TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 1, pp. 39-51, February 2014
DOI: 10.11650/tjm.18.2014.2986
This paper is available online at http://journal.taiwanmathsoc.org.tw

GENERALIZATIONS OF STRONGLY STARLIKE FUNCTIONS

Jacek Dziok

Abstract. By using functions of bounded variation we generalize the class of strongly starlike functions and related classes. The main object is to obtain characterizations and inclusion properties of these classes of functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}_p$ $(p \in \mathbb{N}_0 := \{0, 1, 2, \ldots\})$ denote the class of functions $f \in \mathcal{A}$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (z \in \mathcal{U}) . \tag{1}$$

Let $a, \delta \in \mathbb{C}, |a| < 1,\ \alpha < p,\ 0 < \beta \leq 1,\ k \geq 2, p \in \mathbb{N}, \varphi \in \mathcal{A}_p$.

A function $f \in \mathcal{A}_p$ is said to be in the class S^*_β of *multivalent strongly starlike function of order* β if

$$\left|\mathrm{Arg}\ \frac{zf'(z)}{pf(z)}\right| < \beta\frac{\pi}{2} \quad (z \in \mathcal{U}).$$

We denote by M_k the class of real-valued functions m of bounded variation on $[0, 2\pi]$ which satisfy the conditions

$$\int_0^{2\pi} \mathrm{d}m\,(t) = 2,\ \int_0^{2\pi} |\mathrm{d}m\,(t)| \leq k. \tag{2}$$

It is clear that M_2 is the class of nondecreasing functions on $[0, 2\pi]$ satisfying (2) or equivalently $\int_0^{2\pi} \mathrm{d}m\,(t) = 2$.

Received February 22, 2013, accepted May 16, 2013.
Communicated by Hari M. Srivastava.
2010 *Mathematics Subject Classification*: 30C45, 30C50, 30C55.
Key words and phrases: Analytic functions, Bounded variation, Bounded boundary rotation, Strongly starlike functions.

Let $\mathcal{P}_k(a,\beta)$ denote the class of functions $q \in \mathcal{A}_0$ for which there exists $m \in M_k$ such that

$$q(z) = \frac{1}{2}\int_0^{2\pi}\left(\frac{1+(1-2a)\,ze^{-it}}{1-ze^{-it}}\right)^{\beta} \mathrm{d}m(t) \quad (z \in \mathcal{U}). \tag{3}$$

Here and throughout we assume that all powers denote principal determinations.

Moreover, let us denote

$$\mathcal{P}_k(a) := \mathcal{P}_k(a,1),\ \widetilde{\mathcal{P}}_k(a,\beta) := \left\{q \in \mathcal{A}_0 : q^{1/\beta} \in \mathcal{P}_k(a)\right\}.$$

In particular, $\mathcal{P} := \mathcal{P}_2(0)$ is the well-known class of Carathéodory functions. The classes $\mathcal{P}_k := \mathcal{P}_k(0)$, $\mathcal{P}_k(\rho)$ $(0 \le \rho < 1)$ were investigated by Paatero [19] (see also Pinchuk [23]) and Padmanabhan and Parvatham [21], respectively. We note that

$$f \in S^*_\beta \Longleftrightarrow \frac{zf'(z)}{pf(z)} \in P(\beta),$$

where

$$P(\beta) := \mathcal{P}_2(\beta) = \left\{q \in \mathcal{A}_0 : |\mathrm{Arg}\ q(z)| < \beta\frac{\pi}{2}\right\}.$$

Now, we generalize the class of strongly starlike functions. We denote by $\mathcal{M}_k(a,\beta;\delta,\varphi)$ the class of functions $f \in \mathcal{A}_p$ such that

$$J_{\delta,\varphi}(f)(z) := \frac{\delta}{p}\left(1+\frac{z(\varphi*f)''(z)}{(\varphi*f)'(z)}\right)+(1-\delta)\frac{z(\varphi*f)'(z)}{p(\varphi*f)(z)} \in \mathcal{P}_k(a,\beta),$$

where $*$ denote the Hadamard product (or convolution). Moreover, let us denote

$$\mathcal{M}(a,\beta;\delta,\varphi) := \mathcal{M}_2(a,\beta;\delta,\varphi),\ \mathcal{W}_k(a,\beta;\varphi) := \mathcal{M}_k(a,\beta;0,\varphi),$$
$$\mathcal{W}(a,\beta;\varphi) := \mathcal{W}_2(a,\beta;\varphi),\ \mathcal{W}_k(a,\beta) := \mathcal{W}_k(a,\beta;z^p/(1-z)),$$
$$\mathcal{S}^*_p(\varphi,a) := \mathcal{W}_2(a,1;\varphi).$$

