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INVERSE PROBLEM FOR A CLASS OF DIRAC OPERATOR

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Abstract. In this paper, we consider a problem for the first order canonical Dirac differential equations system with piecewise continuous coefficient and spectral parameter dependent in boundary condition. The asymptotic behavior of eigenvalues, eigenfunctions and normalizing numbers of this system is investigated. The completeness theorem is proved. The spectral expansion formula with respect to eigenvector functions or equivalently Parseval equality is obtained. Weyl solution and Weyl function for the problem are constructed. Uniqueness theorem for inverse problem by the Weyl function and by the spectral data are proved.

1. INTRODUCTION

Let us consider canonical Dirac differential equation

$$BY' + \Omega(x)Y = \lambda\rho(x)Y, \quad 0 < x < \pi \tag{1.1}$$

with boundary conditions

$$\begin{aligned} &U(Y) := y_1(0) = 0, \\ &V(Y) := \lambda(b_1 y_2(\pi) + b_2 y_1(\pi)) + a_1 y_1(\pi) + a_2 y_2(\pi) = 0, \end{aligned} \tag{1.2}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad Y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

$p(x)$, $q(x)$ are real measurable functions, $p(x) \in L_2(0,\pi)$, $q(x) \in L_2(0,\pi)$, λ is a spectral parameter, a_1, a_2, b_1 and b_2 are real numbers,

$$\rho(x) = \begin{cases} 1, & 0 \le x \le a, \\ \alpha, & a < x \le \pi \end{cases}$$

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and $1 \neq \alpha > 0$. Let us define $k := a_2 b_2 - a_1 b_1 > 0$.

In the finite interval, the inverse problem for Dirac operator is widely developed, for example [2, 4, 6, 9, 10, 13, 15, 20]. Inverse problems in the periodic case for Dirac operator was analyzed in [16]. Using the Weyl-Titchmarsh m-function reconstruction of potential of the Dirac operator was examined in [3]. Extensive review of the literature on inverse problem in finite interval is discussed in [1]. Inverse problem according to different spectral data of Dirac operator and Sturm-Liouville operator was given in detail in [5, 7, 8, 12, 14]. In this study as different from other studies, it is used new integral representation (not operator transformation) for the solution of the equation (1.1) (detail in [11]).

In physical events, Dirac differential equations system is frequently encountered. Therefore the applications of this system is widespread, such as [17, 18, 19]. This paper is organized as follows: In section 2, the eigenvalues, the eigenfunctions and normalizing numbers of the problem (1.1), (1.2) are examined. In section 3, the resolvent operator is constructed, completeness theorem is proved and the expansion formula with respect to eigenfunctions is obtained by using contour integration. In section 4, Weyl solution and Weyl function for the problem are given, uniquness theorem for the inverse problem by the Weyl function and by the spectral data are proved by the following the way of [5].

Assume that

$$\int_0^{\pi} \|\Omega(x)\| \, dx < +\infty$$

is satisfied for Euclidean norm of matrix function $\Omega(x)$. Then the integral representation of the solution of equation (1.1) satisfying the initial condition $Y(0) = I$, (I is unite matrix) can be represented (see [11])

$$E(x,\lambda) = e^{-\lambda B \mu(x)} + \int_{-\mu(x)}^{\mu(x)} K(x,t) e^{-\lambda B t} dt,$$

where

$$\mu(x) = \begin{cases} x, & 0 \le x \le a, \\ \alpha x - \alpha a + a, & a < x \le \pi \end{cases}$$

and for a kernel $K(x,t)$ the inequality

$$\int_{-\mu(x)}^{\mu(x)} \|K(x,t)\| \, dt \le e^{\sigma(x)} - 1,$$

where

$$\sigma(x) = \int_0^x \|\Omega(s)\| \, ds$$

is hold. Moreover, if $\Omega(x)$ is differentiable, then $K(x,t)$ satisfy the following relations

