TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 2, pp. 497-507, April 2014
DOI: 10.11650/tjm.18.2014.2794
This paper is available online at http://journal.taiwanmathsoc.org.tw

THE INTEGER PARTS OF A NONLINEAR FORM WITH MIXED POWERS 3 AND k

Baiyun Su and Weiping Li*

Abstract. Using the Davenport-Heilbronn circle method, we show that if $\lambda_1, \cdots, \lambda_5$ are positive real numbers, at least one of the ratios $\lambda_i/\lambda_j (1 \le i < j \le 5)$ is irrational, then, for arbitrary positive integer $k \ge 4$, the integer parts of $\lambda_1 x_1^3 + \lambda_2 x_2^3 + \lambda_3 x_3^3 + \lambda_4 x_4^3 + \lambda_5 x_5^k$ are prime infinitely often for natural numbers $x_1, \cdots, x_5$.

1. INTRODUCTION

The study of additive diophantine inequalities has been one of the guiding themes in diophantine approximation. Davenport and Heilbronn [1] showed that if $\lambda_1, \cdots, \lambda_s$ are nonzero real numbers, not all of the same sign, and not all in rational ratio, then for every $\varepsilon > 0$ the inequality

$$|\lambda_1 x_1^k + \lambda_2 x_2^k + \cdots + \lambda_s x_s^k| < \varepsilon$$

has infinitely many solutions in natural numbers x_j provided that $s \ge 2^k + 1$. Brüdern [2] proved that if $\lambda_1, \cdots, \lambda_8$ are nonzero real numbers such that at least one of the ratios $\lambda_i/\lambda_j (1 \le i < j \le 8)$ is irrational, and μ is real number. Then, for some $\sigma > 0$ the inequality

$$|\lambda_1 x_1^3 + \cdots + \lambda_6 x_6^3 + \lambda_7 x_7^4 + \lambda_8 x_8^4 + \mu| < (\max |x_i|)^{-\sigma}$$

has infinitely many solutions in positive integers $x_1, \cdots, x_8$.

In 2010, Brüdern, Kawada and Wooley [3] proved that if $\lambda_1, \cdots, \lambda_s$ are positive real numbers, λ_1/λ_2 is irrational, all Dirichlet L-functions satisfy the Riemann Hypothesis, $s \ge \frac{8}{3}k + 2$. Then, the integer parts of $\lambda_1 x_1^k + \lambda_2 x_2^k + \cdots + \lambda_s x_s^k$ are prime infinitely often for natural numbers x_j.

Received December 28, 2012, accepted September 13, 2013.
Communicated by Wen-Ching Winnie Li.
2010 *Mathematics Subject Classification*: 11D75, 11P55.
Key words and phrases: Davenport-Heilbronn circle method, Integer variables, Diophantine approximation.
*Corresponding author.

In present paper, using the Davenport-Heilbronn circle method we consider above problems with mixed powers, and our main result is as follows.

Theorem 1.1. *Let* $\lambda_1, \cdots, \lambda_5$ *be positive real numbers, at least one of the ratios* $\lambda_i/\lambda_j (1 \le i < j \le 5)$ *is irrational. Then, for arbitrary positive integer* $k \ge 4$*, the integer parts of* $\lambda_1 x_1^3 + \lambda_2 x_2^3 + \lambda_3 x_3^3 + \lambda_4 x_4^3 + \lambda_5 x_5^k$ *are prime infinitely often for natural numbers* $x_1, \cdots, x_5$.

Note that the above theorem do not assume the Riemann Hypothesis. No doubt, these results will be optimum if the k-th power can be removed. But, at present, using Davenport and Heilbronn method they are out of reach.

2. Notation

Throughout this paper p, with or without subscripts, always denotes a prime number, and x_j denotes a natural number. We denote by δ a sufficiently small positive number, and by ε an arbitrarily small positive number. Constants both explicit and implicit, in Landau or Vinogradov symbols may depend on $\lambda_1, \cdots, \lambda_5$. We write $e(x) = \exp(2\pi i x)$. We use $[x]$ to denote the integer part of real variable x. We take X to be the basic parameter, a large real integer. Since at least one of the ratios $\lambda_i/\lambda_j (1 \le i < j \le 5)$ is irrational, we break into two cases to consider. Case 1: We assume that λ_1/λ_2 is irrational. Case 2: We may assume that one of $\lambda_i/\lambda_5 (i = 1, \cdots, 4)$ is irrational, and without loss of generality we assume λ_1/λ_5 is irrational.

