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RANDOM ATTRACTOR FOR FRACTIONAL GINZBURG-LANDAU EQUATION WITH MULTIPLICATIVE NOISE

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Abstract. In this paper, we consider the asymptotic behavior of solutions to the stochastic fractional complex Ginzburg-Landau equation with multiplicative noise in one spatial dimensions. We first transfer stochastic fractional Ginzburg-Landau equation into random equation which solutions generate a random dynamical system. Then, we consider the existence of a random attractor for the random dynamical system. At last we estimate the Hausdorff dimension of the random attractor by using linearization and Lyapunov exponents.

1. INTRODUCTION

The fractional differential equation is called an equation that contains fractional derivatives or fractional integrals. The physical background of fractional differential equations is profound. At present, the fractional derivative and fractional integral have a wide range of applications in physics, biology, chemistry and other fields of science, such as kinetic theories of systems with chaotic dynamics ([1, 2]); pseudochaotic dynamics ([3]); dynamics in a complex or porous media ([4]); random walks with a memory and flights ([5, 7]); and many other aspects. In recent years, fractional partial differential equations have been proposed in many fields of fluid mechanics, mechanics of materials, biology, plasma physics, finance, chemistry, etc., and are being studied, including the fractional Schrödinger equation ([8, 9, 10]), fractional Landau-Lifshitz equation ([11]), fractional Landau-Lifshitz-Maxwell equation ([12]) and fractional Ginzburg-Landau equation ([13]).

The fractional Ginzburg-Landau equation can be used to describe the dynamical processes in a medium with fractal dispersion. The fractional generalization of the Ginzburg-Landau equation from the variational Euler-Lagrange equation for the fractal

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media is derived in [13]. However, some perturbations may neglect in the derivation of this ideal model (such as molecular collisions in gases and liquids and electric fluctuations in resistors [14]). When considering the perturbations of each microscopic units to the models, which will lead to a very large complex system, people usually represent the micro effects by random perturbations in the dynamics of the macro observable. So stochastic partial differential equation and its application is becoming more and more interesting and important in mathematical physics recently.

One of the most important problems of modern mathematical physics is the asymptotic behavior of random dynamical systems. One way to attack the problem for dissipative random dynamical systems is to study its random attractor. The theory of random attractors developed by Crauel, Flandoli ([15, 16]), which closely parallels the deterministic case ([17]), is becoming very useful for the study of the asymptotic behavior of dissipative random dynamical systems. The random attractor is a random invariant compact set that attracts any orbit starting from $-\infty$. Its geometry is very complicated and it can reflect the complexity of the long-time behavior of the random dynamical systems. However it seems that the asymptotic behavior of the random dynamical systems is governed by a finite number of degrees of freedom. Debussche ([18]) proved that the Hausdorff dimension of the random attractor could be estimated by using global Lyapunov exponents. In this paper, we obtain an upper bound on the Hausdorff dimension by using linearization and Lyapunov exponents.

In this paper, we consider the following stochastic fractional Ginzburg-Landau equation with multiplicative noise of Itô form in $\mathbb{R}$:

$$du+\big((1+\mathrm{i}\nu)(-\triangle)^{\alpha}u+(1+\mathrm{i}\mu)|u|^2u\big)\,dt=\rho u dt+\beta u dW(t),\ x\in\mathbb{R},\ \ t>0 \tag{1.1}$$

with the initial condition and the periodic boundary condition:

$$u(x,0)=u_0,\quad x\in\mathbb{R}, \tag{1.2}$$

$$u(x+2\pi,t)=u(x,t),\ x\in\mathbb{R},\quad t>0, \tag{1.3}$$

$u(x,t)$ is a complex-valued function on $\mathbb{R}\times[0,+\infty)$. In (1.1), i is the imaginary unit, $\nu,\mu,\rho>0,\beta>0$ are real constants, and $\alpha\in(1/2,1)$. The white noise described by a two-sided Wiener process $W(t)$ on a complete probability space results from the fact that small irregularity has to be taken account in some circumstances.

For the deterministic fractional complex Ginzburg-Landau equation, in [19], the authors obtained the well-posedness with the semigroup method under the condition

$$\frac{1}{2}\le\sigma\le\frac{1}{\sqrt{1+\mu^2-1}}. \tag{1.4}$$

The existence of global attractor in L^2 is obtained also under the condition $\sigma=1$.

In this paper, we arrange as follows. In section 2, some preliminaries and notations are shown and the random attractors theory for the random dynamical system are recalled. In section 3, we define a continuous random dynamical system for the stochastic fractional complex Ginzburg-Landau equation. In section 4, the existence of the random attractor for the stochastic Ginzburg-Landau equation is proved. In section 5, we prove that the random attractor has a finite Hausdorff dimension.

2. Preliminaries and Notations

In this section, we first review some basic concepts related to random attractors for stochastic dynamical systems. The reader may refer to [15, 16] for more details.

Let $(X, ||\cdot||_X)$ be a separable Hilbert space with Borel σ-algebra $\mathcal{B}(x)$, and $\{\theta_t : \Omega \longrightarrow \Omega, t \in \mathbb{R}\}$ be a family of measure preserving transformations of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $(t, \omega) \longmapsto \theta_t\omega$ is measurable, $\theta_0 = \mathbb{I}$, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$. Thus $(\theta_t)_{t\in\mathbb{R}}$ is a flow, and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a (measurable) dynamical system. We also denote the mappings $S(t, s; \omega) : X \to X, \ -\infty < s \leq t < \infty$, with explicit dependence on $\omega \in \Omega$.

