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# NEW $L(j,k)$-LABELINGS FOR DIRECT PRODUCTS OF COMPLETE GRAPHS

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**Abstract.** An $L(j,k)$-*labeling* of a graph is a vertex labeling such that the difference between the labels of adjacent vertices is at least $j$ and that between vertices separated by a distance 2 is at least $k$. The minimum of the spans of all $L(j,k)$-labelings of $G$ is denoted by $\lambda_k^j(G)$. Recently, Haque and Jha [16] proved that if $G$ is a multiple direct product of complete graphs, then $\lambda_k^j(G)$ coincides with the trivial lower bound $(N-1)k$, where $N$ is the order of $G$ and $\frac{j}{k}$ is within a certain bound.

In this paper, we suggest a new labeling method for such a graph $G$. With this method, we extend the range of $\frac{j}{k}$ such that $\lambda_k^j(G) = (N-1)k$ holds. Moreover, we obtain the upper bound of $\lambda_k^j(G)$ for the remaining cases in the range $\frac{j}{k}$.

## 1. INTRODUCTION

A channel-assignment problem involves assigning frequencies represented by non-negative integers to radio or television transmitters at various nodes in a region. To minimize interference between transmitters that are close, or very close, to each other, there must be some restrictions when distributing frequencies. The natural restrictions on allocating frequencies to transmitters would be as follows.

For positive integers $j, k (j \geq k)$, we assign frequencies with a difference of at least $j$ to transmitters that are very close, and at least $k$ to those which are close.

Hale [15] first placed this problem in a theoretical graph context as a more generalized form. Vertices of a graph represent the transmitters. Two vertices are adjacent

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if the corresponding transmitters are very close and they have distance 2 if the corresponding transmitters are close. An $L(j,k)$-labeling of a graph $G = (V_G, E_G)$ is a function $f : V_G \to [0, \infty)$ such that

$$|f(u) - f(v)| \geq \begin{cases} j & \text{if } \mathrm{dist}_G(u,v) = 1 \\ k & \text{if } \mathrm{dist}_G(u,v) = 2. \end{cases}$$

The elements of the image of $f$ are called the labels, and the maximum label assigned by $f$ is called the span of $f$. The minimum span over all $f$ is the $\lambda_k^j$-number of $G$, denoted by $\lambda_k^j(G)$. Griggs and Yeh [14] proposed a problem for determining $\lambda_k^j(G)$, in particular, $\lambda_1^2(G)$, and proved that for a graph $G$ of maximum degree $\Delta$, $\lambda_1^2(G) \leq \Delta^2 + 2\Delta$. They also conjectured that $\lambda_1^2(G) \leq \Delta^2$. Their conjecture is settled when $\Delta = 3$ and $G$ is Hamiltonian [20], and when $\Delta$ is sufficiently large [17]. But generally it is still open. Many papers [3, 5, 6, 7, 11, 12, 18, 32] on studies of $L(j,k)$-labelings have been published. Calamoneri [1, 2] and Yeh [35] provided good surveys for $L(j,k)$-labeling of graphs. We can find many results on distance 3 labelings [4, 9, 21, 22] and radio labelings [8, 21, 22, 27, 29, 30, 31]. A radio labeling of a graph $G$ is $L(d, d-1, \cdots, 2, 1)$ labeling when $d = \mathrm{diam}(G)$. Distance two labeling problem can be generalized to the distance $d$ labeling problem. For integers $j_1, j_2, \cdots, j_d$, an $L(j_1, j_2, \cdots, j_d)$ labeling of $G$ is the function $f : V \to \mathbb{N} \bigcup \{0\}$ which satisfies for each $u, v \in V$, $\mid f(u) - f(v) \mid \geq j_l$ when $\mathrm{dist}(u,v) = l$.

For two graphs $G$ and $H$, the Cartesian product of $G$ and $H$ is the graph $G \Box H$ with the vertex set $V_G \times V_H$, in which two vertices $(x,y)$ and $(x',y')$ are adjacent such that (i) $x = x'$ and $(y,y') \in E_H$, or (ii) $y = y'$ and $(x,x') \in E_G$. The direct (or Kronecker) product of $G$ and $H$ is the graph $G \times H$ with the vertex set $V_G \times V_H$, in which two vertices $(x,y)$ and $(x',y')$ are adjacent such that $(x,x') \in E_G$ and $(y,y') \in E_H$. Georges et al. [13] showed that if $K_m$ and $K_n$ are complete graphs of order $m, n$ $(2 \leq n < m)$, respectively, then

$$\lambda_k^j(K_m \Box K_n) = \begin{cases} (m-1)j + (n-1)k & \text{if } \frac{j}{k} > n \\ (mn-1)k & \text{if } \frac{j}{k} \leq n. \end{cases}$$

Whittlesey et al. [34] found $\lambda_1^2(P_m \Box P_n)$ for $k$-path $P_k$. The $\lambda_1^2(P_m \Box C_n)$ is computed by Klavzar and Vesel [24], and independently by Kuo and Yan [26]. Also Schwarz and Troxell [33] determined $\lambda_1^2(C_m \Box C_n)$. More results on the $\lambda_k^j$-number for the Cartesian products of various simple graphs are available [10, 13, 19, 25]. With respect to the $L(j,k)-$labelings of the direct product of two graphs, Lam et al. [27] showed the following theorem.

**Theorem 1.** *If $j, k, m$ and $n$ are positive integers with $3 \leq n \leq m$ and $j \geq k$, then we have*

$$\lambda_k^j(K_m \times K_n) = \begin{cases} (mn-1)k & \text{if } \frac{j}{k} \le 2 \\ (mn-2n+1)k+(n-1)j & \text{if } \frac{j}{k} > 2. \end{cases}$$

Haque and Jha [16] presented the following theorem.

**Theorem 2.** *If* $r \ge 3$, $n_0 \ge n_1 \ge \cdots \ge n_{r-1} \ge 3$, *and* $1 \le \frac{j}{k} \le n_0 n_1 \cdots n_{r-3}+1$, *then*

$$\lambda_k^j(K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-1}}) = (N-1)k,$$

*where* $N = n_0 n_1 \cdots n_{r-1}$.

They also provided the upper bound of $\lambda_k^j(K_n \times K_2)$, which is extremal when $(j,k) = (1,2)$. Recently Lin and Lam [28] and Kim et al. [23] studied the $\lambda$-number of the direct product of a complete graph and a cycle.
In this paper, we extend Theorem 2 and we also find the upper bound of $\lambda_k^j(K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-2}} \times K_2)$, as follows.