In the case that λ_1/λ_2 is irrational, then there are infinitely many pairs of integers q, a with $|\lambda_1/\lambda_2 - a/q| \le q^{-2}, (a, q) = 1, q > 0$ and $a \ne 0$. q is large and is taken to be a convergent to the continued fraction expansion of λ_1/λ_2. We make the following definitions.

$$N \asymp X^3, L = \log N, [N^{1-14\delta}] = q, \tau = N^{-1+\delta},$$
$$Q = (|\lambda_1|^{-1} + |\lambda_2|^{-1}) N^{1-\delta}, P = N^{12\delta}, T = N^{\frac{1}{3}}.$$

For the case λ_1/λ_5 is irrational, we define a, q, N, X, τ, P, T as above, only in place of $Q = (|\lambda_1|^{-1} + |\lambda_5|^{-1}) N^{1-\delta}$. We note that the only difference of dealing with the two cases is in section 4.

Let ν is positive real number, we define

$$\begin{aligned} K_\nu(\alpha) &= \nu \left(\frac{\sin \pi\nu\alpha}{\pi\nu\alpha} \right)^2, \alpha \ne 0,\ K_\nu(0) = \nu, \\ f(\alpha) &= \sum_{1 \le x \le X} e(\alpha x^3),\ g(\alpha) = \sum_{1 \le x \le X^{\frac{3}{k}}} e(\alpha x^k),\ h(\alpha) = \sum_{p \le N} (\log p) e(\alpha p), \\ U(\alpha) &= \int_1^X e(\alpha x^3) dx,\ V(\alpha) = \int_1^{X^{\frac{3}{k}}} e(\alpha x^k) dx,\ T(\alpha) = \int_1^N e(\alpha x) dx. \end{aligned} \tag{2.1}$$

It follows from (2.1) that

$$K_\nu(\alpha) \ll \min(\nu, \nu^{-1}|\alpha|^{-2}), \tag{2.2}$$

$$\int_{-\infty}^{+\infty} e(\alpha y)K_\nu(\alpha)d\alpha = \max(0, 1-\nu^{-1}|y|). \tag{2.3}$$

From (2.3) it is clear that

$$\begin{aligned} J &=: \int_{-\infty}^{+\infty} \prod_{i=1}^{4} f(\lambda_i\alpha)g(\lambda_5\alpha)h(-\alpha)e(-\frac{1}{2}\alpha)K_{\frac{1}{2}}(\alpha)d\alpha \\ &\le \log N \sum_{\substack{|\lambda_1 x_1^3+\cdots+\lambda_4 x_4^3+\lambda_5 x_5^k-p-\frac{1}{2}|<\frac{1}{2} \\ 1\le x_1,\cdots,x_4\le X, 1\le x_5\le X^{3/k}, p\le N}} 1 \\ &=: (\log N)\mathcal{N}(X), \end{aligned}$$

thus

$$\mathcal{N}(X) \ge (\log N)^{-1}J.$$

Next, we shall split the range of infinite integration into three sections, traditional named the neighbourhood of the origin $\mathfrak{C} = \{\alpha \in \mathbb{R} : |\alpha| \le \tau\}$, the intermediate region $\mathfrak{D} = \{\alpha \in \mathbb{R} : \tau < |\alpha| \le P\}$, the trivial region $\mathfrak{c} = \{\alpha \in \mathbb{R} : |\alpha| > P\}$. If we can prove

$$J(\mathfrak{C}) \gg X^{4+\frac{3}{k}}, \quad J(\mathfrak{D}) = o(X^{4+\frac{3}{k}}), \quad J(\mathfrak{c}) = o(X^{4+\frac{3}{k}}),$$

then

$$J \gg X^{4+\frac{3}{k}}, \quad \mathcal{N}(X) \gg X^{4+\frac{3}{k}}L^{-1},$$

namely, under conditions of Theorem 1.1,

$$|\lambda_1 x_1^3 + \cdots + \lambda_4 x_4^3 + \lambda_5 x_5^k - p - \frac{1}{2}| < \frac{1}{2} \tag{2.4}$$

has infinitely many solutions in positive integers $x_1, \cdots, x_5$ and prime p. It is evident from (2.4) that

$$p < \lambda_1 x_1^3 + \cdots + \lambda_4 x_4^3 + \lambda_5 x_5^k < p + 1,$$

and hence

$$[\lambda_1 x_1^3 + \cdots + \lambda_4 x_4^3 + \lambda_5 x_5^k] = p.$$

Theorem 1.1 can be established.