Definition 2.1. For any invariant set A, if $P(A) = 0$ or $P(A) = 1$, we call measure preserving transformation $\theta_t : \Omega \longrightarrow \Omega, t \in \mathbb{R}$ or metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is ergodic.

Definition 2.2. Given $t \in \mathbb{R}$, and $\omega \in \Omega$, $K(t, \omega) \subset X$ is called an attracting set, if for all bounded sets $B \subset X$,

$$d(S(t, s; \omega)B, K(t, \omega)) \to 0 \quad \text{as} \quad s \to -\infty,$$

where $d(Y, Z) = \sup_{y\in Y} \inf_{z\in Z} \|y - z\|_X$, for any $Y, Z \subseteq X$.

Definition 2.3. A set valued map $\mathcal{A}(\omega) : \Omega \to 2^X$ taking valued in the closed subsets of X is called to be measurable, if for all $x \in X$, the mapping $\omega \mapsto d(\mathcal{A}(\omega), x)$ is measurable.

Definition 2.4. The random omega limit set of a bounded set $B \subset X$ at time t by

$$\mathcal{A}(B, t, \omega) = \bigcap_{T\leq t} \overline{\bigcup_{s\leq T} S(t, s; \omega)B}.$$

Definition 2.5. Let $S(t, s; \omega)_{t\geq s, \omega\in\Omega}$ be a stochastic dynamical system, A random set $\mathcal{A}(\omega)$ is called a random attractor if the following conditions are satisfied, for P-a.e. $\omega \in \Omega$,

- It is the minimal closed set such that, for $t \in \mathbb{R}, B \subset X$,

$$d(S(t, s; \omega)B, \mathcal{A}(\omega)) \to 0 \quad \text{as} \quad s \to -\infty,$$

$\mathcal{A}(\omega)$ attracts B (B is a deterministic set);

- $\mathcal{A}(\omega)$ is the largest compact measurable set, which is invariant in the sense that

$$S(t,s;\omega)\mathcal{A}(\theta_s\omega)=\mathcal{A}(\theta_t\omega),\quad s\le t.$$

According to [15], we have the following theorem about existence of random attractors.

Theorem 2.1. *Let $S(t,s;\omega)_{t\ge s,\omega\in\Omega}$ be a stochastic dynamical system satisfying the following conditions:*

- $S(t,r;\omega)S(r,s;\omega)=S(t,s;\omega)x$*, for all* $s\le r\le t$ *and* $x\in X$*;*
- $S(t,s;\omega)$ *is continuous in X, for all* $s\le t$*;*
- *for all* $s<t$ *and* $x\in X$*, the mapping* $\omega\mapsto S(t,s;\omega)x$ *from* $(\Omega,\mathcal{F})$ *to* $(X,\mathcal{B}(x))$ *is measurable;*
- *for all* $t,x\in X$ *and P-a.e.* ω*, the mapping* $s\mapsto S(t,s;\omega)x$ *is right continuous at any point.*

Assume that there exists a group $\theta_t,t\in\mathbb{R}$*, of measure preserving mapping, such that*

$$S(t,s;\omega)x=S(t-s,0;\theta_s\omega)x,\qquad P-a.e.$$

holds, and for P-a.e. ω*, there exists a compact attracting set* $K(\omega)$ *at time 0, for P-a.e.* $\omega\in\Omega$*, we set* $\mathcal{A}(\omega)=\overline{\bigcup_{B\subset X}A(B,\omega)}$*, where the union is taken over all the bounded subsets of X, and* $A(B,\omega)$ *is given by*

$$\mathcal{A}(B,t,\omega)=\bigcap_{T\le 0}\overline{\bigcup_{s\le T}S(t,s;\omega)B}.$$

Then, $\mathcal{A}(\omega)$ *is the random attractor.*

Although the random attractor is not uniformly bounded, it is expected that the theory on the Hausdorff dimension of a global attractor of a deterministic dynamical system can be generalized to the stochastic case under some assumption ([18]). Due to [18], we have the following conclusion.

Theorem 2.2. *Let* $\mathcal{A}(\omega)$ *be a compact measurable set which is invariant under a random map* $S(\omega),\omega\in\Omega$*, for some ergodic metric dynamical system* $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$*. Assume the following conditions are satisfied.*

(1) $S(\omega)$ *is almost surely uniformly differentiable on* $\mathcal{A}(\omega)$*, that is, for every* $u,u+h\in\mathcal{A}(\omega)$ *there exists* $DS(\omega,u)$ *in* $\mathcal{L}(X)$*, the space of the bounded linear operator from* X *to* X*, such that*

$$\|S(\omega)(u+h)-S(\omega)u-DS(\omega,u)h\|\le\bar{k}(\omega)\|h\|^{1+\delta},$$

where $\delta>0,\bar{k}(\omega)$ *is a random variable satisfying* $\bar{k}(\omega)\ge 1,E(\log\bar{k})<\infty$,

(2) $\omega_d(DS(\omega,u)) \le \bar{\omega}_d(\omega)$ *for* $u \in \mathcal{A}(\omega)$ *and some random variable* $\bar{\omega}_d(\omega)$ *satisfying* $E(\log(\bar{\omega}_d)) < 0$, *where*

$$\omega_d(DS(\omega,u)) = \alpha_1(DS(\omega,u)) \cdots \alpha_d(DS(\omega,u)),$$

$$\alpha_d(DS(\omega,u)) = \sup_{\substack{G\subset X,\\ \dim G\le d}} \inf_{\substack{\varphi\in G,\\ \|\varphi\|_X=1}} \|DS(\omega,u)\varphi\|,$$

(3) $\alpha_1(DS(\omega,u)) \le \bar{\alpha}_1(\omega)$, *for* $u \in \mathcal{A}(\omega)$ *and a random variable* $\bar{\alpha}_1(\omega) \ge 1$ *with* $E(\log \bar{\alpha}_1) < \infty$.

