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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF DEGENERATE CHEMOTAXIS SYSTEM

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Abstract. In this paper we establish the existence and uniqueness of a weak solution of the strongly coupled chemotaxis model with Dirichlet boundary conditions.

1. Introduction

Chemotaxis model was first proposed by Keller and Segel [1] and further it has been studied by many researchers in the last few decades (see [2, 3, 4, 5, 6]) and the references there in. Chemotaxis is a chemosensitive movement of cells which direct towards the gradient of a chemical contained in the environment. For example, when a bacterial infection invades the body it may be attacked by movement of cells towards the source due to chemotaxis. Convincing evidence suggests that leukocyte cells in the blood move towards a region of bacterial inflammation by moving up a chemical gradient caused by the infection.

In the literature, there has been increasing biological interest in the qualitative analysis of solutions of differential equations of chemotaxis model and related models (see [2, 7, 8, 9, 10, 11, 12, 5, 6]). The review article by Hillen and Painter [13] explores in detail a number of variations of the original Keller-Segel model, contrasts their patterning properties, summarizes the key results on their analytical properties and classifies their solution form. In its original form, this model consists of four coupled reaction-advection-diffusion equations. These can be reduced under quasi-steady-state assumptions to a model involves two unknown functions which would be the focus of our study in this article.

Let the space and time evolution of the cell density be denoted by u and the space and time evolution of the chemo-attractant density by v; then the model represented in

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[4] reads as

$$
(1.1)\qquad \left.\begin{aligned}
&\partial_t u = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(g(u)\nabla v) \quad \text{in} \quad Q_T = \Omega\times(0,T),\\
&\partial_t v - d\Delta v = \alpha u - \beta v \;\; \text{in} \quad Q_T,\\
&(u,v)(x,0) = (u_0(x), v_0(x)) \;\; \text{in} \quad \Omega,\\
&(u,v) = 0 \;\; \text{in} \quad \Gamma_T = \partial\Omega\times(0,T),
\end{aligned}\right\}
$$

where Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. $D(u)$ denotes the density-dependent diffusion co-efficient and $g(u) = ug_1(u)$ where $g_1(u)$ is commonly referred to as the chemotactic sensitivity function. Here the linear function $\alpha u - \beta v$, where $\alpha, \beta \geq 0$, involves the rates of production and degradation of the chemo-attractant. Suppose we assume that there exists a maximal population density of cells u_m such that $g(u_m) = 0$. This corresponds to a switch to repulsion at high densities, known as prevention of overcrowding, volume-filling effect or density control, for example, see [14]. It means that cells stop to accumulate at a given point of Ω after their density attains a certain threshold value, and the chemotactic cross-diffusion term $g(u)$ vanishes identically when $u \geq u_m$. We also assume that the diffusion coeffcient $D(u)$ vanishes at 0 and u_m, so that, (1.1) degenerates for $u = 0$ and $u = u_m$, while $D(u) > 0$ for $0 < u < u_m$. Therefore the above type of chemotaxis model is the special case of the classical Keller-Segel model and it is called as degenerate chemotaxis model according to the strong degeneracy of density-dependent coefficient $D(u)$. This means that there is no diffusion when u approaches values close to a threshold value in the absence of cell-population. For example in [15], this interpretation was proposed where the diffusion coefficient takes the form $D(u) = \varepsilon u(1-u)$ for $\varepsilon > 0$. However for more details regarding the degenerate chemotaxis model, see [4, 14, 16]. The main advantage of this nonlinear diffusion model seem to be related to the finite speed of propagation (which is more realistic in biological applications) and the asymptotic behavior of solutions.

The basic assumption made here is the existence of a threshold value for the cell density which corresponds to a tight packing state. In otherwords, cells cannot accumulate without bound at a given point of Ω and the corresponding mathematical assumption is that the function g vanishes identically when u crosses the threshold value. The effect of a threshold cell density or a volume-filling effect has been taken into account in the modelling of chemotaxis phenomenon in [9, 4, 14, 16] and the well-posedness and large time behavior of solutions to chemotaxis systems incorporating the volume-filling effect has been studied in [16, 17, 18]. Chemotaxis model has received considerable attention in modeling segregation phenomena in population dynamics and it plays a major role in modern applied mathematics and medicine biology. For example, Kowalczyk and Szymanska considered the chemotaxis model used in angiogenesis, which is the process of the creation of new (blood) vessels from existing ones (see [19]), and established the existence and uniqueness of solutions. While in cancer invasion theory, global existence and uniqueness of classical solutions of chemotaxis-haptotaxis model have been studied by [20, 21, 22] using fixed point technique.

Moreover, in the literature there have appeared few articles in which the authors established the existence of solutions of chemotaxis model with volume filling effect. In particular, Bendahamne et al. [16] established the existence of weak solutions of two-sidedly degenerate chemotaxis model with volume filling effect by using regularization and Schauder's fixed point method. Cieslak and Morales-Rodrigo [23] proved the existence of solutions of quasilinear parabolic-elliptic chemotaxis system using the Schauder fixed point theorem. Existence and uniqueness of solutions and the Holder continuity of solutions of doubly nonlinear chemotaxis model have been studied by Bendahamne et al. [3].

Laurencot and Wrzosek [4] established the existence and uniqueness of weak solutions of chemotaxis model with threshold density and degenerate diffusion using general abstract theory. Andreianov et al. [2] used finite volume method to establish the numerical solutions of the degenerate chemotaxis model. On the other hand Bendahamne et al. [24] established the existence of solutions of reaction diffusion system with L^1 data related to the chemotaxis models and Shangerganesh et al. [12] proved the different notion of weak-renormalized solutions for similar type of degenerate equations. Bendahamne and Karlsen [25] proved the existence of solutions of reaction diffusion and cardiac tissue model using Galerkin's approximation method. In contrast with their works, we prove herein the existence and uniqueness of weak solutions for the degenerate system (1.1) using difference and non-variational methods. In this work, we obtain the weak solutions for the given system under the following hypotheses: the density dependent diffusion $D \in C^2(\mathbb{R})$ and the function $g \in C^2(\mathbb{R})$ vanishes at large density. As for the initial data, we assume that $u_0(x), v_0(x) \in L^2(\Omega)$ with u_0 and v_0 being non-negative a.e in Ω.

