6k+4, 6k+5 \in I(K_{6k+6})$. ■

Lemma 5.4. *Let k be a positive integer, $n = 6k+4$, and $n \geq 40$. Then $I(K_n) \supseteq \{q \,|\, \lceil \frac{n}{4} \rceil \leq q \leq 2k$, q is an integer$\}$.*

Proof.

(1) If k is odd and $k = 2m+1$, where m is an integer, then $n = 12m+10$. Since $n \geq 40$, we have $m \geq 3$. By Theorem 2.2, there is a 3-GDD of type $12^m 10^1$, i.e., there is a PBD($n, \{12, 10, 3\}$). By Examples 3.4 and 3.5, $I(K_{12}) = \{3, 4, 5, \ldots, 22\}$ and $I(K_{10}) = \{3, 4, 5, \ldots, 15\}$. Thus,

$$\begin{aligned} I(K_{12m+10}) &\supseteq m \cdot I(K_{12}) + I(K_{10}) = \{3m+3, 3m+4, \ldots, 22m+15\} \\ &\supseteq \left\{ q \,\middle|\, \left\lceil \frac{n}{4} \right\rceil \leq q \leq 2k \right\}. \end{aligned}$$

(2) If k is even and $k = 2m$, where m is an integer, then $n = 12m+4$. Since $n \geq 40$, we have $m \geq 3$. By Theorem 2.2, there is a 3-GDD of type $12^m 4^1$, i.e., there is a PBD($n, \{12, 4, 3\}$). By Example 3.5, $I(K_{12}) = \{3, 4, 5, \ldots, 22\}$. Thus,

$$I(K_{12m+4}) \supseteq m \cdot I(K_{12}) + I(K_4) = \{3m+1, 3m+2, \ldots, 22m+1\}$$
$$\supseteq \left\{q \,\middle|\, \left\lceil \frac{n}{4} \right\rceil \leq q \leq 2k \right\}. \qquad \blacksquare$$

Lemma 5.5. *Let* $n = 6k + 4$ *and* $16 \leq n \leq 34$*. Then* $I(K_n) = J_n$.

Proof.

(1) $n = 16$. Take $K_{16} = 4K_4 \oplus K_{4(4)}$. By Theorem 2.2, $K_{4(4)}$ has a C_3-decomposition, we conclude that $4 \in I(K_{16})$.

(2) $n = 22$.

(a) By Lemma 5.3 (i), we have $I(K_{22}) \supseteq \{7, 8, \ldots, 77\}$.

(b) K_{22} can be decomposed as follows: S(2; 1, 5, 6), S(3; 1, 7, 8), S(4; 1, 9, 10), S(11; 12, 13, 14), S(15; 16, 17, 18), S(19; 20, 21, 22), (1, 5, 11), (1, 6, 12), (1, 7, 13), (1, 8, 15), (1, 9, 19), (1, 10, 16), (1, 14, 18), (1, 17, 20), (1, 21, 22), (2, 3, 17), (2, 4, 11), (2, 7, 10), (2, 8, 18), (2, 9, 15), (2, 12, 20), (2, 13, 21), (2, 14, 22), (2, 16, 19), (3, 4, 12), (3, 5, 22), (3, 6, 18), (3, 9, 10), (3, 11, 19), (3, 13, 20), (3, 14, 16), (3, 15, 21), (4, 5, 7), (4, 6, 8), (4, 13, 18), (4, 14, 19), (4, 15, 20), (4, 16, 22), (4, 17, 21), (5, 6, 9), (5, 8, 20), (5, 10, 15), (5, 12, 18), (5, 13, 19), (5, 14, 21), (5, 16, 17), (6, 7, 16), (6, 10, 21), (6, 11, 17), (6, 13, 14), (6, 15, 19), (6, 20, 22), (7, 8, 19), (7, 9, 12), (7, 11, 15), (7, 14, 17), (7, 18, 22), (7, 20, 21), (8, 9, 21), (8, 10, 14), (8, 11, 16), (8, 12, 22), (8, 13, 17), (9, 11, 22), (9, 13, 16), (9, 14, 20), (9, 17, 18), (10, 11, 20), (10, 12, 13), (10, 17, 22), (10, 18, 19), (11, 18, 21), (12, 14, 15), (12, 16, 21), (12, 17, 19), (13, 15, 22), (16, 18, 20). Thus $6 \in I(K_{22})$.