3. The Neighbourhood of the Origin

Lemma 3.1. If $\alpha = a/q + \beta$, where $(a, q) = 1$, then

$$\sum_{1\le x\le N^{1/t}} e(\alpha x^t) = q^{-1}\sum_{m=1}^{q} e(am^t/q)\int_1^{N^{1/t}} e(\beta y^t)dy + O(q^{1/2+\varepsilon}(1+N|\beta|)).$$

Proof. This is Theorem 4.1 of Vaughan [4].

If $|\alpha| \in \mathfrak{C}$, by Lemma 3.1, taking $a=0, q=1$, then

$$f(\alpha) = U(\alpha) + O(X^{3\delta}), \quad g(\alpha) = V(\alpha) + O(X^{3\delta}).$$

Lemma 3.2. *Let $\rho = \beta + i\gamma$ be a typical zero of the Riemann zeta function, and write*

$$A(\alpha) = \sum_{|\gamma| \le T, \beta \ge \frac{2}{3}} \sum_{n \le N} n^{\rho-1} e(n\alpha), \quad B(\alpha) = O\left((1+|\alpha|N)N^{\frac{2}{3}}L^C\right),$$

then

$$\begin{aligned} h(\alpha) &= T(\alpha) - A(\alpha) + B(\alpha), \ \int_{-\frac{1}{2}}^{\frac{1}{2}} |A(\alpha)|^2 d\alpha \\ &\ll N \exp(-L^{\frac{1}{5}}), \ \int_{-\tau}^{\tau} |B(\alpha)|^2 d\alpha \ll N \exp(-L^{\frac{1}{5}}). \end{aligned}$$

Proof. These results can be seen in [5].

Lemma 3.3. *We have*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |U(\alpha)|^2 d\alpha \ll X^{-1}L^2, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} |V(\alpha)|^2 d\alpha \ll X^{\frac{6}{k}-3}L^2.$$

Proof. These results are from (5.16) of Vaughan [6].

Lemma 3.4. *We have*

$$\int_{\mathfrak{C}} |\prod_{i=1}^{4} f(\lambda_i\alpha)g(\lambda_5\alpha)h(-\alpha) - \prod_{i=1}^{4} U(\lambda_i\alpha)V(\lambda_5\alpha)T(-\alpha)|K_{\frac{1}{2}}(\alpha)d\alpha \ll X^{4+\frac{3}{k}}L^{-1}.$$

Proof. It is easily seen that

$$\begin{aligned} f(\lambda_i\alpha) &\ll X, \quad U(\lambda_i\alpha) \ll X, \ i=1,2,3,4, \\ g(\lambda_5\alpha) &\ll X^{\frac{3}{k}}, \quad V(\lambda_5\alpha) \ll X^{\frac{3}{k}}, \ h(-\alpha) \ll N, \quad T(-\alpha) \ll N, \end{aligned}$$

$$\begin{aligned} &\prod_{i=1}^{4} f(\lambda_i\alpha)g(\lambda_5\alpha)h(-\alpha) - \prod_{i=1}^{4} U(\lambda_i\alpha)V(\lambda_5\alpha)T(-\alpha) \\ &= (f(\lambda_1\alpha) - U(\lambda_1\alpha))f(\lambda_2\alpha)f(\lambda_3\alpha)f(\lambda_4\alpha)g(\lambda_5\alpha)h(-\alpha) \\ &\quad + U(\lambda_1\alpha)(f(\lambda_2\alpha) - U(\lambda_2\alpha))f(\lambda_3\alpha)f(\lambda_4\alpha)g(\lambda_5\alpha)h(-\alpha) \\ &\quad + \cdots + U(\lambda_1\alpha)U(\lambda_2\alpha)U(\lambda_3\alpha)U(\lambda_4\alpha)(g(\lambda_5\alpha) - V(\lambda_5\alpha))h(-\alpha) \\ &\quad + U(\lambda_1\alpha)U(\lambda_2\alpha)U(\lambda_3\alpha)U(\lambda_4\alpha)V(\lambda_5\alpha)(h(-\alpha) - T(-\alpha)). \end{aligned}$$

