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ON I-STATISTICALLY PRE-CAUCHY SEQUENCES

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Abstract. In this paper we continue our investigation of the recent summability notion of $I-$statistical convergence introduced in [8, 26] (where a new extension of the notion of natural density and statistical convergence was proposed using the notion of ideals of the set of positive integers $\mathbb{N}$) and introduce the notion of I-statistically pre-Cauchy sequences in line of [3]. We mainly show that $I-$statistical convergence implies $I-$statistical pre-Cauchy condition and give certain sufficient conditions for the converse to be true.

1. INTRODUCTION

The idea of convergence of a real sequence was extended to statistical convergence by Fast [12] (see also Schoenberg [28]) as follows : If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then $K(m,n)$ denotes the cardinality of the set $K \cap [m,n]$. The upper and lower natural density of the subset K is defined by

$$\overline{d}(K) = \lim_{n\to\infty} \sup \frac{K(1,n)}{n} \text{ and } \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1,n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$ then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly $d(K) = \lim_{n\to\infty} \frac{K(1,n)}{n}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to L if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\}$ has natural density zero. Statistical convergence turned out to be one of the most active areas of research in summability theory after the work of Fridy [13] and Salát [23]. More initial work

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on this convergence can be found from [3, 1, 2, 14, 15] where other references can be found.

In particular in [3] a very interesting notion was introduced, that of statistically pre-Cauchy sequences and it was shown that statistically convergent sequences are always statistically pre-Cauchy and on the other hand under certain general conditions statistical pre-Cauchy condition implies statistical convergence of a sequence (and so one do not have to guess the limit of the statistically convergent sequence beforehand).

The idea of statistical convergence was further extended to I-convergence in [17] using the notion of ideals of $\mathbb{N}$ with many interesting consequences. More investigations in this direction and more applications of ideals of $\mathbb{N}$ can be found in [4, 5, 6, 7, 8, 11, 20, 21, 22, 24, 25, 26, 27] where many important references can be found.

Recently in [8, 26] we used ideals to introduce the concept of I-statistical convergence which naturally extends the notion of statistical convergence and studied some basic properties of this more general convergence. Very recently further work on such summability methods have been done in [9, 10].

As a natural consequence, in this note, we continue our investigation of $I-$statistical convergence and introduce the notion of I-statistically pre-Cauchy sequences in line of [3]. We mainly show that $I-$statistical convergence implies $I-$statistical pre-Cauchy condition and give certain sufficient conditions for the converse to be true.

2. Main Results

We first recall some basic definitions and notions which will be needed.

Definition 2.1. A non-empty family $I \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if
(a) $A, B \in I$ imply $A \cup B \in I$,
(b) $A \in I,\ B \subset A$ imply $B \in I$.

Definition 2.2. A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be a filter of $\mathbb{N}$ if
(a) $\emptyset \notin F$,
(b) $A, B \in F$ imply $A \cap B \in F$,
(c) $A \in F, A \subset B$ imply $B \in F$.

if I is a proper nontrivial ideal of $\mathbb{N}$ $(i.e\ \mathbb{N} \notin I, I \neq \{\varnothing\})$ then the family of sets $F(I) = \{M \subset \mathbb{N} : \exists\ A \in I,\ M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal I.

Definition 2.3. A proper ideal I is said to be admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.

Throughout I will stand for a proper admissible ideal of $\mathbb{N}$, and by a sequence we always mean a sequence of real numbers.

Definition 2.4. [17]. (i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $I-$convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in I$.

(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be I^*- convergent to $L \in \mathbb{R}$ if there exists $M \in F(I)$ such that $\{x_n\}_{n\in M}$ converges to L.

Definition 2.5. [8, 26]. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $I-$statistically convergent to L or $S(I)-$ convergent to L if, for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \in I.$$

In this case we write $x_k \rightarrow L(S(I))$. The class of all $I-$statistically convergent sequences will be denoted simply by $S(I)$.

We now introduce the main definition of this paper.