(3) $n = 28$.

(a) By Lemma 5.3 (i), we have $I(K_{28}) \supseteq \{9, 10, \ldots, 126\}$.

(b) Take $K_{28} = 4K_4 \oplus K_{12} \oplus K_{4(4),12}$. By Theorem 2.2, there is a 3-GDD of type $4^4 12^1$, i.e., there is a PBD(28, $\{12, 4, 3\}$), Since $4 \in I(4K_4)$ and $3, 4 \in I(K_{12})$ we have $7, 8 \in I(K_{28})$.

(4) $n = 34$.

(a) By Lemma 5.3 (i), we have $I(K_{34}) \supseteq \{11, 12, \ldots, 187\}$.

(b) Take $K_{34} = K_{10} \oplus K_{10,24} \oplus K_{24}$. Partition the difference set $D = \{1, 2, \ldots, 12\}$ of K_{24} into $\{1, 7, 9, 10, 11\}$, $\{2, 3, 5\}$, $\{4, 8\}$, and $\{6, 12\}$. By Theorem 5.2, $\langle\{1, 7, 9, 10, 11\}\rangle_{24}$ can be decomposed into 10 1-factors. Then $\langle\{1, 7, 9, 10, 11\}\rangle_{24} \cup K_{10,24}$ has a C_3-decomposition. Both $\langle\{2, 3, 5\}\rangle_{24}$ and $\langle\{4, 8\}\rangle_{24}$ have a C_3-decomposition. $\langle\{6, 12\}\rangle_{24}$ is a K_4-factor (6 copies of K_4). Since $3 \in I(K_{10})$, we have $9 \in I(K_{34})$.

(c) Take $K_{34} = K_6 \oplus K_{6,28} \oplus K_{28}$. Partition the difference set $D = \{1, 2, \ldots, 14\}$ of K_{28} into $\{1, 12, 13\}$, $\{2, 6, 8\}$, $\{4, 5, 9\}$, $\{3, 10, 11\}$, and $\{7, 14\}$. By Theorem 5.2, $\langle\{3, 10, 11\}\rangle_{28}$ can be decomposed into 6 1-factors. $\langle\{3, 10, 11\}\rangle_{28} \cup K_{6,28}$ has a C_3-decomposition. $\langle\{1, 12, 13\}\rangle_{28}$, $\langle\{2, 6, 8\}\rangle_{28}$, and $\langle\{4, 5, 9\}\rangle_{28}$ have a C_3-decomposition. $\langle\{7, 14\}\rangle_{28}$ is a K_4 factor (7 copies of K_4). Since $3 \in I(K_6)$, we have $10 \in I(K_{34})$. ■

Combine Example 3.4, Lemmas 5.3, 5.4 and 5.5, we get the following result.

Theorem 5.6. *Let n be a positive integer, $n \equiv 4 \pmod 6$ and $n \geq 10$. Then $I(K_n) = J_n = \{q | \lceil \frac{n}{4} \rceil \leq q \leq \frac{n(n-1)}{6}$, q is an integer$\}$.*

Lemma 5.7. *Let k be a positive integer, $n = 6k + 6$, and $n \geq 36$. Then $I(K_n) \supseteq \{\lceil \frac{n}{4} \rceil, \lceil \frac{n}{4} \rceil + 1, \ldots, 6k + 2\}$.*

Proof.

(1) If k is odd and $k = 2m + 1$, where m is an integer, then $n = 12(m + 1)$. Since $n \geq 36$, we have $m \geq 2$. By Theorem 2.2, there is a 3-GDD of type 12^{m+1}, i.e., there is a PBD($n, \{12, 3\}$). By Example 3.5, $I(K_{12}) = \{3, 4, 5, \ldots, 22\}$. Thus $I(K_{12m+12}) \supseteq (m + 1) \cdot I(K_{12}) = \{3m + 3, 3m + 4, \ldots, 22m + 22\} \supseteq \{\lceil \frac{n}{4} \rceil, \lceil \frac{n}{4} \rceil + 1, \ldots, 6k + 2\}$.