Then

$$\begin{aligned}
&\int_{\mathfrak{C}} |(f(\lambda_1\alpha) - U(\lambda_1\alpha))f(\lambda_2\alpha)f(\lambda_3\alpha)f(\lambda_4\alpha)g(\lambda_5\alpha)h(-\alpha)|K_{\frac{1}{2}}(\alpha)d\alpha \\
\ll\ & N^{-1+\delta}X^{3\delta}X^3X^{\frac{3}{k}}N \ll X^{3+\frac{3}{k}+6\delta}, \\
&\cdots \quad \cdots \\
&\int_{\mathfrak{C}} |(U(\lambda_1\alpha)U(\lambda_2\alpha)U(\lambda_3\alpha)U(\lambda_4\alpha)(g(\lambda_5\alpha) - V(\lambda_5\alpha))h(-\alpha)|K_{\frac{1}{2}}(\alpha)d\alpha \\
\ll\ & N^{-1+\delta}X^4X^{3\delta}N \ll X^{4+6\delta}, \\
&\int_{\mathfrak{C}} |U(\lambda_1\alpha)U(\lambda_2\alpha)U(\lambda_3\alpha)U(\lambda_4\alpha)V(\lambda_5\alpha)(h(-\alpha) - T(-\alpha))|K_{\frac{1}{2}}(\alpha)d\alpha \\
\ll\ & X^4\left(\int_{\mathfrak{C}} |V(\lambda_5\alpha)|^2K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{2}}\left(\int_{\mathfrak{C}} |B(-\alpha) - A(-\alpha)|^2K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{2}} \\
\ll\ & X^4\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |V(\lambda_5\alpha)|^2d\alpha\right)^{\frac{1}{2}}\left(\int_{\mathfrak{C}} |B(\alpha)|^2d\alpha + \int_{-\frac{1}{2}}^{\frac{1}{2}} |A(\alpha)|^2d\alpha\right)^{\frac{1}{2}} \\
\ll\ & X^4(X^{\frac{6}{k}-3}L^2)^{\frac{1}{2}}(N\exp(-L^{\frac{1}{5}}))^{\frac{1}{2}} \\
\ll\ & X^{4+\frac{3}{k}}L^{-1}.
\end{aligned}$$

The proof of Lemma 3.4 is completed.

Lemma 3.5. *We have*

$$\int_{|\alpha|>N^{-1+\delta}} |\prod_{i=1}^{4} U(\lambda_i\alpha)V(\lambda_5\alpha)T(-\alpha)|K_{\frac{1}{2}}(\alpha)d\alpha \ll X^{(4+\frac{3}{k})(1-\delta)}.$$

Proof. It follows from Vaughan [4] that for $\alpha \neq 0$,

$$U(\lambda_i\alpha) \ll |\alpha|^{-\frac{1}{3}},\ i = 1, 2, 3, 4, \quad V(\lambda_5\alpha) \ll |\alpha|^{-\frac{1}{k}}, \quad T(-\alpha) \ll |\alpha|^{-1}.$$

Thus

$$\begin{aligned}
&\int_{|\alpha|>N^{-1+\delta}} |\prod_{i=1}^{4} U(\lambda_i\alpha)V(\lambda_5\alpha)T(-\alpha)|K_{\frac{1}{2}}(\alpha)d\alpha \\
\ll &\int_{|\alpha|>N^{-1+\delta}} |\alpha|^{-\frac{7}{3}-\frac{1}{k}}d\alpha \ll X^{(4+\frac{3}{k})(1-\delta)}.
\end{aligned}$$

Lemma 3.6. We have

$$\int_{-\infty}^{+\infty} \prod_{i=1}^{4} U(\lambda_i\alpha)V(\lambda_5\alpha)T(-\alpha)e(-\frac{1}{2}\alpha)K_{\frac{1}{2}}(\alpha)d\alpha \gg X^{4+\frac{3}{k}}.$$

Proof. From (2.3), one has

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\prod_{i=1}^{4}U(\lambda_i\alpha)V(\lambda_5\alpha)T(-\alpha)e(-\frac{1}{2}\alpha)K_{\frac{1}{2}}(\alpha)d\alpha\\
&=\int_1^X\cdots\int_1^X\int_1^{X^{\frac{3}{k}}}\int_1^N\int_{-\infty}^{+\infty}e(\alpha(\sum_{i=1}^{4}\lambda_i x_i^3+\lambda_5 x_5^k-x-\frac{1}{2}))\\
&\quad\cdot K_{\frac{1}{2}}(\alpha)d\alpha dx dx_5\cdots dx_1\\
&=\frac{1}{81k}\int_1^{X^3}\cdots\int_1^{X^3}\int_1^N\int_{-\infty}^{+\infty}x_1^{-\frac{2}{3}}\cdots x_4^{-\frac{2}{3}}x_5^{\frac{1}{k}-1}e(\alpha(\sum_{j=1}^{5}\lambda_j x_j-x-\frac{1}{2}))\\
&\quad\cdot K_{\frac{1}{2}}(\alpha)d\alpha dx dx_5\cdots dx_1\\
&=\frac{1}{81k}\int_1^{X^3}\cdots\int_1^{X^3}\int_1^N x_1^{-\frac{2}{3}}\cdots x_4^{-\frac{2}{3}}x_5^{\frac{1}{k}-1}\\
&\quad\cdot\max(0,\frac{1}{2}-|\sum_{j=1}^{5}\lambda_j x_j-x-\frac{1}{2}|)dx dx_5\cdots dx_1.
\end{aligned}\tag{3.1}
$$