Then the Hausdorff dimension $d_H(\mathcal{A}(\omega))$ *of* $\mathcal{A}(\omega)$ *is less than* d *almost surely.*

In what follows, we redefine some concepts and notations to the fractional derivative and fractional Sobolev space.

Since u is a periodic function, it can be expressed by a Fourier series $u = \sum_{k\in\mathbb{Z}} u_k e^{\mathrm{i}\langle k,x\rangle}$. Then, we have $u_x = \sum_{k\in\mathbb{Z}} \mathrm{i}k u_k e^{\mathrm{i}\langle k,x\rangle}$. So $(-\triangle)^\alpha$ is defined by

$$(-\triangle)^\alpha u = \sum_{k\in\mathbb{Z}} |k|^{2\alpha} u_k e^{\mathrm{i}\langle k,x\rangle},$$

where $\triangle = \partial^2/\partial x^2$. Let H^α denote the complete Sobolev space of order α under the norm:

$$\|u\|_{H^\alpha} = \left(\sum_{k\in\mathbb{Z}} |k|^{2\alpha}|u_k|^2 + \sum_{k\in\mathbb{Z}} |u_k|^2\right)^{\frac{1}{2}}.$$

Let $\mathcal{D} = [0, 2\pi] \subset \mathbb{R}$. Throughout this paper, we denote by $(\cdot,\cdot)$ the usual inner product of $L^2(\mathcal{D})$, $\|\cdot\|_{H^m}$ the norm of Sobolev spaces $H^m(\mathcal{D})$, and $\|\cdot\|_m = \|\cdot\|_{L^m(\mathcal{D})}(m = 1, 2, \cdots, \infty)$. Let $L^2_p(\mathcal{D}) = \{\varphi \in L^2(\mathcal{D}) | \varphi(x + 2\pi) = \varphi(x)\}$ with the norm defined just as that of $L^2(\mathcal{D})$. Let $H^m_p(\mathcal{D}) = \{\varphi \in H^m(\mathcal{D}) | \varphi(x+2\pi) = \varphi(x)\}$ with the norm defined just as that of $H^m(\mathcal{D})$. In the forthcoming discussions, we use T to denote any arbitrary positive constant, and use $c_j (j = 1, 2, \cdots)$ denote different positive constants which depend only on the constants $\rho, \nu, \mu, \alpha, \sigma$. In this paper, we denote $\int_{\mathcal{D}} f dx$ by the notation $\int f$. In addition, the following Gagliardo-Nirenberg inequality([20]) is frequently used.

Lemma 2.1. *Let u belong to* L^q *and its derivatives of order* m, $D^m u$, *belong to* L^r, $1 \le q, r \le \infty$. *For the derivatives* $D^j u$, $0 \le j < m$, *the following inequalities hold*

(2.1)
$$\|D^j u\|_{L^p} \le c\|u\|^\theta_{W^{m,r}} \|u\|^{1-\theta}_{L^q}.$$

where

$$\frac{1}{p} = \frac{j}{n} + \theta(\frac{1}{r} - \frac{m}{n}) + (1-\theta)\frac{1}{q},$$

for all θ in the interval

$$\frac{j}{m} \leq \theta \leq 1$$

(the constant c depending only on n, m, j, q, r, θ), with the following exceptional case

1. *If $j = 0, rm < n, q = \infty$ then we make the additional assumption that either u tends to zero at infinite or $u \in L^{\tilde{q}}$ for some finite $\tilde{q} > 0$.*
2. *If $1 < r < \infty$, and $m - j - n/r$ is a nonnegative integer then* (2.1) *holds only for θ satisfying*
   $j/m \leq \theta < 1$.

The following lemma ([21]) is also needed in this paper.

Lemma 2.2. *Suppose that $s > 0$ and $p \in (1, +\infty)$. If $f, g \in \mathcal{S}$, the Schwartz class, then*

$$\|(-\triangle)^{\frac{s}{2}}(fg)\|_p \leq C(\|f\|_{p_1}\|g\|_{H^{s,p_2}} + \|f\|_{H^{s,p_3}}\|g\|_{p_4})$$

with $p_2, p_3 \in (1, +\infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

3. Stochastic Fractional Complex Ginzburg-landau Equation

In this section, we discuss the existence of a continuous random dynamical system for the stochastic fractional complex Ginzburg-Landau equation perturbed by a multiplicative white noise in the Itô sense. Thanks to the special linear multiplicative noise, the stochastic fractional Ginzburg-Landau equation can be reduced to an equation with random coefficients by a suitable change of variable. The process

$$z(t) = e^{-\beta W(t)}$$

satisfies the stochastic differential equation:

$$dz(t) = \frac{1}{2}\beta^2 z dt - \beta z dW(t).$$

We translate the unknown $v(t) = z(t)u(t)$ to obtain the following random differential equation:

$$v_t = -(1 + \mathrm{i}\nu)(-\triangle)^{\alpha} v + (\rho + \frac{1}{2}\beta^2)v - (1 + \mathrm{i}\mu)z^{-2}|v|^2 v, \quad t > s, \tag{3.1}$$

with the initial data at time s

$$v(x, s) = v_s, \qquad x \in \mathbb{R}, \tag{3.2}$$

and the periodic boundary condition:

$$v(x+2\pi,t)=v(x,t), \qquad x\in\mathbb{R}, \tag{3.3}$$

In what follows, we construct a random dynamical system modeling the stochastic fractional Ginzburg-Landau equation. First, we consider the set of continuous functions with value 0 at 0,

$$\Omega=\{\omega\in C(\mathbb{R},\mathbb{R}):\omega(0)=0\}.$$

Let $W(t,\omega)=\omega(t)$. A family of measure preserving and ergodic transformations of $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ can be defined by

$$\theta_t\omega(s)=\omega(t+s)-\omega(t), \quad s,t\in\mathbb{R}.$$

The existence and uniqueness of the solution for the problem (3.1)-(3.3) can be obtained (see Theorem 3.3 in [19], for example), which defines a stochastic dynamical system $(S(t,s;w))_{t\geq s,\omega\in\Omega}$ by

$$S(t,s;w)u_s=u(t,\omega;s,u_s)=v(t,\omega;s,u_sz(s,\omega))z(t,\omega).$$

4. A Priori Estimates and the Existence of a Random Attractor

In this section, we make some a priori estimates of the solution, which can prove the existence of a compact absorbing set. Then applying the Theorem 2.1, the existence of random attractor is obtained. First, we obtain the following lemmas to prove the existence of a compact absorbing set.

Lemma 4.1. *There exists a random radius $r_1(\omega),r_2(\omega)>0$ such that, for any given $R>0$, there exists $\bar{s}_1(\omega)\leq -1$ such that for all $s\leq\bar{s}_1(\omega)$, $u_s\in L^2_p(\mathcal{D})$ satisfying $\|u_s\|\leq R$, the following inequalities*

$$\|v(t)\|^2\leq r_1^2(\omega), \qquad \forall\ t\in[-1,0], \tag{4.1}$$

and

$$\int_{-1}^{0}\Big(\|(-\triangle)^{\frac{\alpha}{2}}v(\tau)\|^2+z^{-2}(\tau)\|v(\tau)\|_4^4\Big)\,d\tau\leq r_2^2(\omega). \tag{4.2}$$

where

$$r_2^2(\omega)=C_0'\int_{-1}^{0}z^2(\tau)d\tau+r_1^2(\omega).$$

Proof. Taking the inner product in L^2 of (1.1) with v and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2+\|(-\triangle)^{\frac{\alpha}{2}}v\|^2+z^{-2}\int|v|^4=(\rho+\frac{1}{2}\beta^2)\|v\|^2. \tag{4.3}$$

By Young's inequality, we obtain

$$(3\rho+\beta^2)\|v\|^2=(3\rho+\beta^2)\int|v|^2\le z^{-2}\int|v|^4+\pi z^2\cdot\frac{(3\rho+\beta^2)^2}{2}.$$

Then (4.3) can be rewritten as

$$\frac{d}{dt}\|v\|^2+2\|(-\triangle)^{\frac{\alpha}{2}}v\|^2+z^{-2}\|v\|_4^4+\rho\|v\|^2\le\pi z^2\cdot\frac{(3\rho+\beta^2)^2}{2}=z^2\cdot C_0', \tag{4.4}$$

which implies that, for any $t\ge s$,

$$\begin{aligned}\|v(t)\|^2&\le e^{-\rho t}\left(e^{\rho s}\|v_s\|^2+C_0'\int_s^t e^{\rho\tau}z^2(\tau)d\tau\right)\\&=e^{-\rho t}\left(e^{\rho s}z^2(s)\|u_s\|^2+C_0'\int_s^t e^{\rho\tau}z^2(\tau)d\tau\right).\end{aligned}$$

Thanks to

$$\lim_{s\to-\infty}\frac{W(s)}{s}=0,\quad P-a.s.\ ,$$

it is easy to check it,

$$e^{\rho s}z^2(s)=e^{\rho s-2\beta W(s)}\longrightarrow 0,\quad P-a.s.\quad \text{as}\quad s\longrightarrow-\infty, \tag{4.5}$$

Then we infer that, for any $u_s\in L_p^2(\mathcal{D})$ with $\|u_s\|\le R$, there exists a time $s_1(\omega)\le -1$ such that

$$e^{\rho s}z^2(s)\|u_s\|^2\le e^{\rho s}z^2(s)R^2\le 1$$

hold P-a.s. for any $s\le s_1(\omega)$. We also obtain that

$$\forall\ \varepsilon>0,\exists\ \ s_1'(\omega)\le-1,\ \text{as}\ s(\omega)<s_1'(\omega),\quad\text{we have}\ \left|\frac{W(s)}{s}\right|<\varepsilon,$$

then we have

$$e^{-2\beta W(s)}<e^{-2\beta\varepsilon s}.$$

So we deduce that

$$C_0'\int_s^t e^{\rho\tau}z^2(\tau)d\tau\le C_0'\int_{-\infty}^0 e^{\rho\tau}e^{-2\beta W(s)}d\tau\le C_0'\int_{-\infty}^0 e^{\rho\tau}e^{-2\beta\varepsilon\tau}d\tau.$$