(2) If k is even and $k = 2m$, where m is an integer, then $n = 12m+6$. Since $n \geq 36$, we have $m \geq 3$. By Theorem 2.2, there is a 3-GDD of type $12^m 6^1$, i.e., there is a PBD($n, \{12, 6, 3\}$). By Examples 3.5 and 3.1, $I(K_{12}) = \{3, 4, 5, \ldots, 22\}$ and $I(K_6) = \{3, 4, 5\}$. Thus $I(K_{12m+6}) \supseteq m \cdot I(K_{12}) + I(K_6) = \{3m + 3, 3m + 4, \ldots, 22m + 5\}$. Next, we will get $3m + 2 \in I(K_{12m+6})$.

By [1, Theorem 8.3.3], we can get a Skolem triple system of order $4r + 1$, for $r \geq 1$. Let T be a Skolem triple system of order $4r + 1$ where T is $\{\{1, 12r+2, 12r+3\}, \{2t+1, 10r-t, 10r+t+1\}, \{2r+2t-1, 5r-t+1, 7r+t\}, \{4r-1, 5r+1, 9r\}, \{4r+1, 8r, 12r+1\}, \{2r, 10r, 12r\}, \{2t, 6r-t+1, 6r+t+1\}, \{2r + 2t, 9r - t, 11r + t\}, \{4r, 6r + 1, 10r + 1\} | 1 \leq t \leq r - 1\}$.

(a) If $m = 2r + 1$, then $n = 24r + 18$. Let $K_{24r+18} = K_{10} \oplus K_{10,24r+8} \oplus K_{24r+8}$. Since T is a partition of $\{1, 2, \ldots, 12r + 3\}$ into $4r + 1$ triples, K_{24r+7} has a C_3-decomposition. Since the difference set of K_{24r+8} is $D = \{1, 2, \ldots, 12r + 4\}$, D can be partitioned into T and $\{12r + 4\}$. From T, pick two triples $(1, 12r + 2, 12r + 3)$, $(2, 6r, 6r + 2)$ with $\{12r + 4\}$, we can get two sets $\{1, 6r, 12r + 2, 12r + 3, 2\}$ and $\{6r + 2, 12r + 4\}$. By Theorem 5.2, the graphs $\langle\{1, 6r\}\rangle_{24r+8}$ and $\langle\{12r+2, 12r+3\}\rangle_{24r+8}$ can be decomposed into two Hamilton cycles, i.e., four 1-factors respectively. Thus $\langle\{1, 6r, 12r+2, 12r+3, 2\}\rangle_{24r+8} \cup K_{10,24r+8}$ has a C_3-decomposition. The graph $\langle\{6r + 2, 12r + 4\}\rangle_{24r+8}$ is a K_4-factor. Thus there are $6r + 2$ copies of S_3 in the decomposition of one K_4-factor and $3 \in I(K_{10})$. Therefore, $6r + 5 = 3m + 2 = \lceil \frac{n}{4} \rceil \in I(K_{12m+6})$.

(b) If $m = 2r$, then $n = 24r + 6$. $K_{24r+6} = K_{10} \oplus K_{24r-4} \oplus K_{10,24r-4}$. The difference set of K_{24r-4} is $D = \{1, 2, \ldots, 12r - 2\}$. D can be partitioned into triples in T except those triples containing $12r - 1, 12r, \ldots, 12r + 3$ and $R = \{1, 4r+1, 8r, 2r, 10r, 4r-2, 8r+1\}$. Pick two triples $(4, 6r-1, 6r+3)$ and $(4r-4, 8r+2, 12r-2\}$ from D union R to get the set $A = \{1, 4r+1, 8r, 2r, 10r, 4r-2, 8r+1, 4, 6r-1, 6r+3, 4r-4, 8r+2, 12r-2\}$. Then A can be partitioned into 6 subsets: $\{2r, 8r, 10r\}$, $\{1, 8r+1, 8r+2\}$, $\{4r-4, 6r+3\}$, $\{4, 4r+1\}$, $\{4r-2\}$, $\{6r-1, 12r-2\}$. $\langle\{2r, 8r, 10r\}\rangle_{24r-4}$ and $\langle\{1, 8r + 1, 8r + 2\}\rangle_{24r-4}$ have C_3-decomposition. By Theorem 5.2, $\langle\{4r - 4, 6r + 3, 4, 4r + 1, 4r - 2\}\rangle_{24r-4}$ can be decomposed into 10 1-factors, thus $\langle\{4r - 4, 6r + 3, 4, 4r + 1, 4r - 2\}\rangle_{24r-4} \cup K_{10,24r-4}$ has a C_3-decomposition. $\langle\{6r-1, 12r-2\}\rangle_{24r-4}$ is a K_4-factor (contains $6r-1$ copies of K_4) and $3 \in I(K_{10})$. Therefore, $6r + 2 = 3m + 2 = \lceil \frac{n}{4} \rceil \in I(K_{12m+6})$.