Let $|\lambda_1 x_1+\cdots+\lambda_5 x_5-x-\frac{1}{2}|\le\frac{1}{4}$, then

$$\lambda_1 x_1+\cdots+\lambda_5 x_5-\frac{3}{4}\le x\le\lambda_1 x_1+\cdots+\lambda_5 x_5-\frac{1}{4}.$$

Based on

$$\lambda_1 x_1+\cdots+\lambda_5 x_5-\frac{3}{4}>1,\lambda_1 x_1+\cdots+\lambda_5 x_5-\frac{1}{4}<N,$$

one may take

$$\lambda_j X^3(8\sum_{i=1}^{5}\lambda_i)^{-1}\le x_j\le\lambda_j X^3(4\sum_{i=1}^{5}\lambda_i)^{-1},\quad j=1,\cdots,5,$$

hence (3.1)

$$\ge\frac{1}{648k}\prod_{j=1}^{5}\lambda_j(8\sum_{i=1}^{5}\lambda_i)^{-5}X^{4+\frac{3}{k}}.$$

This completes the proof of Lemma 3.6.

From Lemma 3.4-3.6, we therefore conclude that

$$J(\mathfrak{C})\gg X^{4+\frac{3}{k}}.$$

4. The Intermediate Region

Lemma 4.1. *We have*

$$\int_{-\infty}^{+\infty}|f(\lambda_i\alpha)|^8K_{\frac{1}{2}}(\alpha)d\alpha\ll X^{5+\varepsilon},\quad i=1,\cdots,4,\tag{4.1}$$

$$\int_{-\infty}^{+\infty} |g(\lambda_5\alpha)|^{2^k} K_{\frac{1}{2}}(\alpha)d\alpha \ll (X^{\frac{3}{k}})^{2^k-k+\varepsilon}, \tag{4.2}$$

$$\int_{-\infty}^{+\infty} |h(-\alpha)|^{2} K_{\frac{1}{2}}(\alpha)d\alpha \ll NL. \tag{4.3}$$

Proof. By (2.2) and Hua's inequality, for $i = 1, \cdots, 4$, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |f(\lambda_i\alpha)|^8 K_{\frac{1}{2}}(\alpha)d\alpha \\
\ll& \sum_{m=-\infty}^{+\infty} \int_m^{m+1} |f(\lambda_i\alpha)|^8 K_{\frac{1}{2}}(\alpha)d\alpha \\
\ll& \sum_{m=0}^{1} \int_m^{m+1} |f(\lambda_i\alpha)|^8 d\alpha + \sum_{m=2}^{+\infty} m^{-2} \int_m^{m+1} |f(\lambda_i\alpha)|^8 d\alpha \\
\ll& X^{5+\varepsilon} + X^{5+\varepsilon} \sum_{m=2}^{+\infty} m^{-2} \\
\ll& X^{5+\varepsilon}.
\end{aligned}$$

The proofs of (4.2) and (4.3) are similar to (4.1).

Lemma 4.2. *If $\alpha = a/q + \beta$, where $(a, q) = 1$ and $1 \le q \le X^{1-\delta}, \beta \ll q^{-1}X^{1-k-\delta}$, then*

$$\sum_{X<x<2X} e(\alpha x^k) \ll q^{-1/k} X.$$

Proof. This is Lemma 4 of Davenport and Roth [7].

Lemma 4.3. *Suppose thatλ_1/λ_2 is irrational. For every real number $\alpha \in \mathfrak{D}$, let $W(\alpha) = \min(|f(\lambda_1\alpha)|, |f(\lambda_2\alpha)|)$, then*

$$W(\alpha) \ll X^{1-12\delta+\varepsilon}.$$

Proof. For $\alpha \in \mathfrak{D}$ and $j = 1, 2$, we choose a_j, q_j such that

$$|\lambda_j\alpha - a_j/q_j| \le q_j^{-1}Q^{-1} \tag{4.4}$$

with $(a_j, q_j) = 1$ and $1 \le q_j \le Q$.