(1) The range of $\frac{j}{k}$ in which $\lambda_k^j(K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-1}}) = (N-1)k$ can be extended to $[1, A]$, where $A = 2 + n_0 + n_0 n_1 + \cdots + n_0 n_1 \cdots n_{r-3}$ (Theorem 3), which is broader than the range given by Haque and Jha in Theorem 2.

(2) If $\frac{j}{k} > A$, then the upper bound of $\lambda_k^j(K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-1}})$ is $(N-1)k + (j-Ak)(n_{r-1}-1)$ (Theorem 4).

(3) We provide the upper bound of $\lambda_k^j(K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-2}} \times K_2)$, which is extremal when $\frac{j}{k} \le A' = 2 + n_0 + n_0 n_1 + \cdots + n_0 n_1 \cdots n_{r-4}$ (Corollary 1).

## 2. Main Theorems

Let $K_n$ be a complete graph with $V_{K_n} = \{0, 1, \cdots, n-1\}$. For $r \ge 2$, consider the direct product $K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-1}}$ of $K_{n_0}, K_{n_1}, \cdots, K_{n_{r-1}}$. We may assume that $n_0 \ge n_1 \ge \cdots \ge n_{r-1} \ge 3$. Throughout this paper, for simplicity, we use

$$u = (u_0, u_1, \ldots, u_{r-1}),$$

$$v = (v_0, v_1, \ldots, v_{r-1}),$$

$$\tilde{u} = (u_0, u_1, \ldots, u_{r-2}),$$

and

$$\tilde{v} = (v_0, v_1, \ldots, v_{r-2}).$$

So, $u = (\tilde{u}, u_{r-1})$ and $v = (\tilde{v}, v_{r-1})$. It is not difficult to see that the diameter of $K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-1}}$ is 2, and $\text{dist}(u,v) = 1$ if and only if $u_i \ne v_i$ for each $i$, $0 \le i \le r-1$. For $0 \le u_t \le n_t - 1$ with $0 \le t \le r-2$, we define $h(\tilde{u}) = u_0 + n_0 u_1 + \cdots + n_0 n_1 \cdots n_{r-3} u_{r-2}$.

**Theorem 3.** *If* $r \geq 3$, $n_0 \geq n_1 \geq \cdots \geq n_{r-1} \geq 3$ *and* $1 \leq \frac{j}{k} \leq A$*, then*

$$\lambda_k^j(K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-1}}) = (N-1)k,$$

*where* $N = n_0 n_1 \cdots n_{r-1}$ *and* $A = 2 + n_0 + n_0 n_1 + \cdots + n_0 n_1 \cdots n_{r-3}$.

*Proof.* Since the diameter of $K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-1}}$ is 2, the difference between any two vertices of $K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-1}}$ is at least $k$. So,

$$\lambda_k^j(K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-1}}) \geq (N-1)k.$$

We give a labeling

$$f(u) = \begin{cases} (u_{r-1} n_0 n_1 \cdots n_{r-2} + h(\tilde{u}))k & \text{if } u_{r-1} \text{ is even,} \\ ((u_{r-1}+1) n_0 n_1 \cdots n_{r-2} - 1 - h(\tilde{u}))k & \text{if } u_{r-1} \text{ is odd.} \end{cases}$$

This labeling is shown in Table 1 for the example $K_5 \times K_4 \times K_3$ and in Table 2 for the example $K_5 \times K_4 \times K_3 \times K_3$, respectively. In Table 1, the number located in the $(u_0+1)$-th row and $(u_1+1)$-th column of the box over which $u_2$ is indicated denotes the labeling of the vertex $(u_0, u_1, u_2)$ of $K_5 \times K_4 \times K_3$, for example, $f(3,2,1) = 26$. Table 2 is read similarly.

Since $0 \leq h(\tilde{u}) = u_0 + n_0 u_1 + \cdots + n_0 n_1 \cdots n_{r-3} u_{r-2} \leq n_0 n_1 \cdots n_{r-2} - 1$, we have $u_{r-1} n_0 n_1 \cdots n_{r-2} k \leq f(u) \leq ((u_{r-1}+1) n_0 n_1 \cdots n_{r-2} - 1)k$. Let $u$ and $v$ be two vertices of $K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-1}}$. We may assume that $u_{r-1} \leq v_{r-1}$. If the distance from $u$ to $v$ is 2, and $v_{r-1} \geq u_{r-1} + 1$, then $f(v) - f(u) \geq v_{r-1} n_0 n_1 \cdots n_{r-2} k - ((u_{r-1}+1) n_0 n_1 \cdots n_{r-2} - 1)k \geq k$. If the distance from $u$ to $v$ is 2, and $v_{r-1} = u_{r-1}$, then $l$ exists such that $u_l \neq v_l$ and $u_t = v_t$ for all $t = 0, 1, \ldots, l-1$. Then, since $-n_l + 1 \leq v_l - u_l \leq n_l - 1$ and $u_l \neq v_l$, $v_l - u_l$ is not a multiple of $n_l$. So, we have

$$\begin{aligned} h(\tilde{v}) - h(\tilde{u}) &= \sum_{t=l}^{r-2} (v_t - u_t) n_0 n_1 \cdots n_{t-1} \\ &= (v_l - u_l) n_0 n_1 \cdots n_{l-1} \\ &\quad + n_0 n_1 \cdots n_l \Big( \sum_{t=l+1}^{r-2} (v_t - u_t) n_{l+1} n_{l+2} \cdots n_{t-1} \Big) \end{aligned}$$

is not a multiple of $n_0 n_1 \cdots n_l$. So, $h(\tilde{v}) \neq h(\tilde{u})$. Thus, $|f(v) - f(u)| = |h(\tilde{v}) - h(\tilde{u})|k \geq k$. If $u$ is adjacent to $v$, then $u_l \neq v_l$ for all $l = 0, 1, \ldots, r-1$. So, $1 \leq u_l + v_l \leq 2n_l - 3$ for all $l = 0, 1, \ldots, r-1$ and $v_{r-1} \geq u_{r-1} + 1$. If $v_{r-1} \geq u_{r-1} + 2$, then, from

$$u_{r-1} n_0 n_1 \cdots n_{r-2} k \leq f(u) \leq ((u_{r-1}+1) n_0 n_1 \cdots n_{r-2} - 1)k,$$

Table 1. $L(j,k)-$ labeling of $K_5 \times K_4 \times K_3$ for $k=1,\ 1 \le j \le 7$