$$BK_x + \Omega(x)K + \rho(x)\,K_t B = 0,$$

$$\rho(x)\left[BK(x,\mu(x))\right] = -\Omega(x)\,,$$

$$BK\left(x,-\mu(x)\right) = 0.$$

In the Hilbert space $H_\rho = L_{2,\rho}(0,\pi;\mathbb{C}^2)\oplus\mathbb{C}$ an inner product is defined by

$$\left\langle \hat{Y},\hat{Z}\right\rangle := \int_0^\pi \left\{ y_1(x)\overline{z_1(x)} + y_2(x)\overline{z_2(x)}\right\}\rho(x)dx + \frac{1}{k}y_3\overline{z_3}, \tag{1.3}$$

where

$$\hat{Y} = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3 \end{pmatrix} \in H_\rho, \qquad \hat{Z} = \begin{pmatrix} z_1(x) \\ z_2(x) \\ z_3 \end{pmatrix} \in H_\rho.$$

Let us define

$$L(\hat{Y}) := \begin{pmatrix} l(Y) \\ -a_1 y_1(\pi) - a_2 y_2(\pi) \end{pmatrix}$$

with

$$\begin{aligned} D(L) &= \Big\{ \hat{Y} \mid \hat{Y} = (y_1(x), y_2(x), y_3)^T \in H_\rho,\ y_1(x),\ y_2(x) \in AC[0,\pi], \\ y_3 &= b_1 y_2(\pi) + b_2 y_1(\pi),\ l(Y) \in L_{2,\rho}(0,\pi;\mathbb{C}^2),\ y_1(0)=0 \Big\} \end{aligned}$$

where

$$l(Y) = \frac{1}{\rho(x)} \begin{pmatrix} y_2' + p(x)y_1 + q(x)y_2 \\ -y_1' + q(x)y_1 - p(x)y_2 \end{pmatrix}.$$

The boundary value problem (1.1), (1.2) is equivalent to the equation $L\hat{Y} = \lambda\hat{Y}$.

Lemma 1. *The vector-valued eigenfunctions $Y(x,\lambda_1)$ and $Z(x,\lambda_2)$ corresponding different eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal.*

Proof. Since $Y(x,\lambda_1)$ and $Z(x,\lambda_2)$ are eigenfunctions of the problem (1.1), (1.2), we get

$$\begin{aligned} y_2'(x,\lambda_1) + p(x)y_1(x,\lambda_1) + q(x)y_2(x,\lambda_1) &= \lambda_1\rho(x)y_1(x,\lambda_1), \\ -y_1'(x,\lambda_1) + q(x)y_1(x,\lambda_1) - p(x)y_2(x,\lambda_1) &= \lambda_1\rho(x)y_2(x,\lambda_1), \\ z_2'(x,\lambda_2) + p(x)z_1(x,\lambda_2) + q(x)z_2(x,\lambda_2) &= \lambda_2\rho(x)z_1(x,\lambda_2), \\ -z_1'(x,\lambda_2) + q(x)z_1(x,\lambda_2) - p(x)z_2(x,\lambda_2) &= \lambda_2\rho(x)y_2(x,\lambda_2). \end{aligned}$$

Multplying these equalities by $z_1(x,\lambda_2)$, $z_2(x,\lambda_2)$, $-y_1(x,\lambda_1)$, $-y_2(x,\lambda_1)$ respectively and adding together

$$\frac{d}{dx}\left\{z_1(x,\lambda_2)y_2(x,\lambda_1)-y_1(x,\lambda_1)z_2(x,\lambda_2)\right\}$$
$$=(\lambda_1-\lambda_2)\,\rho(x)\left\{y_1(x,\lambda_1)z_1(x,\lambda_2)+y_2(x,\lambda_1)z_2(x,\lambda_2)\right\}$$

is found. Integrating it from 0 to π and using the conditions (1.2), we have

$$(\lambda_1-\lambda_2)\left\{\int_0^\pi\left[y_1(x,\lambda_1)z_1(x,\lambda_2)+y_2(x,\lambda_1)z_2(x,\lambda_2)\right]\rho(x)dx\right.$$
$$\left.+\frac{1}{k}\left[b_1y_2(\pi,\lambda_1)+b_2y_1(\pi,\lambda_1)\right]\left[b_1z_2(\pi,\lambda_2)+b_2z_1(\pi,\lambda_2)\right]\right\}=0.$$