From (a), and (b), we get $I(K_{12m+6}) \supseteq \{3m+2, 3m+3, 3m+4, \ldots, 22m+5\} \supseteq \{\lceil \frac{n}{4} \rceil, \lceil \frac{n}{4} \rceil + 1, \ldots, 6k + 2\}$. ■

Lemma 5.8. *Let* $n \equiv 0 \pmod 6$ *and* $18 \leq n \leq 30$. *Then* $I(K_n) = J_n$.

Proof.

(1) $n = 18$.

(a) By Lemma 5.3 (ii), we have $I(K_{18}) \supseteq \{15, 16, \ldots, 51\}$.

(b) Let $K_{18} = K_6 \oplus K_{6,12} \oplus K_{12}$. Partition the difference set $D = \{1, 2, \ldots, 6\}$ of K_{12} into $\{1, 2, 5\}$, $\{3, 6\}$, $\{4\}$. By Theorem 5.2, $\langle\{1, 2, 5\}\rangle_{12}$ can be decomposed into 6 1-factors, thus $\langle\{1, 2, 5\}\rangle_{12} \cup K_{6,12}$ has a C_3-decomposition. $\langle\{4\}\rangle_{12}$ has a C_3-decomposition. $\langle\{3, 6\}\rangle_{12}$ is a K_4-factor (3 copies of K_4). Since $I(K_6) = \{3, 4, 5\}$, we have $6, 7, 8 \in I(K_{18})$.

(c) Let $K_{18} = 3K_6 \oplus K_{6,6,6}$. By Theorem 2.2, $K_{6,6,6}$ has a C_3-decomposition. Since $I(K_6) = \{3, 4, 5\}$, we have $I(K_{18}) \supseteq 3 \cdot I(K_6) = \{9, 10, 11, \ldots, 15\}$.

(d) K_{18} can be decomposed as follows: S(1; 2, 3, 4), S(4; 8, 9, 10), S(5; 4, 6, 7), S(11; 12, 13, 14), S(15; 16, 17, 18) (1, 5, 9), (1, 6, 11), (1, 7, 13), (1, 8, 15), (1, 10, 16), (1, 12, 18), (1, 14, 17), (2, 3, 18), (2, 4, 17), (2, 5, 16), (2, 6, 12), (2, 7, 14), (2, 8, 13), (2, 9, 15), (2, 10, 11), (3, 4, 12), (3, 5, 17), (3, 6, 14), (3, 7, 15), (3, 8, 16), (3, 9, 11), (3, 10, 13), (4, 6, 15), (4, 7, 18), (4, 11, 16), (4, 13, 14), (5, 8, 10), (5, 11, 15), (5, 12, 14), (5, 13, 18) (6, 7, 16), (6, 8, 18), (6, 9, 13), (6, 10, 17), (7, 8, 11), (7, 9, 17), (7, 10, 12), (8, 9, 14), (8, 12, 17), (9, 10, 18), (9, 12, 16), (10, 14, 15), (11, 17, 18), (12, 13, 15), (13, 16, 17), (14, 16, 18). Thus $5 \in I(K_{18})$. Therefore, $I(K_{18}) \supseteq \{5, 6, 7, \ldots, 51\} = J_{18}$.

(2) $n = 24$.

(a) By Lemma 5.3 (ii), we have $I(K_{24}) \supseteq \{21, 22, \ldots, 92\}$.