We see that $S^*_\beta = \mathcal{W}_2(0,\beta)$ and

$$f \in \mathcal{W}_k(a,\beta;\varphi) \Leftrightarrow \varphi * f \in \mathcal{W}_k(a,\beta). \tag{4}$$

Let $\overrightarrow{a} = (a_1,a_2)$, $\overrightarrow{\beta} = (\beta_1,\beta_2)$. We say that a function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\varphi\right)$, if there exists a function $g \in \mathcal{W}_k(a_2,\beta_2;\varphi)$ such that

$$\frac{\delta}{p}\left(1+\frac{z(\varphi*f)''(z)}{(\varphi*g)'(z)}\right)+(1-\delta)\frac{z(\varphi*f)'(z)}{p(\varphi*g)(z)} \in \mathcal{P}_k(a_1,\beta_1).$$

These classes generalize well-known classes of functions, which were defined in ealier works, see for example [1-10] and [14-26]. We note that

- $\mathcal{M}_k(a,\beta;\alpha,\varphi)$ is related to the class of functions with the bounded Mocanu variation defined by Coonce and Ziegler [4] and intensively investigated by Noor *et al.* [15-18]
- $V_k := \mathcal{W}_k\left(0,1;\frac{z}{(1-z)^2}\right)$ is the well-known class of functions of bounded boundary rotation (for details, see, [2, 7, 14, 21]).
- $$\mathcal{S}_p^*(\alpha) := \mathcal{S}_p^*\left(\frac{z^p}{1-z},\alpha/p\right),\ \mathcal{S}_p^c(\alpha) := \mathcal{S}_p^*\left(\frac{z^p(p+(1-p)z)}{p(1-z)^2},\alpha/p\right)$$

  are the classes of *multivalent starlike* functions of order α and *multivalent convex* functions of order α, respectively.
- $$\mathcal{R}_p(\alpha) := \mathcal{S}_p^*\left(z^p/(1-z)^{2(p-\alpha)},\alpha/p\right) \quad (\alpha<p)$$

  will be called the class of *multivalent prestarlike* functions of order α. In particular, $\mathcal{R}(\alpha) := \mathcal{R}_1(\alpha)$ is the well-know class of *prestarlike* functions of order α introduced by Ruscheweyh [24].
- $CC := \mathcal{CW}_k\left(0,1;0\frac{z}{(1-z)^2}\right)$ is the well-known class of *close-to-convex* functions.

The main object of the paper is to obtain some characterizations and inclusion properties for the defined classes of functions. Some applications of the main results are also considered.

2. Characterization Theorems

Let us define

$$\mathcal{B}_k(a,\beta) := \left\{\left(\frac{k}{4}+\frac{1}{2}\right)q_1 - \left(\frac{k}{4}-\frac{1}{2}\right)q_2 : q_1,q_2 \prec h_{a,\beta}\right\},$$

where

$$h_{a,\beta}(z) := \left(\frac{1+(1-2a)z}{1-z}\right)^\beta,\ h_a := h_{a,1} \quad (z\in\mathcal{U}). \tag{5}$$

From the result of Hallenbeck and MacGregor ([13], pp. 50) we have the following lemma.

Lemma 1. *$q \prec h_{a,\beta}$ if and only if there exists $m \in M_2$ such that*

$$q(z) = \frac{1}{2}\int_0^{2\pi}\left(\frac{1+(1-2a)ze^{-it}}{1-ze^{-it}}\right)^\beta \mathrm{d}m(t) \quad (z\in\mathcal{U}).$$

Theorem 1.

$$\mathcal{B}_{\lambda}(a,\beta)\subset\mathcal{B}_{k}(a,\beta)\quad(2\leq\lambda<k).$$

Proof. Let $q\in\mathcal{B}_{\lambda}(a,\beta)$. Then there exist $q_1,q_2\prec h_{a,\beta}$ such that $q=\left(\frac{\lambda}{4}+\frac{1}{2}\right)q_1-\left(\frac{\lambda}{4}-\frac{1}{2}\right)q_2$ or

$$q=\left(\frac{k}{4}+\frac{1}{2}\right)q_1-\left(\frac{k}{4}-\frac{1}{2}\right)\widetilde{q}_2\quad\left(\widetilde{q}_2=\frac{k-\lambda}{k-2}q_1+\frac{\lambda-2}{k-2}q_2\right).$$

Since $\widetilde{q}_2\prec h_{a,\beta}$, we have $q\in\mathcal{B}_{k}(a,\beta)$. ■

Theorem 2. *The class $\mathcal{B}_{k}(a,\beta)$ is convex.*

Proof. Let $q,r\in\mathcal{B}_{k}(a,\beta)$, $\alpha\in[0,1]$, $\mu:=\frac{k}{4}+\frac{1}{2}$. Then there exist $q_j,r_j\prec h_{a,\beta}$ $(j=1,2)$ such that

$$q=\mu q_1-(1-\mu)q_2,\ r=\mu r_1-(1-\mu)r_2.$$

It follows that

$$\alpha q+(1-\alpha)r=\mu(\alpha q_1+(1-\alpha)r_1)-(1-\mu)(\alpha q_2+(1-\alpha)r_2).$$