A difficulty in the analysis of the system (1.1) is the strong degeneracy of the diffusion term $D(u)$. To handle this difficulty, we replace the diffusion term $D(u)$ by $D_\varepsilon(u) = D(u) + \varepsilon$ and let us first consider, for each fixed $\varepsilon > 0$, the following regularized non-degenerate problem of the system (1.1):

$$
(1.2)\qquad \left.\begin{aligned}
&\partial_t u_\varepsilon = \nabla \cdot (D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon) - \nabla \cdot (g(u_\varepsilon)\nabla v_\varepsilon) \quad \text{in} \quad Q_T,\\
&\partial_t v_\varepsilon - d\Delta v_\varepsilon = \alpha u_\varepsilon - \beta v_\varepsilon \quad \text{in} \quad Q_T,\\
&(u_\varepsilon, v_\varepsilon)(x,0) = (u_0(x), v_0(x)) \quad \text{in} \quad \Omega,\\
&(u_\varepsilon, v_\varepsilon) = 0 \quad \text{in} \quad \Gamma_T.
\end{aligned}\right\}
$$

To prove the existence theorem, first we prove the existence of weak solutions of the regularized problem and then we send the regularization parameter to zero to obtain the weak solutions of the original system (1.1). To attain this, one can use a priori estimates and compactness arguments.

The paper is organized as follows: In Section 2, we prove the existence of weak solutions for the steady-state version of the regularized problem using Galerkin's approximation method. In Section 3, we prove the existence of weak solutions of the regularized system using difference method and further we establish the existence of weak solutions of the original system by bringing the regularization parameter to zero. Moreover, in Section 4, we prove the uniqueness result using the duality approach. Throughout the paper, we use the same generic constant c everywhere instead of different constants.

2. Steady-state Case

In this section, first we consider the existence of solutions of steady-state case of the approximation problem

$$\left.\begin{aligned} -\nabla\cdot(D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon)+\nabla\cdot(g(u_\varepsilon)\nabla v_\varepsilon) &= 0 \ \text{ in } \ \Omega,\\ -d\Delta v_\varepsilon &= \alpha u_\varepsilon-\beta v_\varepsilon \ \text{ in } \ \Omega,\\ (u_\varepsilon,v_\varepsilon) &= 0 \ \text{ on } \ \partial\Omega. \end{aligned}\right\} \tag{2.1}$$

Lemma 2.1. [26, 27]. *Let $F:\mathbb{R}^K\to\mathbb{R}^K$ $(K\in\mathbb{N})$ be a continuous function such that $\langle F(r),r\rangle\ge 0$ on $|r|=\rho$. Then there exists $z\in\bar{B}_\rho(0)$ such that $F(z)=0$ for sufficiently large ρ.*

Theorem 2.1. *Under the assumption for some $d_0>0$ such that $D_\varepsilon(u_\varepsilon)\ge d_0$, the non-degenerate steady-state system (2.1) has a weak solution $(u_\varepsilon,v_\varepsilon)$ such that for any $\phi,\psi\in H_0^1(\Omega)$,*

$$\int_\Omega D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\nabla\phi dx-\int_\Omega g(u_\varepsilon)\nabla v_\varepsilon\nabla\phi dx=0,$$

$$d\int_\Omega\nabla v_\varepsilon\nabla\psi=\int_\Omega(\alpha u_\varepsilon-\beta v_\varepsilon)\psi dx,$$

holds.

Proof. In order to prove the existence of solutions of the system (2.1), we use the Galerkin's method of approximate solutions (see [26]). To use the Galerkin's method, we are in need of the specific basis. Now, let us introduce the spectral problem, find $w\in H_0^1(\Omega)$ and $\lambda\in\mathbb{R}$ such that

$$(\nabla w,\nabla\phi)_{L^2(\Omega),L^2(\Omega)}=\lambda(w,\phi)_{L^2(\Omega),L^2(\Omega)},\quad \text{for all } \ \phi\in H_0^1(\Omega),$$
$$w=0 \ \text{ on } \ \partial\Omega.$$

The above problem gives a sequence of non-decreasing eigenvalues $\{\lambda_l\}_{l=1}^\infty$ and a sequence of corresponding eigenfunctions $\{e_l\}_{l=1}^\infty$, forming an orthogonal basis in

$H_0^1(\Omega)$. For each $n \in \mathbb{N}$, define the subspace $V_n = span\{e_1, \cdots, e_n\}$. It is well known that $(V_n, \|\cdot\|)$ and $(\mathbb{R}^n, |\cdot|)$ are isometrically isomorphic by the natural linear map $T : V_n \to \mathbb{R}^n$ given by $z = \sum_{i=1}^{n} r_i e_i \to T(z) = r = (r_1, \cdots, r_n)$ (see [26]). So $\|z\| = |T(z)| = |r|$, where $|\cdot|$ and $\|\cdot\|$ denote the usual norms in $\mathbb{R}^N$ and $V_n(\Omega)$ respectively.

We look for the functions $(u_{\varepsilon n}, v_{\varepsilon n}) \in H_0^1(\Omega)$ of the form $u_{\varepsilon n} = \sum_{l=1}^{n} r_{n,l} e_l(x)$, $v_{\varepsilon n} = \sum_{l=1}^{n} s_{n,l} e_l(x)$, where we need to determine the coefficients $(r_{n,l}, s_{n,l})$, so that, for $k = 1, 2, \cdots, n$,

$$\begin{aligned} \int_\Omega D_\varepsilon(u_{\varepsilon n}) \nabla u_{\varepsilon n} \nabla e_k dx - \int_\Omega g(u_{\varepsilon n}) \nabla v_{\varepsilon n} \nabla e_k dx = 0 \quad \text{in} \quad \Omega, \\ d \int_\Omega \nabla v_{\varepsilon n} \nabla e_k dx = \int_\Omega (\alpha u_{\varepsilon n} - \beta v_{\varepsilon n}) e_k dx \quad \text{in} \quad \Omega. \end{aligned}$$