$u_2 = 0$

| | | | |
|---|---|---|---|
| 0 | 5 | 10 | 15 |
| 1 | 6 | 11 | 16 |
| 2 | 7 | 12 | 17 |
| 3 | 8 | 13 | 18 |
| 4 | 9 | 14 | 19 |

$u_2 = 1$

| | | | |
|---|---|---|---|
| 39 | 34 | 29 | 24 |
| 38 | 33 | 28 | 23 |
| 37 | 32 | 27 | 22 |
| 36 | 31 | 26 | 21 |
| 35 | 30 | 25 | 20 |

$u_2 = 2$

| | | | |
|---|---|---|---|
| 40 | 45 | 50 | 55 |
| 41 | 46 | 51 | 56 |
| 42 | 47 | 52 | 57 |
| 43 | 48 | 53 | 58 |
| 44 | 49 | 54 | 59 |

Table 2. $L(j,k)-$ labeling of $K_5 \times K_4 \times K_3 \times K_3$ for $k=1,\ 1 \le j \le 27$

$(u_2, u_3) = (0,0)$

| | | | |
|---|---|---|---|
| 0 | 5 | 10 | 15 |
| 1 | 6 | 11 | 16 |
| 2 | 7 | 12 | 17 |
| 3 | 8 | 13 | 18 |
| 4 | 9 | 14 | 19 |

$(u_2, u_3) = (0,1)$

| | | | |
|---|---|---|---|
| 119 | 114 | 109 | 104 |
| 118 | 113 | 108 | 103 |
| 117 | 112 | 107 | 102 |
| 116 | 111 | 106 | 101 |
| 115 | 110 | 105 | 100 |

$(u_2, u_3) = (0,2)$

| | | | |
|---|---|---|---|
| 120 | 125 | 130 | 135 |
| 121 | 126 | 131 | 136 |
| 122 | 127 | 132 | 137 |
| 123 | 128 | 133 | 138 |
| 124 | 129 | 134 | 139 |

$(u_2, u_3) = (1,0)$

| | | | |
|---|---|---|---|
| 20 | 25 | 30 | 35 |
| 21 | 26 | 31 | 36 |
| 22 | 27 | 32 | 37 |
| 23 | 28 | 33 | 38 |
| 24 | 29 | 34 | 39 |

$(u_2, u_3) = (1,1)$

| | | | |
|---|---|---|---|
| 99 | 94 | 89 | 84 |
| 98 | 93 | 88 | 83 |
| 97 | 92 | 87 | 82 |
| 96 | 91 | 86 | 81 |
| 95 | 90 | 85 | 80 |

$(u_2, u_3) = (1,2)$

| | | | |
|---|---|---|---|
| 140 | 145 | 150 | 155 |
| 141 | 146 | 151 | 156 |
| 142 | 147 | 152 | 157 |
| 143 | 148 | 153 | 158 |
| 144 | 149 | 154 | 159 |

$(u_2, u_3) = (2,0)$

| | | | |
|---|---|---|---|
| 40 | 45 | 50 | 55 |
| 41 | 46 | 51 | 56 |
| 42 | 47 | 52 | 57 |
| 43 | 48 | 53 | 58 |
| 44 | 49 | 54 | 59 |

$(u_2, u_3) = (2,1)$

| | | | |
|---|---|---|---|
| 79 | 74 | 69 | 54 |
| 78 | 73 | 68 | 63 |
| 77 | 72 | 67 | 62 |
| 76 | 71 | 66 | 61 |
| 75 | 70 | 65 | 60 |

$(u_2, u_3) = (2,2)$

| | | | |
|---|---|---|---|
| 160 | 165 | 170 | 175 |
| 161 | 166 | 171 | 176 |
| 162 | 167 | 172 | 177 |
| 163 | 168 | 173 | 178 |
| 164 | 169 | 174 | 179 |

we have

$$\begin{aligned} f(v)-f(u) &\geq v_{r-1}n_0n_1\cdots n_{r-2}k-((u_{r-1}+1)n_0n_1\cdots n_{r-2}-1)k \\ &\geq (n_0n_1\cdots n_{r-2}+1)k > Ak \geq j. \end{aligned}$$

If $v_{r-1}=u_{r-1}+1$ and $u_{r-1}$ is even, then

$$\begin{aligned} & 0f(v)-f(u) \\ &= ((u_{r-1}+2)n_0n_1\cdots n_{r-2}-h(\tilde{v})-u_{r-1}n_0n_1\cdots n_{r-2}-h(\tilde{u})-1)k \\ &= (2n_0n_1\cdots n_{r-2}-1-(u_0+v_0)-n_0(u_1+v_1)-\cdots \\ &\quad -n_0n_1\cdots n_{r-3}(u_{r-2}+v_{r-2}))k \\ &\geq (2n_0n_1\cdots n_{r-2}-1-(2n_0-3)-n_0(2n_1-3)-\cdots \\ &\quad -n_0n_1\cdots n_{r-3}(2n_{r-2}-3))k \\ &= (2+n_0+n_0n_1+\cdots+n_0n_1\cdots n_{r-3})k = Ak \geq j. \end{aligned}$$

If $v_{r-1}=u_{r-1}+1$ and $u_{r-1}$ is odd, then

$$\begin{aligned} & f(v)-f(u) \\ &= ((u_{r-1}+1)n_0n_1\cdots n_{r-2}+h(\tilde{v})-(u_{r-1}+1)n_0n_1\cdots n_{r-2}+1+h(\tilde{u}))k \\ &= ((1+(u_0+v_0)+n_0(u_1+v_1)+\cdots+n_0n_1\cdots n_{r-3}(u_{r-2}+v_{r-2}))k \\ &\geq (2+n_0+n_0n_1+\cdots+n_0n_1\cdots n_{r-3})k = Ak \geq j. \end{aligned}$$

So, $f$ is an $L(j,k)$-labeling of $K_{n_0}\times K_{n_1}\times\cdots\times K_{n_{r-1}}$. Since the span of $f$ is $(n_0n_1\cdots n_{r-1}-1)k$, we conclude that $\lambda_k^j(K_{n_0}\times K_{n_1}\times\cdots\times K_{n_{r-1}})=(N-1)k$. ■

**Theorem 4.** *If $r\geq 3$, $n_0\geq n_1\geq\cdots\geq n_{r-1}\geq 3$, and $\frac{j}{k}>A$, then*

$$\lambda_k^j(K_{n_0}\times K_{n_1}\times\cdots\times K_{n_{r-1}})\leq (j-Ak)(n_{r-1}-1)+(N-1)k,$$