We firstly note that $a_1a_2 \ne 0$. For if $a_1 = 0$ or $a_2 = 0$, then $|\lambda_j\alpha| \le Q^{-1}$, and $|\alpha| \le |\lambda_j|^{-1}Q^{-1} < \tau, j = 1$ or 2, this contradicts $\tau < |\alpha| \le P$. Secondly, if $q_1, q_2 \le P$, then

$$|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2| \le |\frac{a_2/q_2}{\lambda_2\alpha}q_1q_2(\lambda_1\alpha - \frac{a_1}{q_1})| + |\frac{a_1/q_1}{\lambda_2\alpha}q_1q_2(\lambda_2\alpha - \frac{a_2}{q_2})| \ll PQ^{-1} < \frac{1}{2q}.$$

We recall that q was chosen as the denominator of a convergent to the continued fraction for λ_1/λ_2. Thus, by Legendre's law of best approximation, we have $|q'\frac{\lambda_1}{\lambda_2} - a'| > \frac{1}{2q}$ for all integers a', q' with $1 \le q' < q$, thus $|a_2q_1| \ge q = [N^{1-14\delta}]$. However, from (4.4) we have $|a_2q_1| \ll q_1q_2P \ll N^{36\delta}$, this is a contradiction. We have thus established that for at least one $j, P < q_j \ll Q$. Hence, by Lemma 4.2 and with $k = 3$, gives the desired inequality for $W(\alpha)$.

If one of $\lambda_i/\lambda_5 (i = 1, \cdots, 4)$ is irrational, without loss of generality we may assume that λ_1/λ_5 is irrational.

Lemma 4.4. *Suppose that*λ_1/λ_5 *is irrational. For every real number* $\alpha \in \mathfrak{D}$*, let* $Z(\alpha) = \min(|f(\lambda_1\alpha)|^{\frac{3}{k}}, |g(\lambda_5\alpha)|)$*, then*

$$Z(\alpha) \ll X^{\frac{3}{k}-\frac{36\delta}{k}+\varepsilon}.$$

Proof. For $\alpha \in \mathfrak{D}$ and $j = 1, 5$, we choose a_j, q_j such that $|\lambda_j\alpha - a_j/q_j| \le q_j^{-1}Q^{-1}$ with $(a_j, q_j) = 1$ and $1 \le q_j \le Q$.

From Lemma 4.3, we know that at least one $j, P < q_j \ll Q$ for $\alpha \in \mathfrak{D}$ and λ_1/λ_5 irrational. Thus, by Lemma 4.2 gives the desired inequality for $Z(\alpha)$.

Lemma 4.5. *We have*

$$\int_{\mathfrak{D}} \prod_{i=1}^{4} f(\lambda_i\alpha)g(\lambda_5\alpha)h(-\alpha)e(-\frac{1}{2}\alpha)K_{\frac{1}{2}}(\alpha)d\alpha \ll X^{4+\frac{3}{k}-\frac{96\delta}{2^k}+\varepsilon}.$$

Proof.

Case 1. If λ_1/λ_2 is irrational, by Lemma 4.1, 4.3 and Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathfrak{D}} \prod_{i=1}^{4} f(\lambda_i\alpha)g(\lambda_5\alpha)h(-\alpha)e(-\frac{1}{2}\alpha)K_{\frac{1}{2}}(\alpha)d\alpha \\
&\ll \max_{\alpha\in\mathfrak{D}} |W(\alpha)|^{\frac{8}{2^k}} \int_{\mathfrak{D}} |f(\lambda_1\alpha)|^{1-\frac{8}{2^k}} \prod_{i=2}^{4} |f(\lambda_i\alpha)||g(\lambda_5\alpha)h(-\alpha)|K_{\frac{1}{2}}(\alpha)d\alpha \\
&\quad + \max_{\alpha\in\mathfrak{D}} |W(\alpha)|^{\frac{8}{2^k}} \int_{\mathfrak{D}} |f(\lambda_2\alpha)|^{1-\frac{8}{2^k}} \prod_{\substack{i=1\\ i\ne 2}}^{4} |f(\lambda_i\alpha)||g(\lambda_5\alpha)h(-\alpha)|K_{\frac{1}{2}}(\alpha)d\alpha \\
&\ll (X^{1-12\delta+\varepsilon})^{\frac{8}{2^k}} \left(\int_{-\infty}^{+\infty} |f(\lambda_1\alpha)|^8 K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{8}-\frac{1}{2^k}} \prod_{i=2}^{4} \left(\int_{-\infty}^{+\infty} |f(\lambda_i\alpha)|^8 K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{8}} \\
&\quad \cdot \left(\int_{-\infty}^{+\infty} |g(\lambda_5\alpha)|^{2^k} K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{2^k}} \left(\int_{-\infty}^{+\infty} |h(-\alpha)|^2 K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{2}}
\end{aligned}$$