Now let us consider the following function $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by

$$F(r,s) = \Big(f_1(r,s), \cdots, f_n(r,s), h_1(r,s), \cdots, h_n(r,s) \Big),$$

where

$$\begin{aligned} f_k(r,s) \;&=\; \int_\Omega \left[D_\varepsilon \left(\sum_{l=1}^{n} r_{n,l} e_l(x) \right) \left(\sum_{l=1}^{n} r_{n,l} \nabla e_l(x) \right) \nabla e_k \right. \\ &\qquad \left. - g \left(\sum_{l=1}^{n} r_{n,l} e_l(x) \right) \left(\sum_{l=1}^{n} s_{n,l} \nabla e_l(x) \right) \nabla e_k \right] dx, \\ h_k(r,s) \;&=\; d \int_\Omega \left[\left(\sum_{l=1}^{n} s_{n,l} \nabla e_l(x) \right) \nabla e_k - \left(\alpha \sum_{l=1}^{n} r_{n,l} e_l(x) - \beta \sum_{l=1}^{n} s_{n,l} e_l(x) \right) e_k \right] dx, \end{aligned}$$

for each point $r = (r_1, \cdots, r_n) \in \mathbb{R}^n$ and $s = (s_1, \cdots, s_n) \in \mathbb{R}^n$. Then

$$\begin{aligned} &\langle F(r,s), (r,s) \rangle \\ &\geq \int_\Omega \left(d_0 \left| \sum_{l=1}^{n} r_{n,l} \nabla e_l(x) \right|^2 - \frac{g_0}{2} \left(\left| \sum_{l=1}^{n} r_{n,l} \nabla e_l(x) \right|^2 + \left| \sum_{l=1}^{n} s_{n,l} \nabla e_l(x) \right|^2 \right) \right) dx \\ &\quad + \int_\Omega \left(d \left| \sum_{l=1}^{n} s_{n,l} \nabla e_l(x) \right|^2 - c \left(\left| \sum_{l=1}^{n} r_{n,l} e_l(x) \right|^2 + \left| \sum_{l=1}^{n} s_{n,l} e_l(x) \right|^2 \right) \right) dx, \\ &\geq c \left(\|u_{\varepsilon n}\|^2 + \|v_{\varepsilon n}\|^2 \right), \end{aligned}$$

where we have used the Holder's and Young's inequality so that $\langle F(r,s),(r,s)\rangle \geq 0$, if $\|(u_{\varepsilon n}, v_{\varepsilon n})\| = \rho$ provided that $\rho > 0$ sufficiently large enough. Hence it follows, from Lemma 2.1, that for each $n \in \mathbb{N}$, there exists $(u_{\varepsilon n}, v_{\varepsilon n}) \in V_n \times V_n$ satisfying

$$F(u_{\varepsilon n}, v_{\varepsilon n}) = (0,0), \quad \|(u_{\varepsilon n}, v_{\varepsilon n})\| \leq \rho.$$

This shows that, given absolutely continuous coefficients $b_{i,n,l},\ i = 1,2$, we set $\phi_n = \sum_{l=1}^{n} b_{1,n,l} e_l(x)$ and $\psi_n = \sum_{l=1}^{n} b_{2,n,l} e_l(x)$ such that

$$\left.\begin{aligned} &\int_\Omega D_\varepsilon(u_{\varepsilon n})\nabla u_{\varepsilon n}\nabla\phi_n dx - \int_\Omega g(u_{\varepsilon n})\nabla v_{\varepsilon n}\nabla\phi_n dx = 0,\\ &\int_\Omega d\nabla v_{\varepsilon n}\nabla\psi_n dx = \int_\Omega (\alpha u_{\varepsilon n} - \beta v_{\varepsilon n})\psi_n dx, \end{aligned}\right\} \tag{2.2}$$

holds with $(\|u_{\varepsilon n}\|, \|v_{\varepsilon n}\|) \leq \rho$, for all $n \in \mathbb{N}$. Let us assume that $u_\varepsilon, v_\varepsilon \in H_0^1(\Omega)$ be the weak limits of $\{u_{\varepsilon n}\}$ and $\{v_{\varepsilon n}\}$ respectively, then there exist subsequences which are also denoted by $\{u_{\varepsilon n}\}$ and $\{v_{\varepsilon n}\}$ such that,

$$\begin{aligned} (u_{\varepsilon n}, v_{\varepsilon n}) &\rightharpoonup (u_\varepsilon, v_\varepsilon) \text{ weakly in } H_0^1(\Omega),\\ (u_{\varepsilon n}, v_{\varepsilon n}) &\to (u_\varepsilon, v_\varepsilon) \text{ in } L^q(\Omega) \text{ for } 1 < q < \frac{2N}{N-2},\\ D_\varepsilon(u_{\varepsilon n})\nabla u_{\varepsilon n} &\rightharpoonup \xi \text{ weakly in } L^2(\Omega),\\ (u_{\varepsilon n}, v_{\varepsilon n}) &\to (u_\varepsilon, v_\varepsilon) \text{ a.e in } \Omega. \end{aligned}$$

By adopting the technique proved in [27] and recalling the definition of monotonicity assumption (see [28]) in the existence theory, that is $\vartheta \to a(x,t,u,\vartheta)$ is monotone if

$$\langle a(x,t,u,\vartheta_1) - a(x,t,u,\vartheta_2), \vartheta_1 - \vartheta_2\rangle \geq 0,$$

for all $\vartheta_i \in \mathbb{R}^n, i = 1,2$, one can easily obtain that $\xi = D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon$. Then taking limit as $n \to \infty$ in (2.2), we get

$$\left.\begin{aligned} &\int_\Omega D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\nabla\phi dx - \int_\Omega g(u_\varepsilon)\nabla v_\varepsilon\nabla\phi dx = 0,\\ &\int_\Omega d\nabla v_\varepsilon\nabla\psi dx = \int_\Omega (\alpha u_\varepsilon - \beta v_\varepsilon)\psi dx. \end{aligned}\right\} \tag{2.3}$$

Equation (2.3) hold for all functions in $H_0^1(\Omega)$, as the functions ϕ and ψ are dense in this space. Hence this proves that $(u_\varepsilon, v_\varepsilon)$ is a weak solution of the system (2.1). ■

3. Existence of Weak Solutions

In this section, we prove the existence of weak solution for the system (1.1) using suitable approximation problem. First let us define the weak solution for the given chemotaxis model (1.1).

Definition 3.1. (Weak solution). A function (u, v) is a weak solution of the system (1.1) if the following conditions hold

$$u \in C([0,T]; L^2(\Omega)) \cap L^\infty(Q_T) \cap L^2(0,T; H_0^1(\Omega)), \quad D(u) \in L^2(0,T; H_0^1(\Omega)),$$
$$v \in C([0,T]; L^2(\Omega)) \cap L^\infty(Q_T) \cap L^p(0,T; W^{2,p}(\Omega)).$$

For any $\phi, \psi \in L^2(0,T; H_0^1(\Omega)) \cap C^1(Q_T)$, with $\phi(\cdot,T) = \psi(\cdot,T) = 0$, we have

$$\left.\begin{aligned} &-\int_\Omega u_0(x)\phi(x)dx - \int_{Q_T} u\phi_t dxdt + \int_{Q_T} D(u)\nabla u \nabla\phi dxdt \\ &-\int_{Q_T} g(u)\nabla v\nabla\phi dxdt = 0, \\ &-\int_\Omega v_0(x)\psi(x)dx - \int_{Q_T} v\psi_t dxdt + d\int_{Q_T} \nabla v\nabla\psi dxdt = \int_{Q_T}(\alpha u - \beta v)\psi dxdt. \end{aligned}\right\} \tag{3.1}$$

For our convenience, here and henceforth we denote the functions $\phi(x,0),\ \psi(x,0)$ by $\phi(x),\ \psi(x)$ respectively. Now we state the main theorem of this section.