Since $\alpha q_j+(1-\alpha)r_j\prec h_{a,\beta}$ $(j=1,2)$, we conclude that $\alpha q+(1-\alpha)r\in\mathcal{B}_{k}(a,\beta)$. Hence, the class $\mathcal{B}_{k}(a,\beta)$ is convex. ■

Theorem 3.

$$\mathcal{P}_{k}(a,\beta)=\mathcal{B}_{k}(a,\beta).$$

Proof. Let $q\in\mathcal{P}_{k}(a,\beta)$. Then q satisfy (3) for some $m\in M_k$. If $m\in M_2$, then by Lemma 1 and Theorem 1 we have $q\in\mathcal{P}_{2}(h)\subset\mathcal{P}_{k}(h)$. Let now $m\in M_k\setminus M_2$. Since m is the function with bounded variation, by the Jordan theorem there exist real-valued functions μ_1,μ_2 which are nondecreasing and nonconstant on $[0,2\pi]$ such that

(6) $$m=\mu_1-\mu_2,\ \int_0^{2\pi}|\mathrm{d}m(t)|=\int_0^{2\pi}\mathrm{d}\mu_1(t)+\int_0^{2\pi}\mathrm{d}\mu_2(t).$$

Thus, putting

$$\alpha_j=\frac{\mu_j(2\pi)-\mu_j(0)}{2},\quad m_j:=\frac{1}{\alpha_j}\mu_j\quad(j=1,2)$$

we get $m_1,m_2\in M_2$ and

(7) $$m=\alpha_1m_1-\alpha_2m_2.$$

Combining (6) and (7) we obtain

$$2\alpha_1 - 2\alpha_2 = \int_0^{2\pi} \mathrm{d}m\,(t) = 2,\ 2\alpha_1 + 2\alpha_2 = \int_0^{2\pi} |\mathrm{d}m\,(t)| \le k,$$

and so

$$\alpha_1 = \left(\frac{\lambda}{4} + \frac{1}{2}\right),\ \alpha_2 = \left(\frac{\lambda}{4} - \frac{1}{2}\right) \quad \left(\lambda = \int_0^{2\pi} |\mathrm{d}m\,(t)| \le k\right).$$

Therefore, by (3) and (7) we obtain

$$q = \left(\frac{\lambda}{4} + \frac{1}{2}\right) q_1 - \left(\frac{\lambda}{4} - \frac{1}{2}\right) q_2,$$

where

$$q_j\,(z) = \frac{1}{2} \int_0^{2\pi} \left(\frac{1 + (1 - 2a)\, ze^{-it}}{1 - ze^{-it}}\right)^{\beta} \mathrm{d}m_j\,(t) \quad (z \in \mathcal{U},\ j = 1, 2)\,.$$

Thus, by Lemma 1 and Theorem 1 we have $q \in \mathcal{B}_\lambda\,(a, \beta) \subset \mathcal{B}_k\,(a, \beta)$. Conversely, let $q \in \mathcal{B}_k\,(a, \beta)$. Then there exist $q_1, q_2 \prec h_{a,\beta}$ such that q is of the form

$$q = \left(\frac{k}{4} + \frac{1}{2}\right) q_1 - \left(\frac{k}{4} - \frac{1}{2}\right) q_2.$$

Thus, by Lemma 1 there exist $m_1, m_2 \in M_2$ such that q is of the form (3) with

$$m = \left(\frac{k}{4} + \frac{1}{2}\right) m_1 - \left(\frac{k}{4} - \frac{1}{2}\right) m_2.$$

Since

$$\int_0^{2\pi} \mathrm{d}m\,(t) = \left(\frac{k}{4} + \frac{1}{2}\right) \int_0^{2\pi} \mathrm{d}m_1 - \left(\frac{k}{4} - \frac{1}{2}\right) \int_0^{2\pi} \mathrm{d}m_2 = 2,$$

$$\int_0^{2\pi} |\mathrm{d}m\,(t)| \le \left(\frac{k}{4} + \frac{1}{2}\right) \int_0^{2\pi} \mathrm{d}m_1 + \left(\frac{k}{4} - \frac{1}{2}\right) \int_0^{2\pi} \mathrm{d}m_2 = k,$$

we have $m \in M_k$ and consequently $q \in \mathcal{P}_k\,(a, \beta)$. ■

Lemma 2. [7]. *Let* $q \in \mathcal{A}_0$. *Then* $q \in \mathcal{P}_k\,(a)$ *if and only if*

$$\int_0^{2\pi} \left| \Re \frac{q\left(re^{it}\right) - a}{1 - a} \right| \mathrm{d}t \le k\pi \quad (0 < r < 1)\,.$$

From Lemma 2 we have the following corollary.