Definition 2.6. A sequence $\{x_k\}_{k\in\mathbb{N}}$ is said to be $I-$ statistically pre-Cauchy if, for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^2} \left|\{(j,k) : |x_k - x_j| \geq \varepsilon,\ j,k \leq n\}\right| \geq \delta\right\} \in I.$$

Theorem 2.1. *An $I-$statistically convergent sequence is $I-$statistically pre-Cauchy.*

Proof. Let $x = \{x_k\}_{k\in\mathbb{N}}$ be $I-$statistically convergent to L. Let $\varepsilon, \delta > 0$ be given. Now

$$C = \left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : |x_k - L| \geq \frac{\varepsilon}{2}\right\}\right| \geq \delta\right\} \in I.$$

Then for all $n \in C^c$ where c stands for the complement,

$$\frac{1}{n} \left|\left\{k \leq n : |x_k - L| \geq \frac{\varepsilon}{2}\right\}\right| < \delta$$

$$i.e.\ \frac{1}{n} \left|\left\{k \leq n : |x_k - L| < \frac{\varepsilon}{2}\right\}\right| > 1 - \delta.$$

Writing $B_n = \left\{k \leq n : |x_k - L| < \frac{\varepsilon}{2}\right\}$ we observe that for $j, k \in B_n$

$$|x_k - x_j| \leq |x_k - L| + |x_j - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence

$$B_n \times B_n \subset \{(j,k) : |x_k - x_j| < \varepsilon, \quad j,k \leq n\}$$

which implies

$$\left[\frac{|B_n|}{n}\right]^2 \leq \frac{1}{n^2} \left|\ \{(j,k) : |x_k - x_j| < \varepsilon, \quad j,k \leq n\}\ \right|.$$

Thus for all $n \in C^c$,

$$\frac{1}{n^2}\left|\{(j,k) : |x_k - x_j| < \varepsilon, \quad j,k \le n\}\right| \ge \left[\frac{|B_n|}{n}\right]^2 > (1-\delta)^2$$

$$i.e. \frac{1}{n^2}\left|\{(j,k) : |x_k - x_j| \ge \varepsilon, \quad j,k \le n\}\right| < 1 - (1-\delta)^2 .$$

Let $\delta_1 > 0$ be given. Choosing $\delta > 0$ so that $1-(1-\delta)^2 < \delta_1$ we see that $\forall n \in C^c$

$$\frac{1}{n^2}\left|\{(j,k) : |x_k - x_j| \ge \varepsilon, \quad j,k \le n\}\right| < \delta_1$$

and so

$$\left\{n \in \mathbb{N} : \frac{1}{n^2}\left|\{(j,k) : |x_k - x_j| \ge \varepsilon, \quad j,k \le n\}\right| \ge \delta_1\right\} \subset C.$$

Since $C \in I$, so

$$\left\{n \in \mathbb{N} : \frac{1}{n^2}\left|\{(j,k) : |x_k - x_j| \ge \varepsilon, \quad j,k \le n\}\right| \ge \delta_1\right\} \in I$$

and this completes the proof of the theorem. ■

In the following we give a necessary and sufficient condition for a sequence to be $I-$statistically pre-Cauchy.

Theorem 2.2. *Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a bounded sequence. Then x is $I-$statistically pre-Cauchy if and only if*

$$I - \lim_n \frac{1}{n^2}\sum_{j,k\le n} |x_k - x_j| = 0.$$

Proof. First suppose that

$$I - \lim_n \frac{1}{n^2}\sum_{j,k\le n} |x_k - x_j| = 0.$$

Note that for any $\varepsilon > 0$ and $n \in \mathbb{N}$ we have

$$\frac{1}{n^2}\sum_{j,k\le n} |x_k - x_j| \ge \varepsilon.\left(\frac{1}{n^2}\left|\{(j,k) : |x_k - x_j| \ge \varepsilon, \quad j,k \le n\}\right|\right).$$

Hence for any $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^2}\left|\{(j,k) : |x_k - x_j| \ge \varepsilon, \quad j,k \le n\}\right| \ge \delta\right\}$$
$$\subset \left\{n \in \mathbb{N} : \frac{1}{n^2}\sum_{j,k\le n} |x_k - x_j| \ge \delta\varepsilon\right\}.$$

Since $I-\lim_n \frac{1}{n^2}\sum_{j,k\le n}|x_k-x_j| = 0$ so the set on the right hand side belongs to I which implies that

$$\left\{n\in\mathbb{N}:\frac{1}{n^2}\left|\{(j,k):|x_k-x_j|\ge\varepsilon,\quad j,k\le n\}\right|\ge\delta\right\}\in I.$$

This proves that x is $I-$statistically pre-Cauchy.