Theorem 3.1. *Under the assumptions* $u_0(x), v_0(x) \in L^2(\Omega)$ *and further assuming that there is a positive constant* d_0 *such that* $D(u) \geq d_0$. *Then the system* (1.1) *admits a unique weak solution in the sense of Definition 3.1.*

Regarding the regularized non-degenerate problem, one can state the following lemma:

Lemma 3.1. *Under the assumptions* $u_0(x), v_0(x) \in L^2(\Omega)$, *the non-degenerate initial-boundary value problem* (1.2) *has the pair of weak solutions* $(u_\varepsilon, v_\varepsilon)$, *such that for any* $\phi, \psi \in L^2(0,T; H_0^1(\Omega)) \cap C^1(Q_T)$ *with* $\phi(\cdot,T) = 0$ *and* $\psi(\cdot,T) = 0$, *the following identities*

$$\left.\begin{aligned} &-\int_\Omega u_0(x)\phi(x)dx - \int_{Q_T} u_\varepsilon\phi_t dxdt \\ &\quad + \int_{Q_T} D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\nabla\phi dxdt - \int_{Q_T} g(u_\varepsilon)\nabla v_\varepsilon\nabla\phi dxdt = 0, \\ &-\int_\Omega v_0(x)\psi(x)dx - \int_{Q_T} v_\varepsilon\psi_t dxdt \\ &\quad + d\int_{Q_T} \nabla v_\varepsilon\nabla\psi dxdt = \int_{Q_T}(\alpha u_\varepsilon - \beta v_\varepsilon)\psi dxdt. \end{aligned}\right\} \tag{3.2}$$

holds.

Proof. To prove the existence of the weak solutions $(u_\varepsilon, v_\varepsilon)$ of (1.2), let us use the difference and non-variational methods. First we denote

$$W = \{(u_\varepsilon, v_\varepsilon) \in H_0^1(\Omega) \cap L^2(\Omega)\},$$

and also

$$\|(u_\varepsilon, v_\varepsilon)\|_W = \|u_\varepsilon\|_{L^2(\Omega)} + \|v_\varepsilon\|_{L^2(\Omega)} + \|u_\varepsilon\|_{H_0^1(\Omega)} + \|v_\varepsilon\|_{H_0^1(\Omega)}.$$

Then it is clear that W is a Banach space.

Now we construct the solutions $\{(u_h)_\varepsilon, (v_h)_\varepsilon\}$ for the regularized problem (1.2). Let n be a positive integer. Let $h = T/n$. Consider the following time-discrete problem of (1.2)

$$\left.\begin{aligned} &\frac{u_{\varepsilon_k} - u_{\varepsilon_{k-1}}}{h} - \nabla \cdot (D_\varepsilon(u_{\varepsilon_k})\nabla u_{\varepsilon_k}) = -\nabla \cdot (g(u_{\varepsilon_k})\nabla v_{\varepsilon_k}) \ \text{ in } \ \Omega,\\ &\frac{v_{\varepsilon_k} - v_{\varepsilon_{k-1}}}{h} - d\Delta v_{\varepsilon_k} = \alpha u_{\varepsilon_k} - \beta v_{\varepsilon_k} \ \text{ in } \ \Omega,\\ &(u_{\varepsilon_k}, v_{\varepsilon_k}) = 0 \ \text{ in } \ \partial\Omega. \end{aligned}\right\} \tag{3.3}$$

Repeating the same procedures as in the previous section, we can obtain the weak solutions $(u_{\varepsilon_k}, v_{\varepsilon_k})$ in W of (3.3) for $k = 1, 2, \cdots, n$. It follows that, for every $(\phi, \psi) \in W$,

$$\left.\begin{aligned} &\frac{1}{h}\int_\Omega (u_{\varepsilon_k} - u_{\varepsilon_{k-1}})\phi dx + \int_\Omega D_\varepsilon(u_{\varepsilon_k})\nabla u_{\varepsilon_k}\nabla\phi dx - \int_\Omega g(u_{\varepsilon_k})\nabla v_{\varepsilon_k}\nabla\phi dx = 0,\\ &\frac{1}{h}\int_\Omega (v_{\varepsilon_k} - v_{\varepsilon_{k-1}})\psi dx + d\int_\Omega \nabla v_{\varepsilon_k}\nabla\psi dx = \int_\Omega (\alpha u_{\varepsilon_k} - \beta v_{\varepsilon_k})\psi dx. \end{aligned}\right\} \tag{3.4}$$

Taking $\phi = u_{\varepsilon_k}, \psi = v_{\varepsilon_k}$ as the test functions in (3.4) respectively and the boundedness of solutions of the time-discretized problem, we obtain a priori estimate for $(u_{\varepsilon_k}, v_{\varepsilon_k}),\ k = 1, 2, \cdots, n$ as

$$\frac{1}{2}\int_\Omega |u_{\varepsilon_k}|^2 dx + \gamma\int_\Omega |\nabla u_{\varepsilon_k}|^2 dx \le c\left(\int_\Omega |\nabla v_{\varepsilon_k}|^2 dx + \int_\Omega |u_{\varepsilon_{k-1}}|^2 dx\right), \tag{3.5}$$

for any $\gamma = h\left(d_0 - \frac{g_0}{2}\right) > 0$ and the constant $c > 0$ independent of ε. Thus considering the left hand side of the above inequality, one can easily find that

$$\|u_{\varepsilon_k}\|^2_{L^2(\Omega)} \le c(\|u_0\|^2_{L^2(\Omega)} + 1). \tag{3.6}$$

Moreover, for the second equation of (3.4), we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega |v_{\varepsilon_k}|^2 dx + dh\int_\Omega |\nabla v_{\varepsilon_k}|^2 dx\\ &\le \frac{\alpha h}{2}\int_\Omega |u_{\varepsilon_k}|^2 dx + \frac{(\alpha - 2\beta)h}{2}\int_\Omega |v_{\varepsilon_k}|^2 dx + \frac{1}{2}\int_\Omega |v_{\varepsilon_{k-1}}|^2 dx. \end{aligned}$$