*where $N=n_0n_1\cdots n_{r-1}$ and $A=2+n_0+n_0n_1+\cdots+n_0n_1\cdots n_{r-3}$.*

*Proof.* We give a labeling

$$f(u)=\begin{cases} u_{r-1}(n_0n_1\cdots n_{r-2}k+j-Ak)+h(\tilde{u})k \text{ if } u_{r-1} \text{ is even,} \\ u_{r-1}(j-Ak)+((u_{r-1}+1)n_0n_1\cdots n_{r-2}-1-h(\tilde{u}))k \text{ if } u_{r-1} \text{ is odd.} \end{cases}$$

This labeling is shown in Tables 3 and 4 for the examples $K_5\times K_4\times K_3$ and $K_5\times K_4\times K_3\times K_3$, respectively. These tables are read in a similar way to Tables 1 and 2. Since $0\leq h(\tilde{u})\leq n_0n_1\cdots n_{r-2}-1$, we have $u_{r-1}(n_0n_1\cdots n_{r-2}k+j-Ak)\leq f(u)\leq u_{r-1}(n_0n_1\cdots n_{r-2}k+j-Ak)+(n_0n_1\cdots n_{r-2}-1)k$. Assume $u$ and $v$ are two vertices of $K_{n_0}\times K_{n_1}\times\cdots\times K_{n_{r-1}}$. We may assume that $u_{r-1}\leq v_{r-1}$. If $v_{r-1}\geq u_{r-1}+1$, then

Table 3. $L(j,k)-$ labeling of $K_5 \times K_4 \times K_3$ for $k=1,\ j>7$

$u_2 = 0$

| | | | |
|---|---|---|---|
| 0 | 5 | 10 | 15 |
| 1 | 6 | 11 | 16 |
| 2 | 7 | 12 | 17 |
| 3 | 8 | 13 | 18 |
| 4 | 9 | 14 | 19 |

$u_2 = 1$

| | | | |
|---|---|---|---|
| $j+32$ | $j+27$ | $j+22$ | $j+17$ |
| $j+31$ | $j+26$ | $j+21$ | $j+16$ |
| $j+30$ | $j+25$ | $j+20$ | $j+15$ |
| $j+29$ | $j+24$ | $j+19$ | $j+14$ |
| $j+28$ | $j+23$ | $j+18$ | $j+13$ |

$u_2 = 2$

| | | | |
|---|---|---|---|
| $2j+26$ | $2j+31$ | $2j+36$ | $2j+41$ |
| $2j+27$ | $2j+32$ | $2j+37$ | $2j+42$ |
| $2j+28$ | $2j+33$ | $2j+38$ | $2j+43$ |
| $2j+29$ | $2j+34$ | $2j+39$ | $2j+44$ |
| $2j+30$ | $2j+35$ | $2j+40$ | $2j+45$ |

Table 4. $L(j,k)-$ labeling of $K_5 \times K_4 \times K_3 \times K_3$ for $k=1,\ j>27$

$(u_2, u_3) = (0,0)$

| | | | |
|---|---|---|---|
| 0 | 5 | 10 | 15 |
| 1 | 6 | 11 | 16 |
| 2 | 7 | 12 | 17 |
| 3 | 8 | 13 | 18 |
| 4 | 9 | 14 | 19 |

$(u_2, u_3) = (0,1)$

| | | | |
|---|---|---|---|
| $j+92$ | $j+87$ | $j+82$ | $j+77$ |
| $j+91$ | $j+86$ | $j+81$ | $j+76$ |
| $j+90$ | $j+85$ | $j+80$ | $j+75$ |
| $j+89$ | $j+84$ | $j+79$ | $j+74$ |
| $j+88$ | $j+83$ | $j+78$ | $j+73$ |

$(u_2, u_3) = (0,2)$

| | | | |
|---|---|---|---|
| $2j+66$ | $2j+71$ | $2j+76$ | $2j+81$ |
| $2j+67$ | $2j+72$ | $2j+77$ | $2j+82$ |
| $2j+68$ | $2j+73$ | $2j+78$ | $2j+83$ |
| $2j+69$ | $2j+74$ | $2j+79$ | $2j+84$ |
| $2j+70$ | $2j+75$ | $2j+80$ | $2j+85$ |

$(u_2, u_3) = (1,0)$

| | | | |
|---|---|---|---|
| 20 | 25 | 30 | 35 |
| 21 | 26 | 31 | 36 |
| 22 | 27 | 32 | 37 |
| 23 | 28 | 33 | 38 |
| 24 | 29 | 34 | 39 |

$(u_2, u_3) = (1,1)$

| | | | |
|---|---|---|---|
| $j+72$ | $j+67$ | $j+62$ | $j+57$ |
| $j+71$ | $j+66$ | $j+61$ | $j+56$ |
| $j+70$ | $j+65$ | $j+60$ | $j+55$ |
| $j+69$ | $j+64$ | $j+59$ | $j+54$ |
| $j+68$ | $j+63$ | $j+58$ | $j+53$ |

$(u_2, u_3) = (1,2)$

| | | | |
|---|---|---|---|
| $2j+86$ | $2j+91$ | $2j+96$ | $2j+101$ |
| $2j+87$ | $2j+92$ | $2j+97$ | $2j+102$ |
| $2j+88$ | $2j+93$ | $2j+98$ | $2j+103$ |
| $2j+89$ | $2j+94$ | $2j+99$ | $2j+104$ |
| $2j+90$ | $2j+95$ | $2j+100$ | $2j+105$ |

$(u_2, u_3) = (2,0)$

| | | | |
|---|---|---|---|
| 40 | 45 | 50 | 55 |
| 41 | 46 | 51 | 56 |
| 42 | 47 | 52 | 57 |
| 43 | 48 | 53 | 58 |
| 44 | 49 | 54 | 59 |

$(u_2, u_3) = (2,1)$

| | | | |
|---|---|---|---|
| $j+52$ | $j+47$ | $j+42$ | $j+37$ |
| $j+51$ | $j+46$ | $j+41$ | $j+36$ |
| $j+50$ | $j+45$ | $j+40$ | $j+35$ |
| $j+49$ | $j+44$ | $j+39$ | $j+34$ |
| $j+48$ | $j+43$ | $j+38$ | $j+33$ |