Conversely suppose that x is $I-$statistically pre-Cauchy. Since x is bounded, $\exists$ a $B>0$ such that $|x_k|\le B\quad\forall k\in\mathbb{N}$. Let $\delta>0$ be given. For each $n\in\mathbb{N}$,

$$\frac{1}{n^2}\sum_{j,k\le n}|x_k-x_j|\le\frac{\varepsilon}{2}+2B\left(\frac{1}{n^2}\left|\left\{(j,k):|x_k-x_j|\ge\frac{\varepsilon}{2},\quad j,k\le n\right\}\right|\right).$$

Since x is $I-$statistically pre-Cauchy, for $\delta>0,$

$$C=\left\{n\in\mathbb{N}:\frac{1}{n^2}\left|\left\{(j,k):|x_k-x_j|\ge\frac{\varepsilon}{2},\quad j,k\le n\right\}\right|\ge\delta\right\}\in I.$$

Then for $n\in C^c$

$$\frac{1}{n^2}\left|\left\{(j,k):|x_k-x_j|\ge\frac{\varepsilon}{2},\quad j,k\le n\right\}\right|<\delta$$

and so

$$\frac{1}{n^2}\sum_{j,k\le n}|x_k-x_j|\le\frac{\varepsilon}{2}+2B\delta.$$

Let $\delta_1>0$ be given. Then choosing $\varepsilon,\delta>0$ so that $\frac{\varepsilon}{2}+2B\delta<\delta_1$ we see that $\forall n\in C^c$,

$$\frac{1}{n^2}\sum_{j,k\le n}|x_k-x_j|<\delta_1$$

i.e.

$$\left\{n\in N:\frac{1}{n^2}\sum_{j,k\le n}|x_k-x_j|\ge\delta_1\right\}\subset C\in I.$$

This proves the necessity of the condition. ■

We now present a sufficient condition under which an $I-$ statistically pre-Cauchy sequence can be $I-$statistically convergent.

For the next theorem we first recall the following definition of $I-$limit inferior [11] (see also [20]).

Definition 2.7. Let I be an admissible ideal of $\mathbb{N}$ and let $x = \{x_n\}_{n\in\mathbb{N}}$ be a real sequence. Let

$$A_x = \{a \in \mathbb{R} : \{k : x_k < a\} \notin I\}.$$

Then the $I-$limit inferior of x is given by

$$I - \liminf\ x = \begin{cases} \inf\ A_x, & if\ A_x \neq \varnothing \\ \infty, & if\ A_x = \varnothing. \end{cases}$$

It is known (Theorem 2, [11]) that $I - \lim\ \inf\ x = \alpha\ (finite)$ if and only if for arbitrary $\varepsilon > 0$,

$$\{k : x_k < \alpha + \varepsilon\} \notin I \ \text{ and } \ \{k : x_k < \alpha - \varepsilon\} \in I.$$

Theorem 2.3. *Suppose $x = \{x_k\}_{k\in\mathbb{N}}$ is $I-$statistically pre-Cauchy. If x has a subsequence $\{x_{p_k}\}_{k\in\mathbb{N}}$ which converges to L and*

$$0 < I - \underset{n}{lim}\inf \frac{1}{n}\left|\{p_k \leq n : k \in \mathbb{N}\}\right| < \infty$$

then x is $I-$statistically convergent to L.

Proof. Let $\varepsilon > 0$ be given. Since $\lim\limits_{k} x_{p_k} = L$, choose $T \in \mathbb{N}$ such that $|x_j - L| < \frac{\varepsilon}{2}$ whenever $j > T$ and $j = p_k$ for some k. Let $A = \{p_k : p_k > T, k \in \mathbb{N}\}$ and $A(\varepsilon) = \{k : |x_k - L| \geq \varepsilon\}$. Now note that

$$\begin{aligned} & \frac{1}{n^2}\left|\left\{(j,k) : |x_k - x_j| \geq \frac{\varepsilon}{2},\quad j,k \leq n\right\}\right| \\ \geq\ & \frac{1}{n^2}\sum_{j,k\leq n} \chi_{A(\varepsilon)\times A}(j,k) \\ =\ & \frac{1}{n}\left|\{p_k \leq n : p_k \in A\}\right| . \frac{1}{n}\left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right|. \end{aligned}$$