Corollary 1. *Let* $q \in \mathcal{A}_0$. *Then* $q \in \widetilde{\mathcal{P}}_k(a,\beta)$ *if and only if*

$$\int_0^{2\pi} \left| \Re \frac{q^{1/\beta}\left(re^{it}\right) - a}{1-a} \right| \mathrm{d}t \le k\pi \quad (0 < r < 1)\,. \tag{8}$$

3. The Main Inclusion Relationships

From now on we make the assumptions: $0 \le \delta \le 1$ and

$$\Re h_{a,\beta}(z) > \alpha \quad (z \in \mathcal{U})\,. \tag{9}$$

Then we have

$$\mathcal{W}_k(a,\beta) \subset \mathcal{S}_p^*(\alpha)\,. \tag{10}$$

Let $\Phi_p(b,c)$ denote the *multivalent incomplete hipergeometric function* defined by

$$\Phi_p(b,c)(z) := z^p\ {}_2F_1(b,1;c;z) = \sum_{n=p}^{\infty} \frac{(b)_{n-p}}{(c)_{n-p}} z^n \quad (z \in \mathcal{U})\,. \tag{11}$$

Lemma 3. [11]. *Let* $h \in \mathcal{K}$, $q \in \mathcal{A}_0$ *and* $\lambda > 0$. *If*

$$q(z) + \lambda \frac{zq'(z)}{q(z)} \prec h(z)\,,$$

then $q \prec h$.

Lemma 4. [5]. *Let* $f \in \mathcal{R}_p(\alpha)$, $g \in \mathcal{S}_p^*(\alpha)$. *Then*

$$\frac{f * (hg)}{f * g}(\mathcal{U}) \subseteq \overline{co}\{h(\mathcal{U})\}\,,$$

where $\overline{co}\{h(\mathcal{U})\}$ *denotes the closed convex hull of* $h(\mathcal{U})$.

Lemma 5. [5]. *Let* $p \in \mathbb{N}$. *If either*

$$\Re[b] \le \Re[c],\ \Im[b] = \Im[c] \ \text{ and } \ \frac{1}{2}(2p + 1 - b - \overline{c}) \le \alpha < p \tag{12}$$

or

$$0 < b \le c \text{ and} \left(p - \frac{c}{2}\right) \le \alpha < p, \tag{13}$$

then

$$\Phi_p(b,c) \in \mathcal{R}_p(\alpha)\,.$$

Theorem 4. *If* $\psi \in \mathcal{R}_p(\alpha)$, *then*

$$\mathcal{W}_k(a,\beta;\varphi) \subset \mathcal{W}_k(a,\beta;\psi*\varphi). \tag{14}$$

Proof. Let $f \in \mathcal{W}_k(a,\beta;\varphi)$. Thus, by Theorem 3 there exist $q_1, q_2 \prec h_{a,\beta}$ such that

$$\frac{z(\varphi*f)'(z)}{p(\varphi*f)(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)q_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)q_2(z) \quad (z \in \mathcal{U}).$$

Moreover, $H = \varphi * f \in \mathcal{W}_k(a,\beta) \subset \mathcal{S}_p^*(\alpha)$. Thus, applying the properties of convolution, we get

$$\begin{aligned}\frac{z[(\psi*\varphi)*f]'(z)}{p[(\psi*\varphi)*f](z)} &= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{\psi(z)*[q_1(z)H(z)]}{\psi(z)*H(z)} \\ &\quad - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{\psi(z)*[q_2(z)H(z)]}{\psi(z)*H(z)} \quad (z \in \mathcal{U}).\end{aligned} \tag{15}$$

By Lemma 4 we conclude that

$$F_j(z) := \frac{\psi(z)*[q_j(z)H(z)]}{\psi(z)*H(z)} \in \overline{co}\{q_j(\mathcal{U})\} \subset \overline{h_{a,\beta}(\mathcal{U})} \quad (z \in \mathcal{U},\ j=1,2).$$

Therefore, $F_j \prec h_{a,\beta}$ and by (15) we have $f \in \mathcal{W}_k(a,\beta;\psi*\varphi)$, which proves the theorem. ■

Theorem 5. *Let* $\psi \in \mathcal{R}_p(\alpha),\ 0 \le \delta \le 1$. *Then*

$$\mathcal{M}_k(a,\beta;\delta,\varphi) \cap \mathcal{W}_k(a,\beta;\varphi) \subset \mathcal{M}_k(a,\beta;\delta,\psi*\varphi). \tag{16}$$