$(u_2, u_3) = (2,2)$

| | | | |
|---|---|---|---|
| $2j+106$ | $2j+111$ | $2j+116$ | $2j+121$ |
| $2j+107$ | $2j+112$ | $2j+117$ | $2j+122$ |
| $2j+108$ | $2j+113$ | $2j+118$ | $2j+123$ |
| $2j+109$ | $2j+114$ | $2j+119$ | $2j+124$ |
| $2j+110$ | $2j+115$ | $2j+120$ | $2j+125$ |

$$\begin{aligned} f(v)-f(u) &\geq (v_{r-1}-u_{r-1})(n_0n_1\cdots n_{r-2}k+j-Ak)-(n_0n_1\cdots n_{r-2}-1)k \\ &\geq j-Ak+k>k. \end{aligned}$$

If $v_{r-1}=u_{r-1}$, then $l$ exists such that $u_l \neq v_l$ and $u_t=v_t$ for all $t=0,1,\ldots,l-1$. Then, since $-n_l+1 \leq v_l-u_l \leq n_l-1$ and $u_l \neq v_l$, we have

$$\begin{aligned} h(\tilde{v})-h(\tilde{u}) &= \sum_{t=l}^{r-2}(v_t-u_t)n_0n_1\cdots n_{t-1} \\ &= (v_l-u_l)n_0n_1\cdots n_{l-1} \\ &\quad +n_0n_1\cdots n_l(\sum_{t=l+1}^{r-2}(v_t-u_t)n_{l+1}n_{1+2}\cdots n_{t-1}) \end{aligned}$$

is not a multiple of $n_0n_1\cdots n_l$. So, $h(\tilde{v}) \neq h(\tilde{u})$, and thus $|f(v)-f(u)|=|h(\tilde{v})-h(\tilde{u})|k \geq k$. If $u$ is adjacent to $v$, then $u_l \neq v_l$ for all $l=0,1,\ldots,r-1$. So, $1 \leq u_l+v_l \leq 2n_l-3$ for all $l=0,1,\ldots,r-1$ and $v_{r-1} \geq u_{r-1}+1$. If $v_{r-1} \geq u_{r-1}+2$, then, from

$$\begin{aligned} &u_{r-1}(n_0n_1\cdots n_{r-2}k+j-Ak) \\ &\leq f(u) \leq u_{r-1}(n_0n_1\cdots n_{r-2}k+j-Ak)+(n_0n_1\cdots n_{r-2}-1)k, \end{aligned}$$

we have

$$\begin{aligned} f(v)-f(u) &\geq (v_{r-1}-u_{r-1})(n_0n_1\cdots n_{r-2}+j-Ak)-(n_0n_1\cdots n_{r-2}-1)k \\ &\geq 2(j-Ak)+n_0n_1\cdots n_{r-2}k+k>j. \end{aligned}$$

If $v_{r-1}=u_{r-1}+1$ and $u_{r-1}$ is even, then

$$\begin{aligned} f(v)-f(u) &= (u_{r-1}+1)(j-Ak)+((u_{r-1}+2)n_0n_1\cdots n_{r-2}-1 \\ &\quad -h(\tilde{v}))k-u_{r-1}(n_0n_1\cdots n_{r-2}k+j-Ak)-h(\tilde{u})k \\ &= j-Ak+(2n_0n_1\cdots n_{r-2}-1-(u_0+v_0) \\ &\quad -n_0(u_1+v_1)-n_0n_1(u_2+v_2)-\cdots-n_0n_1\cdots n_{r-3}(u_{r-2}+v_{r-2}))k \\ &\geq j-Ak+(2n_0n_1\cdots n_{r-2}-1-(2n_0-3)-(2n_1-3)n_0-\cdots \\ &\quad -(2n_{r-2}-3)n_0n_1\cdots n_{r-3})k \\ &= j+(-A+n_0n_1\cdots n_{r-3}+\cdots+n_0+2)k=j. \end{aligned}$$

If $v_{r-1}=u_{r-1}+1$ and $u_{r-1}$ is odd, then

$$\begin{aligned} f(v)-f(u) &= (u_{r-1}+1)(n_0n_1\cdots n_{r-2}k+j-Ak)+h(\tilde{v})k-u_{r-1}(j-Ak) \\ &\quad -((u_{r-1}+1)n_0n_1\cdots n_{r-2}-1-h(\tilde{u}))k \\ &= j-Ak+((u_0+v_0)+n_0(u_1+v_1)+\cdots \\ &\quad +n_0n_1\cdots n_{r-3}(u_{r-2}+v_{r-2})+1)k \\ &\geq j-Ak+(n_0n_1\cdots n_{r-3}+\cdots+n_0+2)k=j. \end{aligned}$$

Table 5. $L(j,k)-$ labeling of $K_4 \times K_3 \times K_2$ for $k=1,\ 1 \le j \le 2$

$u_2 = 0$

| | | |
|---|---|---|
| 0 | 7 | 8 |
| 1 | 6 | 9 |
| 2 | 5 | 10 |
| 3 | 4 | 11 |

$u_2 = 1$

| | | |
|---|---|---|
| 0 | 7 | 8 |
| 1 | 6 | 9 |
| 2 | 5 | 10 |
| 3 | 4 | 11 |

So, $f$ is an $L(j,k)$-labeling of $K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-1}}$. Since the span of $f$ is $(j-Ak)(n_{r-1}-1)+(n_0 n_1 \cdots n_{r-1}-1)k$, we conclude that

$$\lambda_k^j(K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-1}}) \le (j-Ak)(n_{r-1}-1)+(N-1)k. \qquad \blacksquare$$

The following lemma is useful when we deal with the labelings of the graph $K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-2}} \times K_2$.

**Lemma 1.** *If $G$ is a graph, then $\lambda_k^j(G \times K_2) \le \lambda_k^j(G)$.*

*Proof.* Let $f$ be an $L(j,k)$-labeling of $G$ with minimum span. We define a labeling $g$ of $G \times K_2$ such that $g(v,x) = f(v)$. If the distance from $(v,x)$ to $(w,y)$ is 2, $v \ne w$ and the distance from $v$ to $w$ in $G$ is at most 2. So, $|g(v,x)-g(w,y)| = |f(v)-f(w)| \ge k$. If $(v,x)$ and $(w,y)$ are adjacent, $x \ne y$ and $v$ is adjacent to $w$ in $G$. So, $|g(v,x)-g(w,y)| = |f(v)-f(w)| \ge j$. Thus, $g$ is an $L(j,k)$-labeling of $G \times K_2$, whose span equals that of $f$. We have $\lambda_k^j(G \times K_2) \le \lambda_k^j(G)$. $\blacksquare$

The following theorem is a consequence of Theorems 3 and 4 and Lemma 1.