Since x is $I-$statistically pre-Cauchy, for $\delta > 0$,

$$C = \left\{n \in \mathbb{N} : \frac{1}{n^2}\left|\left\{(j,k) : |x_k - x_j| \geq \frac{\varepsilon}{2},\quad j,k \leq n\right\}\right| \geq \delta\right\} \in I.$$

Thus for all $n \in C^c$,

$$\frac{1}{n^2}\left|\left\{(j,k) : |x_k - x_j| \geq \frac{\varepsilon}{2},\quad j,k \leq n\right\}\right| < \delta. \tag{2.1}$$

Again since $I - \underset{n}{lim}\inf \frac{1}{n}\left|\{p_k \leq n : k \in \mathbb{N}\}\right| = b\,(say) > 0$, so

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{p_k \leq n : k \in \mathbb{N}\}\right| < \frac{b}{2}\right\} = D\,(say) \in I$$

and so $\forall n \in D^c$

$$\frac{1}{n}\left|\{p_k \leq n : k \in \mathbb{N}\}\right| \geq \frac{b}{2}. \tag{2.2}$$

From (2.1) and (2.2) it now follows that $\forall n \in C^c \cap D^c = (C \cup D)^c$,

$$\frac{1}{n}\left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| < \frac{2\delta}{b}.$$

Let $\delta_1 > 0$ be given. Then choosing $\delta > 0$ such that $\frac{2\delta}{b} < \delta_1$ we see that $\forall n \in (C \cup D)^c$

$$\frac{1}{n}\left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| < \delta_1$$

$$i.e. \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \subset C \cup D.$$

As $C, D \in I$ so $C \cup D \in I$ and consequently the set on the left hand side also belongs to I. This shows that x is $I-$ statistically convergent to L. ■

In order to give an example of a sequence which is $I-$statistically pre-Cauchy but not $I-$statistically convergent we first observe that every $I-$statistically convergent sequence must have a convergent subsequence which is convergent in the usual sense. As this observation is not at all straightforward as the statistical case, we give a proof below.

Let $x = \{x_k\}_{k\in\mathbb{N}}$ be $I-$ statistically convergent to L. For $\varepsilon = \delta = 1$ we have

$$C = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : |x_k - L| \geq 1\}\right| \geq 1\right\} \in I.$$

Since $C \neq \mathbb{N}$ (as I is non-trivial), choose $n_1 \in \mathbb{N}\backslash C$. Then

$$\frac{1}{n_1}\left|\{k \leq n_1 : |x_k - L| \geq 1\}\right| < 1$$

$$i.e. \frac{1}{n_1}\left|\{k \leq n_1 : |x_k - L| < 1\}\right| > 0.$$

Hence there exists a $k_1 \leq n_1$ such that $|x_{k_1} - L| < 1$. Again taking $\varepsilon = \delta = \frac{1}{2}$ we have

$$D = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : |x_k - L| \geq \frac{1}{2}\right\}\right| \geq \frac{1}{2}\right\} \in I.$$

Since I is admissible, we have

$$D \cup \{1, 2, ..., 3n_1\} \in I.$$

Again as $\mathbb{N} \notin I$, choose $n_2 \in \mathbb{N}$ such that $n_2 \notin D$ and $n_2 > 3n_1$. Then

$$\frac{1}{n_2}\left|\left\{k \leq n_2 : |x_k - L| \geq \frac{1}{2}\right\}\right| < \frac{1}{2}$$

$$i.e. \frac{1}{n_2}\left|\left\{k \leq n_2 : |x_k - L| < \frac{1}{2}\right\}\right| > \frac{1}{2}.$$

Note that if $|x_k - L| \geq \frac{1}{2}$ $\forall k$, $n_1 < k \leq n_2$, then

$$\frac{1}{n_2}\left|\left\{k \leq n_2 : |x_k - L| < \frac{1}{2}\right\}\right| \leq \frac{n_1}{n_2} < \frac{1}{3}.$$

Consequently there must exist a k, $n_1 < k \leq n_2$ for which $|x_k - L| < \frac{1}{2}$. Write this k as k_2. Evidently $k_2 > k_1$. Proceeding in this way we can obtain an increasing sequence of indices $\{k_1 < k_2 < k_3 < ...\}$ such that $\left|x_{k_j} - L\right| < \frac{1}{j}$. Evidently $\left\{x_{k_j}\right\}_{j \in \mathbb{N}}$ is then convergent to L.

In view of the above fact Example 8 [3] actually serves our purpose. We reproduce that example for the sake of completeness.