Proof. Let $f \in \mathcal{M}_k(a,\beta;\delta,\varphi) \cap \mathcal{W}_k(a,\beta;\varphi)$. Then, applying Theorem 4, we obtain $f \in \mathcal{W}_k(a,\beta;\psi*\varphi)$. Thus, we have

$$F_1(z) := \frac{z[(\psi*\varphi)*f]'(z)}{p[(\psi*\varphi)*f](z)},\ F_2(z) := \frac{z(\varphi*f)'(z)}{p(\varphi*f)(z)} \in \mathcal{P}_k(a,\beta).$$

Since the class $\mathcal{P}_k(a,\beta)$ is convex by Theorem 2, we conclude that $(1-\delta)F_1 + \delta F_2 \in \mathcal{P}_k(a,\beta)$. Hence, $f \in \mathcal{M}_k(a,\beta;\delta,\psi*\varphi)$ and, in consequence, we get (16). ■

Lemma 6. *If* $0 \le \gamma \le \delta$, *then*

$$\mathcal{M}(a,\beta;\delta,\varphi) \subset \mathcal{M}(a,\beta;\gamma,\varphi).$$

Proof. Let $f \in \mathcal{M}(a,\beta;\delta,\varphi)$ and let

$$q(z) := \frac{z(\varphi*f)'(z)}{p(\varphi*f)(z)} \quad (z \in \mathcal{U}).$$

Then, we obtain

$$q(z)+\delta\frac{zq'(z)}{q(z)}=J_{\delta,\varphi}(f)(z)\quad(z\in\mathcal{U}).$$

Since $J_{\delta,\varphi}(f)\prec h_{a,\beta}$, we have $q\prec h_{a,\beta}$ by Lemma 3. Moreover,

$$J_{\gamma,\varphi}(f)=\frac{\gamma}{\delta}J_{\delta,\varphi}(f)+\frac{\delta-\gamma}{\delta}q.$$

Because $h_{a,\beta}$ is convex and univalent in $\mathcal{U}$, then we obtain $J_{\gamma\varphi}(f)\prec h_{a,\beta}$ or equivalently $f\in\mathcal{M}(a,\beta;\gamma,\varphi)$. ■

From Theorem 5 and Lemma 6 we have the following corollary.

Corollary 2. *Let* $\psi\in\mathcal{R}_p(\alpha)$, $0\le\delta\le 1$. *Then*

$$\mathcal{M}(a,\beta;\delta,\varphi)\subset\mathcal{M}(a,\beta;\delta,\psi*\varphi).$$

Theorem 6. *If* $\psi\in\mathcal{R}_p(\alpha)$, *then*

$$\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\varphi\right)\subset\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\psi*\varphi\right).\tag{17}$$

Proof. Let $f\in\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\varphi\right)$. Then there exist $g\in\mathcal{W}_k(a_2,\beta_2;\varphi)$ and $q_1,q_2\prec h_{a_1,\beta_1}$ such that

$$\frac{z(\varphi*f)'(z)}{p(\varphi*g)(z)}=\left(\frac{k}{4}+\frac{1}{2}\right)q_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)q_2(z)\quad(z\in\mathcal{U})$$

and $F=\varphi*g\in\mathcal{W}_k(a_2,\beta_2)\subset\mathcal{S}_p^*(\alpha)$. Thus, applying the properties of convolution, we get

$$\begin{aligned}\frac{z[(\psi*\varphi)*f]'(z)}{p[(\psi*\varphi)*g](z)}=&\left(\frac{k}{4}+\frac{1}{2}\right)\frac{\psi*(q_1F)}{\psi*F}(z)\\&-\left(\frac{k}{4}-\frac{1}{2}\right)\frac{\psi*(q_2F)}{\psi*F}(z)\quad(z\in\mathcal{U}).\end{aligned}\tag{18}$$

By Lemma 4 we conclude that

$$F_j(z):=\frac{\psi*(q_jF)}{\psi*F}(z)\in\overline{co}\{q_j(\mathcal{U})\}\subset\overline{h_{a1,\beta_1}(\mathcal{U})}\quad(z\in\mathcal{U},\ j=1,2).$$

Therefore, $F_j\prec h_{a_1,\beta_1}$ and by (18) we have $f\in\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\psi*\varphi\right)$. ■

Combining Theorems 4-6 with Lemma 5 we obtain the following theorem.

Theorem 7. *If either* (12) *or* (13), *then*

$$\begin{aligned}&\mathcal{W}_k(a,\beta;\varphi)\subset\mathcal{W}_k(a,\beta;\Phi_p(b,c)*\varphi),\\&\mathcal{M}(a,\beta;\delta,\varphi)\subset\mathcal{M}(a,\beta;\delta,\Phi_p(b,c)*\varphi),\\&\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\varphi\right)\subset\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\Phi_p(b,c)*\varphi\right).\end{aligned}$$

Since $\Phi_p(b,c) * \Phi_p(c,b) * \varphi = \varphi$, by Theorem 7 we obtain the next result.