Applying Poincare's inequality and the result (3.6), we have, for any $\eta = h[2d+c(2\beta-\alpha)] > 0$,

$$\|v_{\varepsilon_k}\|^2_{L^2(\Omega)} + \eta \int_\Omega |\nabla v_{\varepsilon_k}|^2 dx \le c\left(\|v_{\varepsilon_{k-1}}\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)} + 1\right), \tag{3.7}$$

where $c > 0$ is a constant independent of ε. Thus considering the left hand side of the above inequality, one can easily find that,

$$\|v_{\varepsilon_k}\|^2_{L^2(\Omega)} \le c\left(\|u_0\|^2_{L^2(\Omega)} + \|v_0\|^2_{L^2(\Omega)} + 1\right). \tag{3.8}$$

Next, for every $h = \dfrac{T}{n}$, we have

$$(u_{h_\varepsilon}(x,t), v_{h_\varepsilon}(x,t)) = \begin{cases} (u_0(x), v_0(x)), \quad t=0, \\ (u_{1_\varepsilon}(x), v_{1_\varepsilon}(x)),\ 0<t<h, \\ \quad \vdots, \\ (u_{j_\varepsilon}(x), v_{j_\varepsilon}(x)),\ (j-1)h<t<jh, \\ \quad \vdots, \\ (u_{n_\varepsilon}(x), v_{n_\varepsilon}(x)),\ (n-1)h<t<nh=T. \end{cases} \tag{3.9}$$

For each $t \in [0,T]$, there exists some $k = 1, 2, \cdots, n$ such that $t \in [(k-1)h, kh]$. Thus, recalling (3.6) and (3.8) we conclude that

$$\|u_{h_\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} \le c \text{ and } \|v_{h_\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} \le c. \tag{3.10}$$

where $c>0$ is a constant which depends only on given data and $N, \Omega, \|u_0\|_{L^2(\Omega)}, \|v_0\|_{L^2(\Omega)}$.

By summing up the inequalities (3.5) and (3.7), we have

$$\|\nabla u_{h_\varepsilon}\|_{L^2(0,T;L^2(\Omega))} \le c, \quad \|\nabla v_{h_\varepsilon}\|_{L^2(0,T;L^2(\Omega))} \le c.$$

The above inequalities of $(u_{h_\varepsilon}, v_{h_\varepsilon})$ lead to,

$$\left.\begin{aligned} &\|u_{h_\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} + \|u_{h_\varepsilon}\|_{L^2(0,T;H^1_0(\Omega))} \le c, \\ &\|v_{h_\varepsilon}\|_{L^\infty(Q_T)} + \|v_{h_\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} + \|v_{h_\varepsilon}\|_{L^2(0,T;H^1_0(\Omega))} \le c, \end{aligned}\right\} \tag{3.11}$$

where $c > 0$ is a constant. Therefore we may choose a subsequence (still denoted by $(u_{h_\varepsilon}, v_{h_\varepsilon})$) such that

$$\left.\begin{aligned} &u_{h_\varepsilon} \rightharpoonup u_\varepsilon && \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \\ &u_{h_\varepsilon} \rightharpoonup u_\varepsilon && \text{weakly in } L^2(0,T;H^1_0(\Omega)), \\ &D_\varepsilon(u_{h_\varepsilon})\nabla u_{h_\varepsilon} \rightharpoonup \xi && \text{weakly in } L^2(Q_T), \\ &v_{h_\varepsilon} \rightharpoonup v_\varepsilon && \text{weakly* in } L^\infty(Q_T) \cap L^\infty(0,T;L^2(\Omega)), \\ &v_{h_\varepsilon} \rightharpoonup v_\varepsilon && \text{weakly in } L^2(0,T;H^1_0(\Omega)). \end{aligned}\right\} \tag{3.12}$$

Therefore we conclude that

$$\left.\begin{aligned} &\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \le c, \\ &\|v_\varepsilon\|_{L^\infty(Q_T)} + \|v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|v_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \le c. \end{aligned}\right\} \tag{3.13}$$

Next we show that $(u_\varepsilon, v_\varepsilon)$ is a weak solution of the regularized problem (1.2). For every $\phi, \psi \in C^1(Q_T)$ with $\phi(\cdot, T) = \psi(\cdot, T) = 0$ and $\phi(x,T)|_{\Gamma_T} = \psi(x,T)|_{\Gamma_T} = 0$ and for every $k = \{1, 2, \cdots, n\}$ we solve, $-\Delta\zeta_{1_k}(x) = \phi(x, kh)$ and $-\Delta\zeta_{2_k}(x) = \psi(x, kh)$ to find functions $(\zeta_{1_k}, \zeta_{2_k}) \in W$ and let them be test functions respectively in (3.4) to have

$$\begin{aligned} &\frac{1}{h}\int_\Omega u_{\varepsilon_k}\phi(x,kh)dx - \frac{1}{h}\int_\Omega u_{\varepsilon_{k-1}}\phi(x,kh)dx + \int_\Omega D_\varepsilon(u_{\varepsilon_k})\nabla u_{\varepsilon_k}\nabla\phi(x,kh)dx \\ &= \int_\Omega g(u_{\varepsilon_k})\nabla v_{\varepsilon_k}\nabla\phi(x,kh)dx, \\ &\frac{1}{h}\int_\Omega v_{\varepsilon_k}\psi(x,kh)dx - \frac{1}{h}\int_\Omega v_{\varepsilon_{k-1}}\psi(x,kh)dx + d\int_\Omega \nabla v_{\varepsilon_k}\nabla\psi(x,kh)dx \\ &= \int_\Omega (\alpha u_{\varepsilon_k} - \beta v_{\varepsilon_k})\psi(x,kh)dx. \end{aligned}$$

Summing up all the equalities and recalling the definition of $u_{h_\varepsilon}(x,t)$ in (3.9) and $\phi(\cdot, T) = \phi(\cdot, nh) = 0 = \psi(\cdot, nh) = \psi(\cdot, T)$, we have

$$\left.\begin{aligned} &h\sum_{k=1}^{n-1}\int_\Omega u_{h_\varepsilon}(x,kh)\frac{\phi(x,kh)-\phi(x,(k+1)h)}{h}dx - \int_\Omega u_0(x)\phi(x,h)dx \\ &+h\sum_{k=1}^{n}\int_\Omega D_\varepsilon(u_{h_\varepsilon}(x,kh))\nabla u_{h_\varepsilon}(x,kh)\nabla\phi(x,kh)dx \\ &= h\sum_{k=1}^{n}\int_\Omega g(u_{h_\varepsilon}(x,kh))\nabla v_{h_\varepsilon}(x,kh)\nabla\phi(x,kh)dx, \\ &h\sum_{k=1}^{n-1}\int_\Omega v_{h_\varepsilon}(x,kh)\frac{\psi(x,kh)-\psi(x,(k+1)h)}{h}dx - \int_\Omega v_0(x)\psi(x,h)dx \\ &+h\sum_{k=1}^{n}\int_\Omega d\nabla v_{h_\varepsilon}(x,kh)\nabla\psi(x,kh)dx \\ &= h\sum_{k=1}^{n}\int_\Omega (\alpha u_{h_\varepsilon}(x,kh) - \beta v_{h_\varepsilon}(x,kh))\psi(x,kh)dx. \end{aligned}\right\} \tag{3.14}$$