Let ε be small enough such that $\varepsilon<\frac{\rho}{2\beta}$, and $\bar{s}_1(\omega)=\min\{s_1(\omega),s_1'(\omega)\}$, hence we deduce that

$$\begin{aligned}\|v(t)\|^2&\le e^{-\rho t}\left(1+C_0'\int_{-\infty}^t e^{\rho\tau}z^2(\tau)d\tau\right)\\&\le e^{-\rho t}\left(1+C_0'\int_{-\infty}^0 e^{\rho\tau}z^2(\tau)d\tau\right)\le r_1^2(\omega),\quad t\in[-1,0],\end{aligned}$$

and

$$\int_{-1}^{0}\left(\|(-\triangle)^{\frac{\alpha}{2}}v(\tau)\|^2+z^{-2}(\tau)\|v(\tau)\|_4^4\right)d\tau \le C_0'\int_{-1}^{0}z^2(\tau)d\tau+\|v(-1)\|^2$$
$$\le C_0'\int_{-1}^{0}z^2(\tau)d\tau+r_1^2(\omega)=r_2^2(\omega). \ \blacksquare$$

Lemma 4.2. *There exists a random radius $r_3(\omega)>0$ such that, for any given $R>0$, there exists $\bar{s}_1(\omega)\le -1$ such that for all $s\le \bar{s}_1(\omega)$, $u_s\in L_p^2(\mathcal{D})$ satisfying $\|u_s\|\le R$, the following inequalities*

$$\|(-\triangle)^{\frac{\alpha}{2}}v(t)\|^2\le r_3^2(\omega),\qquad \forall\ t\in[-1,0], \tag{4.6}$$

where

$$r_3^2(\omega)=e^{\sqrt{1+\mu^2}\cdot r_2^2(\omega)+2\rho+\beta^2+c\sqrt{1+\mu^2}\cdot\int_{-1}^{0}z^{-2}(\tau)d\tau}\cdot r_2^2(\omega).$$

Proof. Taking the inner product in L^2 of (1.1) with $(-\triangle)^{\alpha}v$ and taking the real part, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(-\triangle)^{\frac{\alpha}{2}}v\|^2+\|(-\triangle)^{\alpha}v\|^2\\&=(\rho+\frac{1}{2}\beta^2)\|(-\triangle)^{\frac{\alpha}{2}}v\|^2-z^{-2}\mathrm{Re}(1+\mathrm{i}\mu)(|v|^2v,(-\triangle)^{\alpha}v).\end{aligned}\tag{4.7}$$

Integrating by parts, using Hölder inequality and Young inequality, we infer that

$$\begin{aligned}&\left|-2\cdot z^{-2}\mathrm{Re}(1+\mathrm{i}\mu)(|v|^{2\sigma}v,(-\triangle)^{\alpha}v)\right|\\&=2\cdot z^{-2}\left|\mathrm{Re}(1+\mathrm{i}\mu)\int(-\triangle)^{\frac{\alpha+\varepsilon}{2}}\bar{v}(-\triangle)^{\frac{\alpha-\varepsilon}{2}}(|v|^2v)\right|\\&\le 2\cdot z^{-2}\sqrt{1+\mu^2}\|(-\triangle)^{\frac{\alpha+\varepsilon}{2}}v\|_p\|(-\triangle)^{\frac{\alpha-\varepsilon}{2}}(|v|^2v)\|_q\\&\le z^{-2}\sqrt{1+\mu^2}\left(\|(-\triangle)^{\frac{\alpha+\varepsilon}{2}}v\|_p^2+\|(-\triangle)^{\frac{\alpha-\varepsilon}{2}}(|v|^2v)\|_q^2\right),\end{aligned}\tag{4.8}$$

where any ε satisfies $0<\varepsilon<\alpha$, and $p<\infty$ which satisfies $1/p+1/q=1$ is sufficiently large. Let $\varepsilon=1/p$, applying Gagliardo-Nirenberg inequality and Young inequality, we deduce that

$$\begin{aligned}\|(-\triangle)^{\frac{\alpha+\varepsilon}{2}}v\|_p^2&\le c(\|(-\triangle)^{\alpha}v\|^{\frac{1}{\alpha}}\|(-\triangle)^{\frac{\alpha}{2}}v\|^{2-\frac{1}{\alpha}}+\|(-\triangle)^{\frac{\alpha}{2}}v\|^2)\\&\le \xi\|(-\triangle)^{\alpha}v\|^2+(c+c(\xi))\|(-\triangle)^{\frac{\alpha}{2}}v\|^2.\end{aligned}$$

Applying Lemma 2.2 twice, we deduce that

$$\begin{aligned}\|(-\triangle)^{\frac{\alpha-\varepsilon}{2}}(|v|^2v)\|_q&\le c(\|(-\triangle)^{\frac{\alpha-\varepsilon}{2}}|v|^2\|_s\|v\|_4+\|(-\triangle)^{\frac{\alpha-\varepsilon}{2}}v\|_r\||v|^2\|)\\&\le c\|(-\triangle)^{\frac{\alpha-\varepsilon}{2}}v\|_r\|v\|_4^2\\&\le c\|(-\triangle)^{\frac{\alpha}{2}}v\|\|v\|_4^2,\end{aligned}$$

for $1/q = 1/s + 1/4 = 1/r + 1/2$. Let $\xi = z^2/\sqrt{1+\mu^2}$. Then (4.7) can be rewritten as

$$\frac{d}{dt}\|(-\triangle)^{\frac{\alpha}{2}}v\|^2 + \|(-\triangle)^{\alpha}v\|^2 \leq \lambda\|(-\triangle)^{\frac{\alpha}{2}}v\|^2, \tag{4.9}$$

where $\lambda = 2\rho + \beta^2 + z^{-2}\sqrt{1+\mu^2}(c + c(\xi) + \|v\|_4^4)$. Integrating (4.9) from t to s, $-1 \leq s \leq t \leq 0$, we deduce that

$$'\|(-\triangle)^{\frac{\alpha}{2}}v(t)\|^2 \leq e^{\int_s^t \lambda d\tau} \cdot \|(-\triangle)^{\frac{\alpha}{2}}v(s)\|^2 \leq e^{\int_{-1}^0 \lambda d\tau} \cdot \|(-\triangle)^{\frac{\alpha}{2}}v(s)\|^2. \tag{4.10}$$