Example 2.1. Define the sequence $x = \{x_k\}_{k \in \mathbb{N}}$ as follows. For $m, k \in \mathbb{N}$ such that $(m-1)! < k \leq m!$ set $x_k = \sum_{i=1}^{m} \frac{1}{i}$ and $x = \{x_k\}_{k \in \mathbb{N}}$. Since x has no convergent subsequence so it can not be $I-$statistically convergent. However this is statistically pre-Cauchy (see [3] for details) and so $I-$statistically pre-Cauchy since I is admissible.

Next we present an interesting property of $I-$statistically pre-Cauchy sequences in line of Lemma 4 [3].

Theorem 2.4. *Let $x = \{x_k\}_{k \in \mathbb{N}}$ be a sequence and (α, β) is an open interval such that $x_k \notin (\alpha, \beta)$ $\forall k \in \mathbb{N}$. Write $A = \{k : x_k \leq \alpha\}$ and $B = \{k : x_k \geq \beta\}$ and further assume that the following property is satisfied:*

$$(*) \qquad \limsup D_n(A) - \liminf D_n(A) < \nu$$

for some $0 \leq \nu \leq 1$. Then if x is $I-$statistically pre-Cauchy, either

$$I - \lim_n D_n(A) = 0 \;\; or \;\; I - \lim_n D_n(B) = 0.$$

Here as usual $D_n(A) = \frac{1}{n}|\{k \leq n : k \in A\}|$.

Proof. First observe that $B = \mathbb{N} \backslash A$ and so $D_n(B) = 1 - D_n(A)$ $\forall n \in \mathbb{N}$. The proof will be complete if we can show that $I - \lim_n D_n(A) = 0$ or 1.

Note that

$$A \times B \subset \{(j,k) : |x_k - x_j| \geq \beta - \alpha\}.$$

Since x is $I-$statistically pre-Cauchy, it follows that

$$\begin{aligned} & I-\lim_n \frac{1}{n^2}\left|\{(j,k):|x_k-x_j|\geq \beta-\alpha,\ j,k\leq n\}\right| \\ = \ & I-\lim_n D_n(A)\left[D_n(\mathbb{N}\backslash A)\right] \\ = \ & I-\lim_n D_n(A)\left[1-D_n(A)\right]=0. \end{aligned}$$

Now from the definition of $I-$convergence it follows that

$$\left\{n\in\mathbb{N}:D_n(A)\left[1-D_n(A)\right]\geq\frac{1}{9}\right\}\in I$$
$$i.e.\left\{n\in\mathbb{N}:D_n(A)\left[1-D_n(A)\right]<\frac{1}{9}\right\}=M\,(say)\in F(I)\,.$$

Clearly $\forall n\in M$, then either $D_n(A)<\frac{1}{3}$ or $D_n(A)>\frac{2}{3}$.

If $D_n(A)<\frac{1}{3}\ \forall n\in M_1\subset M$ for some $M_1\in F(I)$ then $I-\lim_n D_n(A)=0$. This is because for any $\varepsilon, 0<\varepsilon<\frac{1}{3}$, from the definition of $I-$convergence we get

$$\left\{n\in\mathbb{N}:D_n(A)\left[1-D_n(A)\right]<\varepsilon^2\right\}=M_2\,(say)\in F(I)\,.$$

Taking $M_0=M_1\cap M_2$ we see that $M_0\in F(I)$ and

$$D_n(A)<\varepsilon\quad\forall n\in M_0.$$

Similarly if $D_n(A)>\frac{2}{3}\ \forall n\in M_3\subset M$ for some $M_3\in F(I)$ then we can show that $I-\lim_n D_n(A)=1$. If neither of them holds then we can find an increasing sequence of integers $\{n_1<n_2<n_3<n_4...\}$ from M such that

$$\begin{aligned} D_{n_i}(A) &< \frac{1}{3} \text{ when } i \text{ is odd,} \\ D_{n_i}(A) &> \frac{2}{3} \text{ when } i \text{ is even.} \end{aligned} \tag{2.3}$$