From (3.11)-(3.13) and for $\phi, \psi \in C^1(Q_T)$, we have

$$
\begin{aligned}
& h\sum_{k=1}^{n}\int_{\Omega} D_{\varepsilon}(u_{h_\varepsilon}(x,kh))\nabla u_{h_\varepsilon}(x,kh)\nabla\phi(x,kh)dx \\
&= \int_{Q_T} D_{\varepsilon}(u_{h_\varepsilon}(x,\tau))\nabla u_{h_\varepsilon}(x,\tau)\nabla\phi(x,\tau)dxd\tau \\
&\quad + \sum_{k=1}^{n}\int_{(k-1)h}^{kh}\int_{\Omega} D_{\varepsilon}(u_{h_\varepsilon}(x,\tau))\nabla u_{h_\varepsilon}(x,\tau)\cdot(\nabla\phi(x,kh)-\nabla\phi(x,\tau)dxd\tau \\
&\to \int_{Q_T}\xi\nabla\phi(x,\tau)dxd\tau \quad \text{as} \quad h\to 0.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
& h\sum_{k=1}^{n}\int_{\Omega} g(u_{h_\varepsilon}(x,kh))\nabla v_{h_\varepsilon}(x,kh)\nabla\phi(x,kh)dx \to \int_{Q_T} g(u_\varepsilon)\nabla v_\varepsilon\nabla\phi(x,\tau)dxd\tau, \\
& h\sum_{k=1}^{n}\int_{\Omega} d\nabla v_{h_\varepsilon}(x,kh)\nabla\psi(x,kh)dx \to \int_{Q_T} d\nabla v_\varepsilon\nabla\psi(x,\tau)dxd\tau, \\
& h\sum_{k=1}^{n}\int_{\Omega}\left(\alpha u_{h_\varepsilon}(x,kh)-\beta v_{h_\varepsilon}(x,kh)\right)\psi(x,kh)dx \to \int_{Q_T}\left(\alpha u_\varepsilon-\beta v_\varepsilon\right)\psi(x,\tau)dxd\tau.
\end{aligned}
$$

as $h\to 0$ then we deduce, from (3.14), that

$$
\text{(3.15)} \qquad \left.
\begin{aligned}
& -\int_{Q_T} u_\varepsilon\frac{\partial\phi}{\partial t}dxd\tau-\int_{\Omega}u_0(x)\phi(x,0)dx \\
& \quad +\int_{Q_T}\xi\nabla\phi dxd\tau=\int_{Q_T}g(u_\varepsilon)\nabla v_\varepsilon\nabla\phi dxd\tau, \\
& -\int_{Q_T} v_\varepsilon\frac{\partial\psi}{\partial t}dxd\tau-\int_{\Omega}v_0(x)\psi(x,0)dx \\
& \quad +\int_{Q_T}\nabla v_\varepsilon\nabla\psi dxd\tau=\int_{Q_T}(\alpha u_\varepsilon-\beta v_\varepsilon)\psi dxd\tau,
\end{aligned}
\right\}
$$

which show that $\partial_t u_\varepsilon \in L^2(0,T;H^{-1}(\Omega))$ and $\partial_t v_\varepsilon \in L^2(0,T;H^{-1}(\Omega))$. Thus one can find a large positive integer s such that $H^{-1}(\Omega)\subset H^{-s}(\Omega)$ (see [29]) and we get that $\partial_t u_\varepsilon \in L^2(0,T;H^{-s}(\Omega))$ and $\partial_t v_\varepsilon \in L^2(0,T;H^{-s}(\Omega))$ which follow from [30] that $\partial_t u_\varepsilon \in C([0,T];H^{-s}(\Omega))$ and $\partial_t v_\varepsilon \in C([0,T];H^{-s}(\Omega))$.

For each $\varepsilon>0$ and all $t,t_0\in[0,T]$, together with (3.13) there exists a positive number $\delta>0$ such that

$$
\text{(3.16)} \qquad \delta\|\nabla u_\varepsilon(t)-\nabla u_\varepsilon(t_0)\|_{L^2(\Omega)}\le\frac{\varepsilon}{2},\quad \delta\|\nabla v_\varepsilon(t)-\nabla v_\varepsilon(t_0)\|_{L^2(\Omega)}\le\frac{\varepsilon}{2}.
$$

From the compact imbedding, $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega)$, we have, for all $t, t_0 \in [0, T]$,

$$\begin{aligned}\|u_\varepsilon(t) - u_\varepsilon(t_0)\|_{L^2(\Omega)} &\leq \delta \|u_\varepsilon(t) - u_\varepsilon(t_0)\|_{H_0^1(\Omega)} + c(\delta)\|u_\varepsilon(t) - u_\varepsilon(t_0)\|_{H^{-s}(\Omega)} \\ &\leq \frac{\varepsilon}{2} + c(\delta)\|u_\varepsilon(t) - u_\varepsilon(t_0)\|_{H^{-s}(\Omega)}.\end{aligned}$$

Similarly

$$\|v_\varepsilon(t) - v_\varepsilon(t_0)\|_{L^2(\Omega)} \leq \frac{\varepsilon}{2} + c(\delta)\|v_\varepsilon(t) - v_\varepsilon(t_0)\|_{H^{-s}(\Omega)}.$$

Therefore we conclude that

$$(u_\varepsilon, v_\varepsilon) \in C([0, T]; L^2(\Omega)).$$

To complete the proof of the existence of weak solutions of the regularized problem (1.2), we have to prove that $\xi = D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon$ a.e in Q_T. Taking $\phi = u_\varepsilon$ as a test function in the first equation of (3.15), we get

$$\frac{1}{2}\left(\|u_\varepsilon(T)\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)}\right) + \int_{Q_T} \xi \nabla u_\varepsilon dx d\tau = \int_{Q_T} g(u_\varepsilon)\nabla v_\varepsilon \nabla u_\varepsilon dx d\tau. \tag{3.17}$$

Taking $Au_\varepsilon = D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon$ and using the definition of monotonicity assumption, we have

$$\int_\Omega (Au_{\varepsilon_k} - Az(\tau))(\nabla u_{\varepsilon_k} - \nabla z(\tau)) \geq 0, \tag{3.18}$$

for each $k = 1, 2, \cdots, n$ and every $z \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q_T)$.