**Theorem 5.** *For $r \ge 3$ and $n_0 \ge n_1 \ge \cdots \ge n_{r-2} \ge 3$, let $H = K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-2}} \times K_2$. Then we have*

$$\lambda_k^j(H) \begin{cases} = (N'-1)k \text{ if } \frac{j}{k} \le A', \\ \le (n_{r-2}-1)(j-A'k)+(N'-1)k \text{ if } \frac{j}{k} > A', \end{cases}$$

*where $A' = n_0 n_1 \cdots n_{r-4} + n_0 n_1 \cdots n_{r-5} + \cdots + n_0 + 2$ and $N' = n_0 n_1 \cdots n_{r-2}$.*

Tables 5, 6, 7, and 8 show the explicit labelings of $K_{n_0} \times K_{n_1} \times \cdots \times K_{n_{r-2}} \times K_2$ for the examples $K_4 \times K_3 \times K_2$ and $K_5 \times K_4 \times K_3 \times K_2$.

Table 6. $L(j,k)$ – labeling of $K_4 \times K_3 \times K_2$ for $k=1,\ 2<j\leq 10$

$u_2 = 0$

| | | |
|---|---|---|
| 0 | $j$+5 | 2$j$+4 |
| 1 | $j$+4 | 2$j$+5 |
| 2 | $j$+3 | 2$j$+6 |
| 3 | $j$+2 | 2$j$+7 |

$u_2 = 1$

| | | |
|---|---|---|
| 0 | $j$+5 | 2$j$+4 |
| 1 | $j$+4 | 2$j$+5 |
| 2 | $j$+3 | 2$j$+6 |
| 3 | $j$+2 | 2$j$+7 |

Table 7. $L(j,k)$ – labeling of $K_5 \times K_4 \times K_3 \times K_2$ for $k=1,\ 1\leq j\leq 7$

$(u_2,u_3)=(0,0)$

| | | | |
|---|---|---|---|
| 0 | 5 | 10 | 15 |
| 1 | 6 | 11 | 16 |
| 2 | 7 | 12 | 17 |
| 3 | 8 | 13 | 18 |
| 4 | 9 | 14 | 19 |

$(u_2,u_3)=(0,1)$

| | | | |
|---|---|---|---|
| 0 | 5 | 10 | 15 |
| 1 | 6 | 11 | 16 |
| 2 | 7 | 12 | 17 |
| 3 | 8 | 13 | 18 |
| 4 | 9 | 14 | 19 |

$(u_2,u_3)=(1,0)$

| | | | |
|---|---|---|---|
| 39 | 34 | 29 | 24 |
| 38 | 33 | 28 | 23 |
| 37 | 32 | 27 | 22 |
| 36 | 31 | 26 | 21 |
| 35 | 30 | 25 | 20 |

$(u_2,u_3)=(1,1)$

| | | | |
|---|---|---|---|
| 39 | 34 | 29 | 24 |
| 38 | 33 | 28 | 23 |
| 37 | 32 | 27 | 22 |
| 36 | 31 | 26 | 21 |
| 35 | 30 | 25 | 20 |

$(u_2,u_3)=(2,0)$

| | | | |
|---|---|---|---|
| 40 | 45 | 50 | 55 |
| 41 | 46 | 51 | 56 |
| 42 | 47 | 52 | 57 |
| 43 | 48 | 53 | 58 |
| 44 | 49 | 54 | 59 |

$(u_2,u_3)=(2,1)$

| | | | |
|---|---|---|---|
| 40 | 45 | 50 | 55 |
| 41 | 46 | 51 | 56 |
| 42 | 47 | 52 | 57 |
| 43 | 48 | 53 | 58 |
| 44 | 49 | 54 | 59 |

Table 8. $L(j,k)-$ labeling of $K_5\times K_4\times K_3\times K_2$ for $k=1,\ 8<j\leq 47$

$(u_2,u_3)=(0,0)$

| | | | |
|---|---|---|---|
| 0 | 5 | 10 | 15 |
| 1 | 6 | 11 | 16 |
| 2 | 7 | 12 | 17 |
| 3 | 8 | 13 | 18 |
| 4 | 9 | 14 | 19 |

$(u_2,u_3)=(0,1)$

| | | | |
|---|---|---|---|
| 0 | 5 | 10 | 15 |
| 1 | 6 | 11 | 16 |
| 2 | 7 | 12 | 17 |
| 3 | 8 | 13 | 18 |
| 4 | 9 | 14 | 19 |

$(u_2,u_3)=(1,0)$

| | | | |
|---|---|---|---|
| $j$+32 | $j$+27 | $j$+22 | $j$+17 |
| $j$+31 | $j$+26 | $j$+21 | $j$+16 |
| $j$+30 | $j$+25 | $j$+20 | $j$+15 |
| $j$+29 | $j$+24 | $j$+19 | $j$+14 |
| $j$+28 | $j$+23 | $j$+18 | $j$+13 |

$(u_2,u_3)=(1,1)$

| | | | |
|---|---|---|---|
| $j$+32 | $j$+27 | $j$+22 | $j$+17 |
| $j$+31 | $j$+26 | $j$+21 | $j$+16 |
| $j$+30 | $j$+25 | $j$+20 | $j$+15 |
| $j$+29 | $j$+24 | $j$+19 | $j$+14 |
| $j$+28 | $j$+23 | $j$+18 | $j$+13 |

$(u_2,u_3)=(2,0)$

| | | | |
|---|---|---|---|
| 2$j$+26 | 2$j$+31 | 2$j$+36 | 2$j$+41 |
| 2$j$+27 | 2$j$+32 | 2$j$+37 | 2$j$+42 |
| 2$j$+28 | 2$j$+33 | 2$j$+38 | 2$j$+43 |
| 2$j$+29 | 2$j$+34 | 2$j$+39 | 2$j$+44 |
| 2$j$+30 | 2$j$+35 | 2$j$+40 | 2$j$+45 |

$(u_2,u_3)=(2,1)$

| | | | |
|---|---|---|---|
| 2$j$+26 | 2$j$+31 | 2$j$+36 | 2$j$+41 |
| 2$j$+27 | 2$j$+32 | 2$j$+37 | 2$j$+42 |
| 2$j$+28 | 2$j$+33 | 2$j$+38 | 2$j$+43 |
| 2$j$+29 | 2$j$+34 | 2$j$+39 | 2$j$+44 |
| 2$j$+30 | 2$j$+35 | 2$j$+40 | 2$j$+45 |