After integration with respect to s on $[-1, 0]$, we obtain that

$$\begin{aligned} &\|(-\triangle)^{\frac{\alpha}{2}}v(t)\|^2 \\ &\leq e^{\int_{-1}^0 \lambda d\tau} \cdot \int_{-1}^0 \|(-\triangle)^{\frac{\alpha}{2}}v(s)\|^2 \\ &\leq e^{\sqrt{1+\mu^2}\cdot r_2^2(\omega)+2\rho+\beta^2+c\sqrt{1+\mu^2}\cdot\int_{-1}^0 z^{-2}(\tau)d\tau} \cdot r_2^2(\omega) = r_3^2(\omega). \end{aligned} \tag{4.11}$$

■

By the Lemma 4.2, we deduce that, for given $R > 0$, there exists an $\bar{s}_1(\omega) \leq -1$ such that for any $s \leq \bar{s}_1(\omega)$,

$$\|(-\triangle)^{\frac{\alpha}{2}}v(0)\| = \|(-\triangle)^{\frac{\alpha}{2}}u(0)\| \leq r_3(\omega)$$

holds P-a.e.. Let $K(\omega)$ be the ball in $H_p^{\alpha}(\mathcal{D})$ of radius $r_3(\omega)$. It is shown that, for any B bounded in $L_p^2(\mathcal{D})$, there exist an $\bar{s}_1(\omega)$ such that, for any $s \leq \bar{s}_1(\omega)$,

$$S(0, s; \omega)B \subset K(\omega) \quad \text{holds} \quad P - a.e..$$

This clearly implies that $K(\omega)$ ia an attracting set at time 0 since it is compact in $L_p^2(\mathcal{D})$, and applying Theorem 2.1 we obtain the following results.

Theorem 4.1. *The stochastic dynamical system associated with the fractional Ginzburg-Landau equation with multiplicative noise has a compact stochastic attractor in* $L_p^2(\mathcal{D})$.

5. Hausdorff Dimension of the Random Attractor $\mathcal{A}(\omega)$

In this section, we show that the Hausdorff dimension of the maximal attractor $\mathcal{A}$ is finite. Let

$$S(\omega) = S(1, 0; \omega), \quad T(\omega) = T(1, 0; \omega)$$

and

$$u(t) = S(t, 0; \omega)u_0 = e^{\beta W(t)}T(t, 0; \omega)v_0.$$

Hence, it is easy to check that, if $T(\omega)$ ia almost surely uniformly differentiable with the Fr$\acute{e}$chet derivative $DT(w)$, then $S(\omega)$ is also almost surely uniformly differentiable with the Fr$\acute{e}$chet derivative $DS(w) = e^{\beta W(1)}DT(w)$

Lemma 5.1. *The mapping T is almost surely uniformly differentiable on $\mathcal{A}(\omega)$ and there exist a linear operator $DT(\omega, v)$ in $\mathcal{L}(L^2_p(\mathcal{D}))$, the space of continuous linear operator from $L^2_p(\mathcal{D})$ to $L^2_p(\mathcal{D})$, such that if v and $v+h$ are in $\mathcal{A}(\omega)$:*

$$\|T(\omega)(v+h) - T(\omega)(v) - DT(\omega, v)h\| \leq \mathcal{K}(\omega)\|h\|^{1+\chi}, \tag{5.1}$$

where $\mathcal{K}(\omega)$ is random variable such that

$$\mathcal{K}(\omega) \geq 1, \;\; E(\ln \mathcal{K}) < \infty, \;\; \omega \in \Omega$$

and $\chi > 0$ is a number such that $\chi > 0$. For any $v_0 \in \mathcal{A}(\omega)$, $DT(\omega, v_0)h = V(1)$ where V is the solution of

$$\frac{dV}{dt} = F'(t, v)V, \tag{5.2}$$

$$V(0) = h, \tag{5.3}$$

where

$$\begin{aligned} v(t) &= e^{-\beta W(t)} S(t, 0; \omega) u_0, \\ F'(t, v)V &= (\rho + \frac{1}{2}\beta^2)V - (1 + \mathrm{i}\nu)(-\triangle)^{\alpha} V - (1 + \mathrm{i}\mu) z^{-2} f'(v)V, \\ f'(v)V &= (|v|^{2\sigma} v)'_t = 2|v|^2 V + v^2 \bar{V}. \end{aligned}$$

Proof. Let $e(t) = v_1(t) - v_2(t) - V(t)$, where $v_j(t) (j = 1, 2)$ be two solutions of (3.1) with $v_j(0) = v^0_j$ and $V(t)$ satisfies (5.2)-(5.3) with $F'(t, v_2)$ and $h = v^0_1 - v^0_2$. Then $e(t)$ satisfies the equation

$$\begin{aligned} \frac{de}{dt} &= (\rho + \frac{1}{2}\beta^2)e - (1 + \mathrm{i}\nu)(-\triangle)^{\alpha} e \\ &\quad -(1 + \mathrm{i}\mu) z^{-2} (f(v_1) - f(v_2) - f'(v_2)(v_1 - v_2 - e)). \end{aligned} \tag{5.4}$$

where

$$\Phi = -(1 + \mathrm{i}\mu) z^{-2} \left(|v_1|^2 v_1 - |v_2|^2 v_2 - 2|u|^2 (v_1 - v_2) - v_2^2 (\bar{v}_1 - \bar{v}_2) \right),$$