Since $\lambda_1\neq\lambda_2$,

$$\int_0^\pi\left[y_1(x,\lambda_1)z_1(x,\lambda_2)+y_2(x,\lambda_1)z_2(x,\lambda_2)\right]\rho(x)dx+\frac{1}{k}y_3z_3=0. \qquad \blacksquare$$

Corollary 2. *The eigenvalues of the boundary value problem (1.1), (1.2) are real.*

Let $S(x,\lambda)$ and $\psi(x,\lambda)$ be solutions of (1.1), (1.2) boundary value problem satisfying the initial conditions

$$S(0,\lambda)=\begin{pmatrix}0\\-1\end{pmatrix}\quad and\quad \psi(\pi,\lambda)=\begin{pmatrix}\lambda b_1+a_2\\-\lambda b_2-a_1\end{pmatrix}.$$

Clearly,

$$\begin{aligned}&U(S):=S_1(0,\lambda)=0,\\&V(\psi):=\lambda\left(b_1\psi_2(\pi,\lambda)+b_2\psi_1(\pi,\lambda)\right)+a_1\psi_1(\pi,\lambda)+a_2\psi_2(\pi,\lambda)=0.\end{aligned}\tag{1.4}$$

Let $Y(x,\lambda)$ and $Z(x,\lambda)$ be vector solutions of the equations system (1.1). The expression

$$W\left[Y(x,\lambda),Z(x,\lambda)\right]=y_2(x,\lambda)z_1(x,\lambda)-y_1(x,\lambda)z_2(x,\lambda)$$

is called Wronskian of the vector solutions $Y(x,\lambda)$ and $Z(x,\lambda)$. Denote

$$\Delta(\lambda)=W[S(x,\lambda),\psi(x,\lambda)].\tag{1.5}$$

The function $\Delta(\lambda)$ is called characteristic function of the problem (1.1),(1.2) and substituting $x=0$ and $x=\pi$ into (1.5), we get

$$\Delta(\lambda)=V(S)=-U(\psi)\tag{1.6}$$

or

$$\Delta(\lambda)=\lambda\left(b_1S_2(\pi,\lambda)+b_2S_1(\pi,\lambda)\right)+a_1S_1(\pi,\lambda)+a_2S_2(\pi,\lambda)=-\psi_1(0,\lambda).$$

Lemma 3. *The zeros λ_n of characteristic function coincide with the eigenvalues of the boundary value problem (1.1), (1.2). The function $S(x, \lambda_n)$ and $\psi(x, \lambda_n)$ are eigenfunctions and there exist a sequence β_n such that*

$$\psi(x, \lambda_n) = \beta_n S(x, \lambda_n), \qquad \beta_n \neq 0.$$

Proof. Let λ_0 be a zero of $\Delta(\lambda)$. Then, because of (1.4)-(1.6), $\psi(x, \lambda_0) = \beta_0 S(x, \lambda_0)$ and the function $\psi(x, \lambda_0)$ and $S(x, \lambda_0)$ satisfy the boundary condition (1.2). Thus, λ_0 is eigenvalue and $\psi(x, \lambda_0)$, $S(x, \lambda_0)$ are corresponding eigenfunctions. On the other hand, let λ_0 be an eigenvalue of the problem (1.1), (1.2) and $Y_0(x) = \begin{pmatrix} y_1^0(x) \\ y_2^0(x) \end{pmatrix} \neq 0$ be a corresponding eigenfunction. Then $Y_0(x)$ satisfies the boundary condition (1.2). Without loss of generality, we put $y_1^0(\pi) = \lambda b_1 + a_2$. Then $y_2^0(\pi) = -\lambda b_2 - a_1$ and consequently $Y_0(x) \equiv \psi(x, \lambda_0)$. Hence from (1.6), $\Delta(\lambda_0) = 0$ and for each eigenvalue there exist only one eigenfunction. ∎