Theorem 8. *If either* (12) *or* (13), *then*

$$\begin{aligned}
&\mathcal{W}_k\left(a,\beta;\Phi_p(c,b)*\varphi\right)\subset\mathcal{W}_k\left(a,\beta;\varphi\right),\\
&\mathcal{M}\left(a,\beta;\delta,\Phi_p(c,b)*\varphi\right)\subset\mathcal{M}\left(a,\beta;\delta,\varphi\right),\\
&\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\Phi_p(c,b)*\varphi\right)\subset\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\varphi\right).
\end{aligned}$$

Let us define the linear operators $J_\lambda : \mathcal{A}_p \longrightarrow \mathcal{A}_p$,

$$J_\lambda\left(f\right)\left(z\right) := \lambda\frac{zf'\left(z\right)}{p} + \left(1-\lambda\right)f\left(z\right), \quad \left(z\in\mathcal{U},\ \Re(\lambda)>0\right). \tag{19}$$

Since $J_\lambda\left(f\right) = \Phi_p(\frac{p}{\lambda}+1,\frac{p}{\lambda}) * f$, putting $b=\frac{p}{\lambda},\ c=\frac{p}{\lambda}+1$ in Theorem 8, we have the following theorem.

Theorem 9. *If* $p-\Re\left[\frac{p}{\lambda}\right]\le\alpha<p$, *then*

$$\begin{aligned}
&\mathcal{W}_k\left(a,\beta;J_\lambda\left(\varphi\right)\right)\subset\mathcal{W}_k\left(a,\beta;\varphi\right),\\
&\mathcal{M}\left(a,\beta;\delta,J_\lambda\left(\varphi\right)\right)\subset\mathcal{M}\left(a,\beta;\delta,\varphi\right),\\
&\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,J_\lambda\left(\varphi\right)\right)\subset\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\varphi\right).
\end{aligned}$$

In particular, for $\lambda=1$ we get the following theorem.

Theorem 10. *If* $0\le\alpha<p$, *then*

$$\begin{aligned}
&\mathcal{W}_k\left(a,\beta;z\varphi'\left(z\right)\right)\subset\mathcal{W}_k\left(a,\beta;\varphi\right),\\
&\mathcal{M}\left(a,\beta;\delta,z\varphi'\left(z\right)\right)\subset\mathcal{M}\left(a,\beta;\delta,\varphi\right),\\
&\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,z\varphi'\left(z\right)\right)\subset\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\varphi\right).
\end{aligned}$$

4. Applications to Classes Defined by Linear Operators

For real numbers $\lambda, t\ \ (\lambda>-p)$, we define the function

$$\Psi(a_1,b_1,t)\left(z\right) := \left(z^p\,{}_qF_s(a_1,\dots,a_q;\ b_1,\dots,b_s;z)\right) * f_{\lambda,t}\left(z\right) \quad (z\in\mathcal{U}), \tag{20}$$

where ${}_qF_s(a_1,\dots,a_q;\ b_1,\dots,b_s;z)$ is the *generalized hypergeometric function* and

$$f_{\lambda,t}\left(z\right)=\sum_{n=p}^{\infty}\left(\frac{n+\lambda}{p+\lambda}\right)^t z^n \quad (z\in\mathcal{U}).$$

It is easy to verify that

$$b\Psi(b+1,c,t)=z\Psi'(b,c,t)+(b-p)\Psi(b,c,t), \tag{21}$$

(22) $$b\Psi(b,c,t) = z\Psi'(b,c+1,t) + (b-p)\Psi(b,c+1,t),$$

(23) $$(p+\lambda)\,\Psi(b,c,t+1) = z\Psi'(b,c,t) + \lambda\Psi(b,c,t),$$

(24) $$\Psi(b,c,t) = \Phi_p(b,d) * \Psi(d,c,t).$$

where $\Phi_p(b,d)$ is defined by (11).

Corresponding to the function $\Psi(b,c,t)$ we consider the following classes of functions:

$$\mathcal{V}_k(a,\beta;b,c,t) := \mathcal{W}_k\left(a,\beta;\Psi(b,c,t)\right),$$
$$\mathcal{CV}_k(\overrightarrow{a},\overrightarrow{\beta};b,c,t) := \mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\Psi(b,c,t)\right).$$

By using the linear operator

(25) $$\Theta_p\,[b,c,t]\,f = \Psi(b,c,t) * f \quad (\,f \in \mathcal{A}_p)$$

we can define the class $\mathcal{V}_k\,(a,\beta;b,c,t)$ alternatively in the following way:

$$f \in \mathcal{V}_k\,(a,\beta;b,c,t) \Longleftrightarrow b\frac{\Theta_p\,[b+1,c,t]\,f(z)}{\Theta_p\,[b,c,t]\,f(z)} + p - b \in \mathcal{P}_k\,(a,\beta)\,.$$