and

$$\Psi = -(1 + \mathrm{i}\mu) z^{-2} \left(2|v_2|^2 e + v_2^2 \bar{e} \right).$$

Applying Taylor's formula for function $G(v_1, \bar{v}_1) = |v_1|^2 v_1$ at the point $(v_2, \bar{v}_2)$, we deduce that

$$|\Phi| \leq c|v_1 - v_2|^2.$$

Taking the inner product in L^2 of (5.4) with $e(t)$ and taking the real part, we obtain

$$\frac{d}{dt}\|e\|^2 + 2\|(-\triangle)^{\frac{\alpha}{2}}e\|^2 - (2\rho+\beta^2)\|e\|^2 = -2\mathrm{Re}\int \Phi\cdot\bar{e} - 2\mathrm{Re}\int \Psi\cdot\bar{e}. \tag{5.5}$$

For the first term on the right-side of (5.5), using Hölder inequality and Young inequality, we estimate that

$$-2\mathrm{Re}\int \Phi\cdot\bar{e} \le \beta^2\|e\|^2 + \frac{c}{2\beta^2}\sqrt{1+\mu^2}\cdot z^{-2}\cdot\|v_1-v_2\|^4. \tag{5.6}$$

For the second term on the right-hand side of (5.5), applying Gagliardo-Nirenberg inequality and Lemma 4.1 and 4.2, we deduce that

$$\begin{aligned} -2\mathrm{Re}\int \Psi\cdot\bar{e} &\le 6\sqrt{1+\mu^2}\cdot z^{-2}\cdot\|v_2\|_\infty^2\|e\|^2 \\ &\le 6c\sqrt{1+\mu^2}\cdot z^{-2}\cdot(\|(-\triangle)^{\frac{\alpha}{2}}v_2\|^2+\|v_2\|^2)\|e\|^2 \\ &\le C(\omega)\|e\|^2. \end{aligned} \tag{5.7}$$

Put (5.6) and (5.7) into (5.5), we obtain that

$$\frac{d}{dt}\|e\|^2 \le C(\omega)\|e\|^2 + \frac{c}{2\beta^2}\sqrt{1+\mu^2}\cdot z^{-2}\cdot\|v_1-v_2\|^4. \tag{5.8}$$

By Gronwall's inequality, we infer that

$$\|e(1)\|^2 \le C(\omega)\cdot\frac{c}{2\beta^2}\sqrt{1+\mu^2}\int_0^1 z^{-2}\|v_1-v_2\|^4 dt. \tag{5.9}$$

$v_j(t)(j=1,2)$ be two solutions of (3.1) with $v_j(0)=v_j^0$, so we have

$$\begin{aligned} (v_1-v_2)_t &= -(1+\mathrm{i}\nu)(-\triangle)^\alpha(v_1-v_2) + (\rho+\frac{1}{2}\beta^2)(v_1-v_2) \\ &\quad -(1+\mathrm{i}\mu)z^{-2}(|v_1|^2v_1-|v_2|^2v_2). \end{aligned} \tag{5.10}$$

Taking the inner product in L^2 of (5.10) with v_1-v_2 and taking the real part, we obtain

$$\begin{aligned} \frac{d}{dt}\|v_1-v_2\|^2 &\le (2\rho+\beta^2)\|v_1-v_2\|^2 \\ &\quad -2\mathrm{Re}(1+\mathrm{i}\mu)z^{-2}\int(|v_1|^2v_1-|v_2|^2v_2)(\bar{v}_1-\bar{v}_2). \end{aligned} \tag{5.11}$$

From Taylor's formula, we deduce that

$$\begin{aligned} |v_1|^2v_1-|v_2|^2v_2 &= 2|v_2+\theta(v_1-v_2)|^2(v_1-v_2) + (v_2+\theta(v_1-v_2))^2(\bar{v}_1-\bar{v}_2) \\ &\le 3|v_2+\theta(v_1-v_2)|^2|v_1-v_2| \qquad \theta\in(0,1). \end{aligned}$$

Then the second term on the right-hand side of (5.11) is bounded by

$$-2\mathrm{Re}(1+\mathrm{i}\mu)z^{-2}\int(|v_1|^2v_1-|v_2|^2v_2)(\bar{v}_1-\bar{v}_2)\le C(\omega)\|v_1-v_2\|^2. \tag{5.12}$$

Combining (5.11) and (5.12), we obtain that

$$\frac{d}{dt}\|v_1-v_2\|^2\le C(\omega)\|v_1-v_2\|^2. \tag{5.13}$$

Applying Gronwall's inequality, we infer that

$$\|v_1-v_2\|^2\le C(\omega)\|v_1^0-v_2^0\|^2. \tag{5.14}$$

Hence, (5.9) can be written as

$$\|e(1)\|^2\le C(\omega)\cdot\frac{c}{2\beta^2}\sqrt{1+\mu^2}\cdot\sup_{0\le t\le 1}z^{-2}(t)\cdot h^4. \tag{5.15}$$

Let

$$\sup_{0\le t\le 1}z^{-2}(t)=M,\quad \mathcal{K}_1^2(\omega)=C(\omega)\cdot\frac{c}{2\beta^2}\sqrt{1+\mu^2}\cdot M,$$

and chose $\mathcal{K}(\omega)=\max\{\mathcal{K}_1(\omega),1\}$, which satisfies $E(\log\mathcal{K}(\omega))<\infty$. Hence, we complete the proof of Lemma 5.1. ∎

In what follows, we check the conditions (2), (3) of Theorem 2.2.