Clearly (1.3) implies that

$$\limsup_n D_n(A)-\liminf_n D_n(A)>\frac{1}{3}. \tag{2.4}$$

We again start the above process and see that

$$\left\{n\in\mathbb{N}:D_n(A)\left[1-D_n(A)\right]<\frac{1}{16}\right\}=M\,(say)\in F(I)$$

which consequently implies as above that either $I-\lim_n D_n(A)=0$ or $I-\lim_n D_n(A)=1$ or if neither holds then

$$\limsup_n D_n(A)-\liminf_n D_n(A)>\frac{2}{4}. \tag{2.5}$$

Proceeding in this way we observe that the process stops only when we get either $I-\lim_n D_n(A)=0$ or $I-\lim_n D_n(A)=1$ and if it does not stop at a finite step then we will have

$$\limsup_n D_n(A)-\liminf_n D_n(A)\geq \lim_n \frac{n-2}{n}=1$$

which is a contradiction to our assumption $(*)$. This completes the proof of the theorem. ∎

Remark 1. For $A\subset\mathbb{N}$ if $I-\lim_n \frac{1}{n}|\{k\leq n: k\in A\}|$ exists we can say that the $I-$asymptotic density of A exists and denote it by $d_I(A)$.Therefore the above result can be re-phrased as:

if $x=\{x_k\}_{k\in\mathbb{N}}$ is $I-$statistically pre-Cauchy and $x_k\notin(\alpha,\beta)$ $\forall k\in\mathbb{N}$ and for the set $A=\{k:x_k\leq\alpha\}$, $lim\sup_n D_n(A)-\liminf_n D_n(A)<\nu$ for some $0\leq\nu\leq 1$ then $d_I(A)=0$ or 1.

It should be mentioned in this context that for $I=I_{fin}$, the above result holds without any additional assumption (see [3]). For our final result we assume that I is such an ideal and $x=\{x_k\}_{k\in\mathbb{N}}$ is such that the above result holds without any additional assumption i.e.

$(**)$ If $x=\{x_k\}_{k\in\mathbb{N}}$ is $I-$ statistically pre-Cauchy and $x_k\notin(\alpha,\beta)$ $\forall k\in\mathbb{N}$ then either $d_I(\{k:x_k\leq\alpha\})=0$ or $d_I(\{k:x_k\geq\beta\})=0$.

Before we prove our final result, we introduce the following definition.

Definition 2.8. $\lambda\in\mathbb{R}$ is said to be an $I-$statistical cluster point of a sequence $x=\{x_k\}_{k\in\mathbb{N}}$ of real numbers if for any $\varepsilon>0$

$$d_I(\{k:|x_k-\lambda|<\varepsilon\})\neq 0.$$

For $I=I_{fin}$, $I-$statistical cluster points become statistical cluster points [14] and as in [14] it can be shown that the collection of all $I-$statistical cluster points is a closed subset of $\mathbb{R}$.

Theorem 2.5. *Let $x=\{x_k\}_{k\in\mathbb{N}}$ be an $I-$statistically pre-Cauchy sequence having an $I-$statistical cluster point. If the set of limit points of x is no-where dense then x is $I-$ statistically convergent under the hypothesis $(**)$.*

Proof. We follow the line of proof of Theorem 5 [3]. Let $\lambda\in\mathbb{R}$ be an $I-$statistical cluster point of x. Then for any $\varepsilon>0$, $d_I(\{k:|x_k-\lambda|<\varepsilon\})\neq 0$. Assume that x is $I-$statistically pre-Cauchy satisfying the hypothesis $(**)$ but x is not $I-$statistically convergent. Since x is not $I-$ statistically convergent so there is an $\hat{\varepsilon}$ such that $d_I\left(\left\{k:|x_k-\lambda|<\hat{\varepsilon}\right\}\right)\neq 0$. Without any boss of generality we can assume that $d_I\left(\left\{k:x_k\leq\lambda-\hat{\varepsilon}\right\}\right)\neq 0$.

We claim that every point of $\left(\lambda - \hat{\varepsilon}, \lambda\right)$ is a limit point of x. If not, then we can find an interval $(\alpha, \beta) \subset \left(\lambda - \hat{\varepsilon}, \lambda\right)$ such that $x_k \notin (\alpha, \beta)\ \forall k \in \mathbb{N}$. From above it immediately follows that both $\{k : x_k \leq \alpha\}$ and $\{k : x_k \geq \beta\}$ have subsets whose I-density are not zero and so $d_I(\{k : x_k \leq \alpha\}) \neq 0$ and $d_I(\{k : x_k \geq \beta\}) \neq 0$. But this contradicts the hypothesis $(**)$. Hence x must be $I-$statistically convergent. ■

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