Choosing $\phi = u_{\varepsilon_k}$ as a test function in (3.4) and, from (3.18), we get

$$\begin{aligned}&\frac{1}{h}\int_\Omega (u_{\varepsilon_k} - u_{\varepsilon_{k-1}})u_{\varepsilon_k} dx + \int_\Omega Au_{\varepsilon_k}\nabla z(\tau) dx \\ &\quad + \int_\Omega Az(\nabla u_{\varepsilon_k} - \nabla z(\tau))dx - \int_\Omega g(u_{\varepsilon_k})\nabla v_{\varepsilon_k}\nabla u_{\varepsilon_k} dx \geq 0.\end{aligned} \tag{3.19}$$

Integrating (3.19) over $((k-1)h, kh)$ and using the Cauchy's inequality, we have

$$\begin{aligned}&\frac{1}{2}\left(\|u_{\varepsilon_k}\|^2_{L^2(\Omega)} - \|u_{\varepsilon_{k-1}}\|^2_{L^2(\Omega)}\right) + \int_{(k-1)h}^{kh}\int_\Omega Au_{\varepsilon_k}\nabla z(\tau) dx d\tau \\ &\quad + \int_{(k-1)h}^{kh}\int_\Omega Az(\nabla u_{\varepsilon_k} - \nabla z(\tau)) dx d\tau - \int_{(k-1)h}^{kh}\int_\Omega g(u_{\varepsilon_k})\nabla v_{\varepsilon_k}\nabla u_{\varepsilon_k} dx \geq 0.\end{aligned}$$

Summing up the above inequalities for $k = 1, 2, \cdots, n$, we have

$$\begin{aligned}&\frac{1}{2}\left(\|u_{h_\varepsilon}(T)\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)}\right) + \int_{Q_T} Au_{h_\varepsilon}\nabla z(\tau) dx d\tau \\ &\quad + \int_{Q_T} Az(\tau)(\nabla u_{h_\varepsilon} - \nabla z(\tau)) dx d\tau - \int_{Q_T} g(u_{h_\varepsilon})\nabla v_{h_\varepsilon}\nabla u_{h_\varepsilon} dx d\tau \geq 0.\end{aligned}$$

Passing the limit as $h \to 0$ and noting that $\|u_\varepsilon(T)\|_{L^2(\Omega)} \le \liminf_{h\to 0} \|u_{h_\varepsilon}(T)\|_{L^2(\Omega)}$, we obtain

$$\begin{aligned} &\frac{1}{2}\left(\|u_\varepsilon(T)\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)}\right) + \int_{Q_T} \xi \nabla z(\tau) dx d\tau \\ &\quad + \int_{Q_T} Az(\tau)(\nabla u_\varepsilon - \nabla z(\tau)) dx d\tau - \int_{Q_T} g(u_\varepsilon)\nabla v_\varepsilon \nabla u_\varepsilon dx d\tau \ge 0. \end{aligned} \tag{3.20}$$

Combining (3.20) with (3.17), we get

$$\int_{Q_T} (\xi - Az(\tau))(\nabla u_\varepsilon - \nabla z(\tau)) dx d\tau \ge 0.$$

Choosing $z = u_\varepsilon - \lambda w$ for any $\lambda > 0, w \in L^2(0,T;H^1_0(\Omega))$ in the above inequality, we get

$$\int_{Q_T} (\xi - A(u_\varepsilon - \lambda w)\nabla w dx d\tau \ge 0.$$

Taking $\lambda \to 0$, and using Lebesgue's dominated convergence theorem, we have

$$\int_{Q_T} (\xi - Au_\varepsilon)\nabla \psi dx d\tau \ge 0,$$

for every $\psi \in L^2(0,T;H^1_0(\Omega))$. Therefore we conclude that $\xi = Au_\varepsilon$ a.e in Q_T and hence we have proved the existence of weak solutions of the regularized problem (1.2). ∎

Proof of Theorem 3.1. In the above, we have proved that there exist a weak solution $(u_\varepsilon, v_\varepsilon)$ for the regularized problem (1.2). To prove the existence of a weak solution of the original system (1.1), let us make the regularization parameter $\varepsilon \to 0$. From (3.13) and the classical L^p regularity theory, one can write the following result

$$\|\partial_t v_\varepsilon\|_{L^p(Q_T)} + \|v_\varepsilon\|_{L^p(0,T;W^{2,p}(\Omega))} \le c, \quad 1 \le p < \infty. \tag{3.21}$$

Proving the non-negativity of solutions $(u_\varepsilon, v_\varepsilon)$ is an easy task under the multiplication of first and second equations of (1.2) by $u_\varepsilon^- = \sup(-u_\varepsilon, 0)$ and $v_\varepsilon^- = \sup(-v_\varepsilon, 0)$ respectively. Now let us show that $D_\varepsilon(u^\varepsilon) \in L^2(0,T;H^1_0(\Omega))$. To prove this, first let us take $\mathcal{D}(r) := \int_0^r D(s)ds$. Now we multiply the first equation of (1.2) by $\mathcal{D}_\varepsilon(u^\varepsilon)$ and integrate over Ω, to get

$$\begin{aligned} \frac{d}{dt}\int_\Omega \hat{\mathcal{D}}_\varepsilon(u_\varepsilon)dx + \int_\Omega |\nabla \mathcal{D}_\varepsilon(u_\varepsilon)|^2 dx &\le \|g\|_{L^\infty(\Omega)}\|\nabla \mathcal{D}_\varepsilon(u_\varepsilon)\|_{L^2(\Omega)}\|\nabla v_\varepsilon\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\|\nabla \mathcal{D}_\varepsilon(u_\varepsilon)\|_{L^2(\Omega)} + \frac{1}{2}\|g\|_{L^\infty(\Omega)}\|\nabla v_\varepsilon\|_{L^2(\Omega)} \end{aligned}$$

where $\frac{d^2}{dr^2}\hat{\mathcal{D}}_\varepsilon = \frac{d}{dr}\mathcal{D}_\varepsilon = D_\varepsilon$ where we have used the Young's inequality. Now integrating from 0 to T and using (3.21), we get

$$\int_{Q_T} |\nabla \mathcal{D}_\varepsilon(u_\varepsilon)|^2 dx \leq c \tag{3.22}$$

which proves the desired estimate.