Corollary 3. *If* $p - \Re\,[b] \le \alpha < p,\ m \in \mathbb{N}$*, then*

(26) $$\mathcal{V}_k\,(a,\beta;b+m,c,t) \subset \mathcal{V}_k\,(a,\beta;b,c,t)\,,$$

(27) $$\mathcal{CV}_k\,(a,\beta;b+m,c,t) \subset \mathcal{CV}_k\,(a,\beta;b,c,t)\,.$$

Proof. It is clear that it is sufficient to prove the corollary for $m = 1$. Let J_λ and $\Psi(b,c,t)$ be defined by (19) and (20), respectively. Then, by (21) we have $\Psi(b+1,c,t) = J_{\frac{p}{b}}\,(\Psi(b,c,t))$. Hence, by using Theorem 9 we conclude that

$$\mathcal{W}_k\,(a,\beta;\Psi(b+1,c,t)) \subset \mathcal{W}_k\,(a,\beta;\Psi(b,c,t))\,,$$
$$\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\Psi(b+1,c,t)\right) \subset \mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\Psi(b,c,t)\right).$$

This clearly forces the inclusion relations (26) and (27) for $m = 1$. ■

Analogously to Corollary 3, we prove the following corollary.

Corollary 4. *Let* $m \in \mathbb{N}$. *If* $p - \Re\,[c] \le \alpha < p$*, then*

$$\mathcal{V}_k\,(a,\beta;b,c,t) \subset \mathcal{V}_k\,(a,\beta;b,c+m,t)\,,$$
$$\mathcal{CV}_k\left(\overrightarrow{a},\overrightarrow{\beta};b,c,t\right) \subset \mathcal{CV}_k\left(\overrightarrow{a},\overrightarrow{\beta};b,c+m,t\right).$$

If $-\Re\,[\lambda] \le \alpha < p$*, then*

$$\mathcal{V}_k\,(a,\beta;b,c,t+m) \subset \mathcal{V}_k\,(a,\beta;b,c,t)\,,$$
$$\mathcal{CV}_k\left(\overrightarrow{a},\overrightarrow{\beta};b,c,t+m\right) \subset \mathcal{CV}_k\left(\overrightarrow{a},\overrightarrow{\beta};b,c,t\right).$$

It is natural to ask about the inclusion relations in Corollaries 3 and 4 when m is positive real. Using Theorems 4 and 6, we shall give a partial answer to this question.

Corollary 5. *If the multivalent incomplete hipergeometric function $\Phi_p(b,d)$ defined by (11) belongs to the class $\mathcal{R}_p(\alpha)$, then*

(28) $$\mathcal{V}_k(a,\beta;d,c,t)\subset\mathcal{V}_k(a,\beta;b,c,t),\ \mathcal{CV}_k\left(\overrightarrow{a},\overrightarrow{\beta};d,c,t\right)\subset\mathcal{CV}_k\left(\overrightarrow{a},\overrightarrow{\beta};b,c,t\right),$$

(29) $$\mathcal{V}_k(a,\beta;c,b,t)\subset\mathcal{V}_k(a,\beta;c,d,t),\ \mathcal{CV}_k\left(\overrightarrow{a},\overrightarrow{\beta};c,b,t\right)\subset\mathcal{CV}_k\left(\overrightarrow{a},\overrightarrow{\beta};c,d,t\right).$$

Proof. Let us put $\psi=\Phi_p(b,d),\ \varphi=\Psi(d,c,t)$. Then, by (22) and Theorems 4 and 6 we obtain

$$\mathcal{W}_k(a,\beta;\Psi(d,c,t))\subset\mathcal{W}_k(a,\beta;\Psi(b,c,t)),$$
$$\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\Psi(d,c,t)\right)\subset\mathcal{CW}_k\left(\overrightarrow{a},\overrightarrow{\beta};\delta,\Psi(b,c,t)\right).$$

Thus, we get the inclusion relations (28). Analogously, we prove the inclusions (29). ■

Combining Corollary 5 with Lemma 5, we obtain the following result.

Corollary 6. *If either (12) or (13), then the inclusion relations (28) and (29) hold true.*

The linear operator $\Theta_p[b,c,t]$ defined by (25) includes (as its special cases) other linear operators of geometric function theory which were considered in earlier works. In particular, we can mention here the Dziok-Srivastava operator, the Hohlov operator, the Carlson-Shaffer operator, the Ruscheweyh derivative operator, the generalized Bernardi-Libera-Livingston operator, the fractional derivative operator, and so on for the precise relationships, see, Dziok and Srivastava ([10], pp. 3-4). Moreover, the linear operator $\Theta_p[b,c,t]$ includes also the Sălăgean operator, the Noor operator, the Choi-Saigo-Srivastava operator, the Kim-Srivastava operator, and others (for the precise relationships, see, Cho *et al.* [3]). By using these linear operators we can consider several subclasses of the classes $\mathcal{V}_k(a,\beta;b,c,t)$ and $\mathcal{CV}_k(\overrightarrow{a},\overrightarrow{\beta};b,c,t)$, see for example [1-10, 12, 20, 22, 26]. Also, the obtained results generalize several results obtained in these classes of functions.