$$+(X^{1-12\delta+\varepsilon})^{\frac{8}{2^k}}\left(\int_{-\infty}^{+\infty}|f(\lambda_2\alpha)|^8K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{8}-\frac{1}{2^k}}\prod_{\substack{i=1\\i\neq 2}}^{4}\left(\int_{-\infty}^{+\infty}|f(\lambda_i\alpha)|^8K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{8}}$$

$$\cdot\left(\int_{-\infty}^{+\infty}|g(\lambda_5\alpha)|^{2^k}K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{2^k}}\left(\int_{-\infty}^{+\infty}|h(-\alpha)|^2K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{2}}$$

$$\ll(X^{1-12\delta+\varepsilon})^{\frac{8}{2^k}}(X^{5+\varepsilon})^{\frac{1}{2}-\frac{1}{2^k}}((X^{\frac{3}{k}})^{2^k-k+\varepsilon})^{\frac{1}{2^k}}(NL)^{\frac{1}{2}}$$

$$\ll X^{4+\frac{3}{k}-\frac{96\delta}{2^k}+\varepsilon}.$$

Case 2. If λ_1/λ_5 is irrational, by Lemma 4.1, 4.4 and Hӧlder's inequality, we have

$$\int_{\mathfrak{D}}\prod_{i=1}^{4}f(\lambda_i\alpha)g(\lambda_5\alpha)h(-\alpha)e(-\frac{1}{2}\alpha)K_{\frac{1}{2}}(\alpha)d\alpha$$

$$\ll\max_{\alpha\in\mathfrak{D}}|Z(\alpha)|\int_{\mathfrak{D}}\prod_{i=1}^{4}|f(\lambda_i\alpha)|h(-\alpha)|K_{\frac{1}{2}}(\alpha)d\alpha$$

$$+\max_{\alpha\in\mathfrak{D}}|Z(\alpha)|^{\frac{8k}{2^k3}}\int_{\mathfrak{D}}|f(\lambda_1\alpha)|^{1-\frac{8}{2^k}}\prod_{i=2}^{4}|f(\lambda_i\alpha)||g(\lambda_5\alpha)h(-\alpha)|K_{\frac{1}{2}}(\alpha)d\alpha$$

$$\ll X^{\frac{3}{k}-\frac{36\delta}{k}+\varepsilon}\prod_{i=1}^{4}\left(\int_{-\infty}^{+\infty}|f(\lambda_i\alpha)|^8K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{8}}\left(\int_{-\infty}^{+\infty}|h(-\alpha)|^2K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{2}}$$

$$+(X^{\frac{3}{k}-\frac{36\delta}{k}+\varepsilon})^{\frac{8k}{2^k3}}\left(\int_{-\infty}^{+\infty}|f(\lambda_1\alpha)|^8K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{8}-\frac{1}{2^k}}\prod_{i=2}^{4}\left(\int_{-\infty}^{+\infty}|f(\lambda_i\alpha)|^8K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{8}}$$

$$\cdot\left(\int_{-\infty}^{+\infty}|g(\lambda_5\alpha)|^{2^k}K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{2^k}}\left(\int_{-\infty}^{+\infty}|h(-\alpha)|^2K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{2}}$$

$$\ll X^{\frac{3}{k}-\frac{36\delta}{k}+\varepsilon}(X^{5+\varepsilon})^{\frac{1}{2}}(NL)^{\frac{1}{2}}+(X^{\frac{3}{k}-\frac{36\delta}{k}+\varepsilon})^{\frac{8k}{2^k3}}(X^{5+\varepsilon})^{\frac{1}{2}-\frac{1}{2^k}}((X^{\frac{3}{k}})^{2^k-k+\varepsilon})^{\frac{1}{2^k}}(NL)^{\frac{1}{2}}$$

$$\ll X^{4+\frac{3}{k}-\frac{96\delta}{2^k}+\varepsilon}.$$

From Lemma 4.5, it follows that

$$J(\mathfrak{D})=o(X^{4+\frac{3}{k}})$$

regardless of irrational.