**Theorem 6.** *For $r\geq 3$ and $n_0\geq n_1\geq\cdots\geq n_{r-2}\geq 3$, let $H=K_{n_0}\times K_{n_1}\times\cdots\times K_{n_{r-2}}\times K_2$, $A=2+n_0+n_0n_1+\cdots+n_0n_1\cdots n_{r-3}$ and $N'=n_0n_1\cdots n_{r-2}$. Then we have*

$$\lambda_k^j(H)\leq j+(2N'-A-1)k.$$

*Proof.* Define

$$f(u)=\begin{cases} h(\tilde{u})k \text{ if } u_{r-1}=0\\ j+(2n_0n_1\cdots n_{r-2}-A-1-h(\tilde{u}))k \text{ if } u_{r-1}=1.\end{cases}$$

This labeling is shown in Tables 9 and 10 for the examples $K_4\times K_3\times K_2$ and $K_5\times K_4\times K_3\times K_2$. Assume $u$ and $v$ are vertices of $H=K_{n_0}\times K_{n_1}\times\cdots\times K_{n_{r-2}}\times K_2$. If the distance between $u$ and $v$ is 2, since $H$ is a bipartite graph with two parts $U_0=\{(p_0,p_1,\ldots,p_{r-1})|p_{r-1}=0\}$ and $U_1=\{(p_0,p_1,\ldots,p_{r-1})|p_{r-1}=1\}$, we have $u_{r-1}=v_{r-1}$. Since $u\neq v$, we let $t$ be the smallest number such that $u_t\neq v_t$. Then, since $0\leq u_t,v_t\leq n_t-1$, we have $|u_t-v_t|\leq n_t-1$. So, $u_t-v_t$ is not a multiple of $n_t$. Since $|h(\tilde{u})-h(\tilde{v})|=|(u_t-v_t)n_0n_1\cdots n_{t-1}+\cdots+(u_{r-2}-v_{r-2})n_0n_1\cdots n_{r-3}|$, which is not a multiple of $n_0n_1\cdots n_t$, we have $|f(u)-f(v)|\geq k$.

If the distance between $u$ and $v$ is 1, $u_{r-1}\neq v_{r-1}$. We may assume that $u_{r-1}=0$ and $v_{r-1}=1$. For each $t=0,\cdots,r-2$, since $u_t\neq v_t$, we have $1\leq u_t+v_t\leq 2n_t-3$.

Table 9. $L(j,k)-$labeling of $K_4 \times K_3 \times K_2$ for $k=1,\ j>10$

$u_2 = 0$

| | | |
|---|---|---|
| 0 | 4 | 8 |
| 1 | 5 | 9 |
| 2 | 6 | 10 |
| 3 | 7 | 11 |

$u_2 = 1$

| | | |
|---|---|---|
| $j$+17 | $j$+13 | $j$+9 |
| $j$+16 | $j$+12 | $j$+8 |
| $j$+15 | $j$+11 | $j$+7 |
| $j$+14 | $j$+10 | $j$+6 |

Table 10. $L(j,k)-$labeling of $K_5 \times K_4 \times K_3 \times K_2$ for $k=1,\ j>47$

$(u_2,u_3)=(0,0)$

| | | | |
|---|---|---|---|
| 0 | 5 | 10 | 15 |
| 1 | 6 | 11 | 16 |
| 2 | 7 | 12 | 17 |
| 3 | 8 | 13 | 18 |
| 4 | 9 | 14 | 19 |

$(u_2,u_3)=(0,1)$

| | | | |
|---|---|---|---|
| $j$+92 | $j$+87 | $j$+82 | $j$+77 |
| $j$+91 | $j$+86 | $j$+81 | $j$+76 |
| $j$+90 | $j$+85 | $j$+80 | $j$+75 |
| $j$+89 | $j$+84 | $j$+79 | $j$+74 |
| $j$+88 | $j$+83 | $j$+78 | $j$+73 |

$(u_2,u_3)=(1,0)$

| | | | |
|---|---|---|---|
| 20 | 25 | 30 | 35 |
| 21 | 26 | 31 | 36 |
| 22 | 27 | 32 | 37 |
| 23 | 28 | 33 | 38 |
| 24 | 29 | 34 | 39 |

$(u_2,u_3)=(1,1)$

| | | | |
|---|---|---|---|
| $j$+72 | $j$+67 | $j$+62 | $j$+57 |
| $j$+71 | $j$+66 | $j$+61 | $j$+56 |
| $j$+70 | $j$+65 | $j$+60 | $j$+55 |
| $j$+69 | $j$+64 | $j$+59 | $j$+54 |
| $j$+68 | $j$+63 | $j$+58 | $j$+53 |

$(u_2,u_3)=(2,0)$

| | | | |
|---|---|---|---|
| 40 | 45 | 50 | 55 |
| 41 | 46 | 51 | 56 |
| 42 | 47 | 52 | 57 |
| 43 | 48 | 53 | 58 |
| 44 | 49 | 54 | 59 |

$(u_2,u_3)=(2,1)$

| | | | |
|---|---|---|---|
| $j$+52 | $j$+47 | $j$+42 | $j$+37 |
| $j$+51 | $j$+46 | $j$+41 | $j$+36 |
| $j$+50 | $j$+45 | $j$+40 | $j$+35 |
| $j$+49 | $j$+44 | $j$+39 | $j$+34 |
| $j$+48 | $j$+43 | $j$+38 | $j$+33 |

So, $f(v) - f(u) = j + (2n_0n_1 \cdots n_{r-2} - A - 1)k - ((u_0 + v_0) + \cdots + (u_{r-2} + v_{r-2})n_0n_1 \cdots n_{r-3})k \geq j$. So, $f$ is an $L(j, k)$-labeling of $H$. Since the span of $f$ is $j + (2n_0n_1 \cdots n_{r-2} - A - 1)k$, we have $\lambda_k^j(H) \leq j + (2n_0n_1 \cdots n_{r-2} - A - 1)k$. ∎

As a consequence of Theorems 5 and 6, we have the following corollary.

**Corollary 1.** *Let $H$, $A$, $A'$, and $N'$ be the same as in Theorems 3 and 5. We have the following:*

(1) $\lambda_k^j(H) = (N' - 1)k$ *if* $\frac{j}{k} \leq A'$.

(2) $\lambda_k^j(H) \leq (n_{r-2} - 1)(j - A'k) + (N' - 1)k$ *if* $A' < \frac{j}{k} \leq A' + \frac{n_0n_1 \cdots n_{r-3}(n_{r-2}-1)}{n_{r-2}-2}$.