(b) Let $K_{24} = 6K_4 \oplus K_{6(4)}$. By Theorem 2.2, $K_{6(4)}$ can be decomposed into C_3. Thus $6 \in I(K_{24})$.

(c) Let $K_{24} = K_{12} \oplus K_{1,12} \oplus (K_{11,12} \oplus K_{12})$. Since K_{12} can be decomposed into 11 1-factors, $K_{11,12} \oplus K_{12}$ has a C_3-decomposition. By Example 3.5, $I(K_{12}) = \{3, 4, 5, \ldots, 22\}$ and $4 \in I(K_{1,12})$, we have $7, 8, 9, \ldots, 26 \in I(K_{24})$. Therefore $I(K_{24}) \supseteq \{6, 7, 8, \ldots, 92\} = J_{24}$.

(3) $n = 30$.

(a) By Lemma 5.3 (ii), we have $I(K_{30}) \supseteq \{27, 28, \cdots, 145\}$.

(b) Let $K_{30} = 3K_{10} \oplus K_{10,10,10}$. By Example 3.4, $I(K_{10}) = \{3, 4, 5, \ldots, 15\}$. Thus $9, 10, 11, \ldots, 45 \in I(K_{30})$.

(c) Let $K_{30} = K_{10} \oplus K_{10,20} \oplus K_{20}$. Partition the difference set $D = \{1, 2, \ldots, 10\}$ of K_{20} into $\{1, 3, 4\}$, $\{2, 6, 7, 8, 9\}$, $\{5, 10\}$. $\langle\{1, 3, 4\}\rangle_{20}$ has a C_3-decom-position. $\langle\{2, 6, 7, 8, 9\}\rangle_{20}$ can be decomposed into 10 1-factors, thus $\langle\{1, 3, 7, 8, 9\}\rangle_{20} \cup K_{10,20}$ has a C_3-decomposition. $\langle\{5, 10\}\rangle_{20}$ is a K_4-factor (5 copies of K_4). Since $3 \in I(K_{10})$, we have $8 \in I(K_{30})$. Therefore, $I(K_{30}) \supseteq \{8, 9, \ldots, 145\} = J_{30}$. ■

Combine Examples 3.1, 3.5, Lemmas 5.3, 5.7 and 5.8, we get the following result.

Theorem 5.9. *Let* $n \equiv 0 \pmod 6$ *and* $n \geq 6$. *Then*

$$I(K_n) = J_n = \left\{ q \,\middle|\, \max\left\{3, \left\lceil \frac{n}{4} \right\rceil\right\} \leq q \leq \frac{n(n-1)}{6},\ q \text{ is an integer} \right\}.$$

References

1. I. Anderson, *Combinatorial Designs: Construction Methods*, Ellis Horwood Limited, England, 1990.

2. J. C. Bermond, O. Favaron and M. Matheo, Hamiltonian decomposition of Cayley graphs of degree 4, *J. Combin. Theory Ser. B*, **46** (1989), 142-153.

3. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan Press, London, 1976.

4. C. J. Colbourn, D. G. Hoffman and R. Rees, A new class of group divisible designs with block size three, *J. Combin. Theory Ser. A*, **59** (1992), 73-89.

5. T. P. Kirkman, On a problem in combinations, *Cambridge and Dublin Math. Journal*, **2** (1847), 191-204.

6. C. C. Lindner and C. A. Rodger, *Design Theory*, Boca Raton, Florida, CRC Press, 1997.

7. T. W. Shyu, *Decomposition of complete graphs into cycles and stars*, Graphs and Combinatorics, 2011, pp. 1-13, doi:10.1007/s00373-011-1105-3.

8. T. Skolem, On certain distributions of integers in pairs with given differences, *Math. Scand.*, **5** (1957), 57-68.

9. M. Tarsi, Decomposition of complete multigraph into stars, *Discrete Math.*, **26** (1979), 273-278.

10. S. Yamamoto, H. Ikeda, S. Shige-ede, K. Ushio and N. Hamada, On claw decomposition of complete graphs and complete bipartite graphs, *Hiroshima Math. J.*, **5** (1975), 33-42.

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