2. The Spectral Properties

Lemma 4. *The eigenvalues $\{\lambda_n\}_{n=-\infty}^{\infty}$ of the boundary value problem (1.1), (1.2) are in the form*

$$\lambda_n = \lambda_n^0 + \epsilon_n,$$

where $\lambda_n^0 = \left(n\pi + \arctan \frac{b_1}{b_2}\right) \frac{1}{\mu(\pi)}$ *and* $\{\epsilon_n\} \in l_2$.

Proof. Using $S(x, \lambda) = E(x, \lambda) \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, we obtain the integral representation of the solution $S(x, \lambda)$ in the form

$$S(x, \lambda) = \begin{pmatrix} sin\lambda\mu(x) \\ -\cos\lambda\mu(x) \end{pmatrix} + \int_0^{\mu(x)} A(x, t) \begin{pmatrix} sin\lambda t \\ -\cos\lambda t \end{pmatrix} dt, \tag{2.1}$$

where

$$\mu(x) = \begin{cases} x, & 0 \leq x \leq a, \\ \alpha x - \alpha a + a, & a < x \leq \pi, \end{cases}$$

$A_{ij}(x, .) \in L_2(0, \pi)$, $i, j = 1, 2$ and

$$A_{11}(x, t) = K_{11}(x, t) + K_{11}(x, -t), \quad A_{12}(x, t) = K_{12}(x, t) - K_{12}(x, -t),$$

$$A_{21}(x, t) = K_{21}(x, t) + K_{21}(x, -t), \quad A_{22}(x, t) = K_{22}(x, t) - K_{22}(x, -t).$$

From (2.1), it follows that

$$\begin{aligned}\Delta(\lambda) = \; & \lambda\left(b_2 \sin\lambda\mu(\pi) - b_1\cos\lambda\mu(\pi)\right) + a_1\sin\lambda\mu(\pi) - a_2\cos\lambda\mu(\pi) + \\ & +\lambda\int_0^{\mu(\pi)} [b_2A_{11}(\pi,t) + b_1A_{21}(\pi,t)]\sin\lambda t dt - \\ & -\lambda\int_0^{\mu(\pi)} [b_2A_{12}(\pi,t) + b_1A_{22}(\pi,t)]\cos\lambda t dt + \\ & +\int_0^{\mu(\pi)} [a_1A_{11}(\pi,t) + a_2A_{21}(\pi,t)]\sin\lambda t dt - \\ & -\int_0^{\mu(\pi)} [a_1A_{12}(\pi,t) + a_2A_{22}(\pi,t)]\cos\lambda t dt.\end{aligned}$$

Since the zeros of $\Delta(\lambda)$ are the eigenvalues,

$$\begin{aligned}b_2\sin\lambda\mu(\pi) \; & -b_1\cos\lambda\mu(\pi) + \frac{a_1\sin\lambda\mu(\pi)}{\lambda} - \frac{a_2\cos\lambda\mu(\pi)}{\lambda} \\ & +\int_0^{\mu(\pi)} [b_2A_{11}(\pi,t) + b_1A_{21}(\pi,t)]\sin\lambda t dt \\ & -\int_0^{\mu(\pi)} [b_2A_{12}(\pi,t) + b_1A_{22}(\pi,t)]\cos\lambda t dt \\ & +\frac{1}{\lambda}\int_0^{\mu(\pi)} [a_1A_{11}(\pi,t) + a_2A_{21}(\pi,t)]\sin\lambda t dt \\ & -\frac{1}{\lambda}\int_0^{\mu(\pi)} [a_1A_{12}(\pi,t) + a_2A_{22}(\pi,t)]\cos\lambda t dt = 0\end{aligned} \tag{2.2}$$