(3) $\lambda_k^j(H) \leq j + (2N' - A - 1)k$ *if* $A' + \frac{n_0n_1 \cdots n_{r-3}(n_{r-2}-1)}{n_{r-2}-2} < \frac{j}{k}$.

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## References

1. T. Calamoneri, The $L(h, k)$-labeling problem: A survey and annotated bibliography, *Comp. J.*, **49** (2006), 585-608.

2. T. Calamoneri and R. Petreschi, The $L(h, k)$-labeling problem: an updated survey and annotated bibliography, *Comp. J.*, **54** (2011), 1344-1371.

3. T. Calamoneri, $L(h, 1)$-labeling subclasses of planar graphs, *J. Parallel Distrib. Comp.*, **54** (2011), 1344-1371.

4. T. Calamoneri, E. G. Fusco, R. B. Tan and P. Vocca, $L(h, 1, 1)$-labeling of outerplanar graphs, *Math. Methods Oper. Res.*, **69** (2009), 307-321.

5. G. J. Chang and C. Lu, Distance-two labelings of graphs, *Europ. J. Combin.*, **24** (2003), 53-58.

6. G. J. Chang and D. Kuo, The $L(2, 1)$-labeling problem on graphs, *SIAM J. Disc. Math.*, **9** (1996), 309-316.

7. G. J. Chang, W. Ke, D. Kuo, D. Liu and R. K. Yeh, On $L(d, 1)$-labelings of graphs, *Disc. Math.*, **220** (2000), 57-66.

8. G. Chartrand, D. Erwin and P. Zhang, A graph labeling problem suggested by FM channel restrictions, *Bull. ICA*, **43** (2005), 43-57.

9. M. L. Chia, D. Kuo, H. Liao, C. H. Yang and R. K. Yeh, $L(3, 2, 1)$ labeling of graphs, *Taiwanese J. Math.*, **15** (2011), 2439-2457.

10. S. H. Chiang and H. H. Yan, On $L(d, 1)$-labeling of Cartesian product of a cycle and a path, *Disc. Appl. Math.*, **156(15)** (2008), 2867-2881.

11. J. P. Georges and D. W. Mauro, Generalized vertex labelings with a condition at distance two, *Congr. Numer.*, **109** (1995), 141-159.

12. J. P. Georges and D. W. Mauro, Labeling trees with a condition at distance two, *Disc. Math.*, **269** (2003), 127-148.

13. J. P. Georges, D. W. Mauro and M. I. Stein, Labeling products of complete graphs with a condition at a distance two, *SIAM J. Disc. Math.*, **14** (2000), 28-35.

14. J. R. Griggs and R. K. Yeh, Labeling graphs with a condition at distance two, *SIAM J. Disc. Math.*, **5** (1992), 586-595.

15. W. K. Hale, Frequency assignment: theory and applications, *Proc. IEEE*, **68** (1980), 1497-1514.

16. E. Haque and P. K. Jha, $L(j,k)$-labelings of Kronecker products of complete graphs, *IEEE Trans. Circuits Syst. II: Express Briefs*, **55(1)** (2008), 70-73.

17. F. Havet, B. Reed and J. S. Sereni, Griggs and Yeh's Conjecture and $L(p,1)$-labelings, *SIAM J. Disc. Math.*, **26(1)** (2012), 145-168.

18. P. K. Jha, Optimal $L(2,1)$-labeling of strong products of cycles, *IEEE Trans. Circuits Syst. I: Fund. Theory Appl.*, **48** (2001), 498-500.

19. P. K. Jha, S. Klavzar and A. Vesel, Optimal $L(d,1)$-labelings of certain direct products of cycles and Cartesian product of cycles, *Disc. Appl. Math.*, **152(1-3)** (2005), 257-265.

20. J. H. Kang, $L(2,1)$-labeling of Hamiltonian graphs with maximum degree $3$, *SIAM J. Disc. Math.*, **22(1)** (2008), 213-230.

21. B. M. Kim, B. C. Song and W. Hwang, Distance three labelings for direct products of 3 complete graphs, *Taiwanese J. of Math.*, **17** (2013), 207-219.

22. B. M. Kim, B. C. Song and W. Hwang, Distance three labelings for $K_n \times K_2$, *Intern. J. Comp. Math.*, **90** (2013), 906-911.

23. B. M. Kim, B. C. Song and Y. Rho, $L(2,1)$-labelings for ditect products of a triangle and a cycle, *Intern. J. Comp. Math.*, **90** (2013), 475-482.

24. S. Klavzar and A. Vesel, Computing graph invariants on rotagraphs using dynamic algorithm approach: the case of $(2,1)$-colorings and independence numbers, *Disc. Appl. Math.*, **129** (2003), 449-460.

25. S. R. Kola and P. Panigrahi, An improved lower bound for the radio $k$-chromatic number of the hypercube $Q_n$, *Computers* $\&$ *Math. with Appl.*, **60** (2010), 2131-2140.

26. D. Kuo and J. H. Yan, On $L(2,1)$-labelings of Cartesian products of paths and cycles, *Disc. Math.*, **283** (2004), 137-144.

27. P. C. B. Lam, W. Lin and J. Wu, $L(j,k)$- and circular $L(j,k)$-labelings for the products of complete graphs, *J. Comp. Optim.*, **14** (2007), 219-227.

28. W. Lin and P. C. B. Lam, Distance two labeling and direct products of graphs, *Disc. Math.*, **308** (2008), 3805-3815.

29. D. D. Liu, Radio number for trees, *Disc. Math.*, **308(7)** (2008), 1153-1164.

30. D. D. Liu and M. Xie, Radio number for square paths, *Ars. Combinatoria*, **90** (2009), 307-319.

31. D. D. Liu and X. Zhu, Multilevel distance labelings for paths and cycles, *SIAM J. Disc. Math.*, **19** (2005), 610-621.

32. D. Sakai, Labeling chordal graphs with a condition at distance two, *SIAM J. Disc. Math.*, **7** (1994), 133-140.

33. C. Schwarz and D. S. Troxell, $L(2, 1)$-labelings of Cartesian products of two cycles, *Disc. Appl. Math.*, **154(10)** (2006), 1522-1540.

34. M. A. Whittlesey, J. P. Georges and D. W. Mauro, On the $\lambda$-number of $Q_n$ and related graphs, *SIAM J. Disc. Math.*, **8** (1995), 499-506.

35. R. K. Yeh, A survey on labeling graphs with a condition at distance two, *Disc. Math.*, **306(12)** (2006), 1217-1231.

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