References

1. M. K. Aouf, Some inclusion relationships associated with Dziok Srivastava operator, *Appl. Math. Comput.*, **216** (2010), 431-437.
2. S. Bhargava and R. S. Nanjunda, Convexity of a class of functions related to classes of starlike functions and functions with boundary rotation, *Ann. Polon. Math.*, **49(3)** (1989), 229-235.

3. N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, *J. Math. Anal. Appl.*, **292** (2004), 470-483.

4. H. B. Coonce and M. R. Ziegler, Functions with bounded Mocanu variation, *Rev. Roumaine Math. Pures Appl.*, **19** (1974), 1093-1104.

5. J. Dziok, Applications of multivalent prestarlike functions, *Appl. Math. Comput.*, **221** (2013), 230-238.

6. J. Dziok, Applications of the Jack lemma, *Acta Math. Hungar.*, **105** (2004), 93-102.

7. J. Dziok, Characterizations of analytic functions associated with functions of bounded variation, *Ann. Polon. Math.*, to appear.

8. J. Dziok, Classes of functions defined by certain differential-integral operators, *J. Comput. Appl. Math.*, **105(1-2)** (1999), 245-255.

9. J. Dziok, Inclusion relationships between classes of functions defined by subordination, *Ann. Polon. Math.*, **100** (2011), 193-202.

10. J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Integral Transforms Spec. Funct.*, **14** (2003), 7-18.

11. P. J. Eenigenburg, S. S. Miller, P. T. Mocanu and O. M. Reade, Second order differential inequalities in the complex plane, *J. Math. Anal. Appl.*, **65** (1978), 289-305.

12. J.-L. Liu and K. I. Noor, On subordinations for certain analytic functions associated with Noor integral operator, *Appl. Math. Comput.*, **187** (2007), 1453-1460.

13. D. J. Hallenbeck and T. H. MacGregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*, Pitman Advanced Publishing Program, Boston, Pitman, 1984.

14. E. J. Moulis, Generalizations of the Robertson functions, *Pacific J. Math.*, **81** (1979), 167-174.

15. K. I. Noor and S. Hussain, On certain analytic functions associated with Ruscheweyh derivatives and bounded Mocanu variation, *J. Math. Anal. Appl.*, **340(2)** (2008), 1145-1152.

16. K. I. Noor and S. N. Malik, On generalized bounded Mocanu variation associated with conic domain, *Math. Comput. Modelling*, **55(3-4)** (2012), 844-852.

17. K. I. Noor and A. Muhammad, On analytic functions with generalized bounded Mocanu variation, *Appl. Math. Comput.*, **196(2)** (2008), 802-811.

18. K. I. Noor and W. Ul-Haq, On some implication type results involving generalized bounded Mocanu variations, *Comput. Math. Appl.*, **63(10)** (2012), 1456-1461.

19. V. Paatero, Über die konforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind, *Ann. Acad. Sei. Fenn. Ser A*, **33** (1931), 1-79.

20. J. Patel, A. K. Mishra and H. M. Srivastava, Classes of multivalent analytic functions involving the Dziok-Srivastava operator, *Comput. Math. Appl.*, **54** (2007), 599-616.

21. K. Padmanabhan and R. Parvatham, Properties of a class of functions with bounded boundary rotation, *Ann. Polon. Math.*, **31** (1975), 311-323.

22. K. Piejko and J. Sokół, On the Dziok-Srivastava operator under multivalent analytic functions, *Appl. Math. Comp.*, **177** (2006), 839-843.

23. B. Pinchuk, Functions with bounded boundary rotation, *Isr. J. Math.*, **10** (1971), 7-16.

24. S. Ruscheweyh, Linear Operators Between Classes of Prestarlike Functions, *Comm. Math. Helv.*, **52** (1977), 497-509.

25. H. M. Srivastava and A. Y. Lashin, Subordination properties of certain classes of multivalently analytic functions, *Math. Comput. Modelling*, **52** (2010), 596-602.

26. Z.-G. Wang, G.-W. Zhang and F.-H. Wen, Properties and characteristics of the Srivastava-Khairnar-More integral operator, *Appl. Math. Comput.*, **218** (2012), 7747-7758.

Jacek Dziok
Institute of Mathematics
University of Rzeszów
35-310 Rzeszów
Poland
E-mail: jdziok@ur.edu.pl