5. The Trivial Region

Lemma 5.1. *Let $F(\alpha)=\sum e(\alpha f(x_1,\cdots,x_m))$, where f is any real function and the summation is over any finite set of values of $x_1,\cdots,x_m$. Then, for any $A>4$,* *we have*

$$\int_{|\alpha|>A} |F(\alpha)|^2 K_\nu(\alpha)d\alpha \leq \frac{16}{A}\int_{-\infty}^{\infty} |F(\alpha)|^2 K_\nu(\alpha)d\alpha.$$

Proof. This is Lemma 2 of Davenport and Roth [7].

Lemma 5.2. *We have*

$$\int_{\mathfrak{c}} \prod_{i=1}^{4} f(\lambda_i\alpha)g(\lambda_5\alpha)h(-\alpha)e(-\frac{1}{2}\alpha)K_{\frac{1}{2}}(\alpha)d\alpha \ll X^{4+\frac{3}{k}-36\delta+\varepsilon}.$$

Proof. By Lemma 5.1, Schwarz's inequality, (4.1) and (4.3), we have

$$\begin{aligned}
&\int_{\mathfrak{c}} \prod_{i=1}^{4} f(\lambda_i\alpha)g(\lambda_5\alpha)h(-\alpha)e(-\frac{1}{2}\alpha)K_{\frac{1}{2}}(\alpha)d\alpha \\
\ll& \int_{\mathfrak{c}} |\prod_{i=1}^{4} f(\lambda_i\alpha)g(\lambda_5\alpha)h(-\alpha)|K_{\frac{1}{2}}(\alpha)d\alpha \\
\ll& \frac{1}{P}\int_{-\infty}^{+\infty} |\prod_{i=1}^{4} f(\lambda_i\alpha)g(\lambda_5\alpha)h(-\alpha)|K_{\frac{1}{2}}(\alpha)d\alpha \\
\ll& N^{-12\delta}\max|g(\lambda_5\alpha)|\prod_{i=1}^{4}\left(\int_{-\infty}^{+\infty} |f(\lambda_i\alpha)|^8 K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{8}} \\
&\cdot\left(\int_{-\infty}^{+\infty} |h(-\alpha)|^2 K_{\frac{1}{2}}(\alpha)d\alpha\right)^{\frac{1}{2}} \\
\ll& N^{-12\delta}X^{\frac{3}{k}}(X^{5+\varepsilon})^{\frac{1}{2}}(NL)^{\frac{1}{2}} \\
\ll& X^{4+\frac{3}{k}-36\delta+\varepsilon}.
\end{aligned}$$

We therefore find from Lemma 5.2 that

$$J(\mathfrak{c}) = o(X^{4+\frac{3}{k}}).$$

ACKNOWLEDGMENTS

We would like to express our thanks to the referee for detailed suggestions and corrections on the manuscript.

This work is supported by the National Natural Science Foundation of China (Grant No. 11071070, 11371122), the Natural Science Research of Henan Province Education Department (Grant No. 2011B110002), the Innovation Scientists and Technicians Troop Construction Projects of Henan Province (China).

REFERENCES

1. H. Davenport and H. Helbronn, On indefine quadratic forms in five variables, *J. London Math. Soc.*, **21** (1946), 185-193.

2. J. Brüdern, Additive diophantine inequalities with mixed powers, III, *J. Number Theory*, **37** (1991), 199-210.

3. J. Brüdern, K. Kawada and T. D. Wooley, Additive representation in thin sequences, VIII: Diophantine inequalities in review, *Series on Number Theory and Its Applications*, **6** (2010), 20-79.

4. R. C. Vaughan, *The Hardy-Littlewood Method*, 2nd ed., Cambridge Tracts in Mathematics, Vol. 125. Cambridge University Press, Cambridge, 1997.

5. R. C. Vaughan, Diophantine approximation by prime numbers I, *Proc. London Math. Soc.*, **28** (1974), 373-384.

6. R. C. Vaughan, Diophantine approximation by prime numbers, II. *Proc. London Math. Soc.*, **28** (1974), 385-401.

7. H. Davenport and K. F. Roth, The solubility of certain diophantine inequalities, *Mathematika*, **2** (1955), 81-96.

Baiyun Su and Weiping Li
Department of Mathematics and Information Science
Henan University of Economics and Law
Zhengzhou 450011
P. R. China
E-mail: wpliyh@163.com