From (5.2), we obtain that

$$\frac{d}{dt}\|V\|^2+2\|(-\triangle)^{\frac{\alpha}{2}}V\|^2=(2\rho+\beta^2)\|V\|^2-2\mathrm{Re}(1+\mathrm{i}\mu)z^{-2}\int f'(v)|V|^2.$$

The second term of the right-side is bounded by

$$\begin{aligned}-2\mathrm{Re}(1+\mathrm{i}\mu)z^{-2}\int f'(v)|V|^2 &\le 6\sqrt{1+\mu^2}\cdot M\cdot\int|v|^2|V|^2\\ &\le 6\sqrt{1+\mu^2}\cdot M\cdot\|v\|_\infty^2\|V\|^2\\ &\le 6\cdot C(\omega)\sqrt{1+\mu^2}\cdot M\cdot\|V\|^2.\end{aligned}$$

Hence, we deduce that $\|V(t)\|\le\|V(0)\|\cdot e^{(2\rho+\beta^2+6\cdot C(\omega)\sqrt{1+\mu^2}\cdot M)t}$. Since $\alpha_1(DT(\omega,v))$ is equal to the norm of $DT(\omega,v)\in\mathcal{L}(X)$, it is not difficult, choosing

$$\bar{\alpha}_1(\omega)=\max\{e^{\beta W(1)+2\rho+\beta^2+6\cdot C(\omega)\sqrt{1+\mu^2}\cdot M},1\},$$

to get

$$\alpha_1(DS(\omega,v))\le\bar{\alpha}_1(\omega),$$

and $E(\log \bar{\alpha}_1) < \infty$.

Note that we can write

$$DT(\omega, v) = \exp\left\{\int_0^1 F'(s, v(s))ds\right\}$$

and

$$DS(\omega, u) = \exp\left\{\beta W(1) + \int_0^1 F'(s, v(s))ds\right\}.$$

By 2.3 of Chapter V [17], we obtain that

$$\omega_d(DS(\omega, u)) = \sup_{\substack{\eta_i \in L^2, \\ \|\eta_i\| \le 1, i=1,2,\cdots,d}} \exp\left\{\beta W(1) + \int_0^1 \mathrm{Re}Tr(F'(s, v(s)) \circ Q_d(s))ds\right\},$$

where $Q_d(s)$ is the orthogonal projector in L^2 onto the space spanned by $V_1(s)$, $\cdots, V_d(s)$, and $V_i(s)$ is the solution of (5.2) with $V_i(0) = \eta_i$.

Let $\psi_i(s),\ i \in \mathbf{N}$ be an orthonormal basis of L^2 such that $Q_d(s)L^2 = span\{\psi_1(s), \cdots, \psi_d(s)\}$, then we obtain

$$\begin{aligned}
&\mathrm{Re}Tr(F'(s, v(s)) \circ Q_d(s)) \\
&= \sum_{i=1}^d \mathrm{Re}\left(F'(s, v(s))\psi_i(s), \psi_i(s)\right) \\
&\le \left(\rho + \frac{1}{2}\beta^2 + 3\sqrt{1+\mu^2} \cdot M \cdot \|v\|_\infty^2\right)\sum_{i=1}^d \|\psi_i\|^2 - \sum_{i=1}^d \|(-\triangle)^{\frac{\alpha}{2}}\psi_i\|^2.
\end{aligned}$$

Since $\{\psi_i\}(i = 1, 2, \cdots, d)$ ia an orthonormal basis in L^2, we have

$$\sum_{i=1}^d \|\psi_i\|^2 = d.$$

It follows from the Sobolev-Lieb-Thirring inequality([17]) that

$$\sum_{j=1}^d \|(-\triangle)^{\frac{\alpha}{2}}\psi_j\|^2 \ge \kappa(2\pi)^\alpha d^{1+\alpha} - d,$$

where constant κ is independent of the family ψ_i, d and the parameters of the equation. So we infer that

$$\begin{aligned}
&\mathrm{Re}TrF'(u(\tau)) \circ Q_m(\tau) \\
&\le \left(1 + \rho + \frac{1}{2}\beta^2 + 3\sqrt{1+\mu^2} \cdot M \cdot \|v\|_\infty^2\right)d - \kappa(2\pi)^\alpha d^{1+\alpha} = \kappa_1 d - \kappa_2 d^{1+\alpha},
\end{aligned}$$

where $\kappa_1 = 1 + \rho + \frac{1}{2}\beta^2 + 3\sqrt{1+\mu^2} \cdot M \cdot \|v\|_\infty^2$, $\kappa_2 = \kappa(2\pi)^\alpha$. Denoting

$$\bar{\omega}_d(\omega) = \exp\{\beta W(1) + \kappa_1 d - \kappa_2 d^{1+\alpha}\}$$

and choosing

$$d = \left[\left(\frac{\kappa_1}{\kappa_2} \right)^{\frac{1}{\alpha}} \right] + 1.$$

Then we have $\omega_d(DS(\omega, u)) \leq \bar{\omega}_d(\omega)$ and $E(\log(\bar{\omega}_d)) < 0$.

Hence, by the Theorem 2.2, we obtain our main results as follows:

Theorem 5.1. *Let $\mathcal{A}(\omega)$ be the random attractor of* (1.1)-(1.3) *which is invariant under a random map $S(\omega), \omega \in \Omega$, for some ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. Then the Hausdorff dimension $d_H(\mathcal{A}(\omega))$ of $\mathcal{A}(\omega)$ is less than d almost surely.*

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