Now from (3.13)-(3.21) and the standard compactness arguments (see [31]), we can extract a subsequences such that as ε tends to 0

$$\begin{aligned}
u_\varepsilon &\rightharpoonup u \quad \text{and} \quad v_\varepsilon \rightharpoonup v \quad \text{weakly in } L^\infty(Q_T),\\
u_\varepsilon &\rightharpoonup u \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)),\\
v_\varepsilon &\rightharpoonup v \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)),\\
\partial_t u_\varepsilon &\rightharpoonup \partial_t u \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)),\\
\partial_t v_\varepsilon &\rightharpoonup \partial_t v \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)).
\end{aligned}$$

$u_\varepsilon \in C([0,T];L^2(\Omega))$ and the compact imbedding $L^2(0,T;H_0^1(\Omega)) \subset L^2(Q_T)$ (see [32]) lead to

$$u_\varepsilon \to u \quad \text{strongly in } L^2(Q_T) \text{ and } a.e \text{ in } Q_T.$$

From the above convergence results and (3.22) in (3.2), as $\varepsilon \to 0$, we conclude that, for some $\phi, \psi \in L^2(0,T;H_0^1(\Omega))$,

$$-\int_\Omega u_0(x)\phi(x)dx - \int_{Q_T} u\phi_t dxdt + \int_{Q_T} D(u)\nabla u \nabla\phi dxdt - \int_{Q_T} g(u)\nabla v\nabla\phi dxdt = 0,$$

$$-\int_\Omega v_0(x)\psi(x)dx - \int_{Q_T} v\psi_t dxdt + d\int_{Q_T} \nabla v\nabla\psi dxdt = \int_{Q_T} (\alpha u - \beta v)\psi dxdt.$$

This proves the existence of weak solutions of the original system (1.1). ■

4. Uniqueness of Solutions

Theorem 4.1. *The solution of the system (1.1) is unique.*

Proof. Let us assume that (u_1, u_2) and (v_1, v_2) are the two solutions of the system (1.1). To prove the uniqueness result, we use here the duality approach [4]. Taking $u = u_1 - u_2$ and $v = v_1 - v_2$, u and v satisfy,

$$\left.\begin{aligned}
&u_t - \nabla\cdot(D(u_1)\nabla u_1 - D(u_2)\nabla u_2)\\
&= -\nabla\cdot(g(u_1)\nabla v_1 - g(u_2)\nabla v_2) \text{ in } Q_T,\\
&v_t - d\Delta v = \alpha u - \beta v \quad \text{in } Q_T,\\
&u(x,0) = v(x,0) = 0 \quad \text{in } \Omega,\\
&u(x,t) = v(x,t) = 0 \quad \text{on } \partial\Omega.
\end{aligned}\right\} \tag{4.1}$$

Define by ϕ the solution of the problem $-\Delta\phi(\cdot,t) = u(\cdot,t)$ in Ω and $\phi(\cdot,t) = 0$ on $\partial\Omega$ for a.e $t \in (0,T)$. Since u_1 and u_2 are bounded solutions, from the theory of linear elliptic equations, the solution ϕ satisfies $\phi \in C([0,T];H^2(\Omega))$ with $\int_\Omega \phi(\cdot,t)dx = 0$. From $u(\cdot,0) = 0$ we get $\phi(\cdot,t) = 0$ in $L^2(\Omega)$. Now

$$\begin{aligned}\int_0^t < u_t, \phi > ds &= \int_{Q_t} D(u_1)\nabla u \nabla\phi dxds \\ &- \int_{Q_t} (D(u_1) - D(u_2))\nabla u_2 \nabla \phi dxds \\ &- \int_{Q_t} g(u_1)\nabla v \nabla\phi dxds + \int_{Q_t} (g(u_1) - g(u_2))\nabla v_2 \nabla\phi dxds,\end{aligned} \tag{4.2}$$

where $Q_t = \Omega \times (0,t)$. Since u_1 and u_2 are bounded solutions, there exists a constant $c > 0$ depending on $\|u_1\|_{L^\infty(\Omega)}, \|u_2\|_{L^\infty(\Omega)}$ such that

$$|D(u_1) - D(u_2)| \leq c|u_1 - u_2| \quad \text{and} \quad |g(u_1) - g(u_2)| \leq c|u_1 - u_2|.$$

$$2\int_0^t < u_t, \phi > ds = \int_\Omega |\nabla\phi(x,t)|^2 dx - \int_\Omega |\nabla\phi(x,0)|^2 dx = \int_\Omega |\nabla\phi(x,t)|^2 dx.$$

With the above results, the equation (4.2) becomes

$$\begin{aligned}&\int_\Omega |\nabla\phi(x,t)|^2 dx \leq 2\int_0^t \|D(u_1)\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} ds \\ &+2\int_0^t \|\nabla u_2\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} ds \\ &+2\int_0^t \|g(u_1)\|_{L^\infty(\Omega)}\|\nabla v\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} ds \\ &+2\int_0^t \|u\|_{L^2(\Omega)}\|\nabla v_2\|_{L^\infty(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} ds \\ &\leq c\left(\int_0^t \|\nabla u\|^2_{L^2(\Omega)} ds + \int_0^t \|u\|^2_{L^2(\Omega)} ds + \int_0^t \|\nabla\phi\|^2_{L^2(\Omega)} ds\right)\end{aligned}$$

Similarly one can get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{Q_t} |v|^2 dxds &= -d\int_{Q_t} |\nabla v|^2 dxds + \int_{Q_t} (\alpha u - \beta v) v dxds \\ &\leq -d\int_{Q_t} |\nabla v|^2 dxds + c\int_{Q_t} |u^2| dxds + c\int_{Q_t} |v|^2 dxds,\end{aligned}$$

where $c > 0$ is constant.
Finally, from the above estimates, we have

$$\frac{1}{2}\int_{\Omega}|v(x,t)|^2dx + \int_{\Omega}|\nabla\phi(x,t)|^2dx \le c\int_0^T \|\nabla\phi\|^2_{L^2(\Omega)}ds + c\int_{Q_t}|v|^2dxds,$$

since $\nabla v \in L^p(0,T;L^\infty(\Omega))$ and Gronwall's lemma show that $v = 0$ and $\nabla\phi = 0$ a.e in Q_T. This proves the uniqueness of solutions. ∎

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