is valid. Denote by

$$b_2\sin\lambda\mu(\pi) - b_1\cos\lambda\mu(\pi) := \chi(\lambda),$$

$$\widetilde{\Delta}(\lambda) := \frac{\Delta(\lambda)}{\lambda}$$

and

$$G_\delta := \left\{\lambda : \left|\lambda - \left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{1}{\mu(\pi)}\right| \geq \delta, \quad n = 0, \pm 1 \pm 2...\right\},$$

where δ is a sufficiently small positive number. For $\lambda \in G_\delta$,

$$|\chi(\lambda)| \geq C_\delta exp\left(|Im\lambda|\,\mu(\pi)\right)$$

is valid, where C_δ is a positive number (see [5]). Taking into account the following contour

$$\Gamma_n := \left\{\lambda : |\lambda| = \left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{1}{\mu(\pi)} + \frac{\pi}{2\mu(\pi)}, \quad n = 0, \pm1, \pm2, ...\right\},$$

for sufficiently large values n, we get

$$\left|\widetilde{\Delta}(\lambda) - \chi(\lambda)\right| \le C_\delta exp\left(|Im\lambda|\,\mu(\pi)\right), \quad \lambda \in \Gamma_n.$$

Thus

$$\left|\widetilde{\Delta}(\lambda) - \chi(\lambda)\right| \le |\chi(\lambda)|\,.$$

Applying the Rouche theorem, it is obtained that the number of zeros of the function $\left\{\widetilde{\Delta}(\lambda) - \chi(\lambda)\right\} + \chi(\lambda) = \widetilde{\Delta}(\lambda)$ inside the contour Γ_n coincides with the number of zeros of function $\chi(\lambda)$. Moreover, using the Rouche theorem, there exist only one zero λ_n of the function $\widetilde{\Delta}(\lambda)$ in the circle

$$\gamma_n(\delta) = \left\{\lambda : \left|\lambda - \left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{1}{\mu(\pi)}\right| < \delta\right\}$$

is concluded. Hence, the eigenvalues of the boundary value problem (1.1),(1.2) are obtained as follows

$$\lambda_n = \left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{1}{\mu(\pi)} + \epsilon_n, \quad \lim_{n\to\pm\infty}\epsilon_n = 0. \tag{2.3}$$

Substituting (2.3) into (2.2), using $\chi(\lambda_n) = \dot{\chi}(\lambda_n^0)\epsilon_n + o(\epsilon_n)$ and the relations (see [14, p. 66]),

$$\int_0^{\mu(\pi)} [b_2A_{11}(\pi,t) + b_1A_{21}(\pi,t)]\sin\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{1}{\mu(\pi)} + \epsilon_n\right] tdt \in l_2,$$

$$\int_0^{\mu(\pi)} [b_2A_{12}(\pi,t) + b_1A_{22}(\pi,t)]\cos\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{1}{\mu(\pi)} + \epsilon_n\right] tdt \in l_2,$$

we have $\{\epsilon_n\} \in l_2$. ■

Lemma 5. *Eigenvector functions of boundary value problem* (*1.1*), (*1.2*) *can be expressed in the following form*

$$S(x,\lambda_n) = \begin{pmatrix} \sin\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{\mu(x)}{\mu(\pi)}\right] \\ -\cos\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{\mu(x)}{\mu(\pi)}\right] \end{pmatrix} + \begin{pmatrix} v_n(x) \\ z_n(x) \end{pmatrix}, \tag{2.4}$$

$v_n(x) \in l_2$ *and* $z_n(x) \in l_2$, *for all* $x \in [0,\pi]$.

Proof. Putting the eigenvalues (2.3) in the representation (2.1), we get

$$S_1(x,\lambda_n) = \sin\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{\mu(x)}{\mu(\pi)}\right] + v_n(x),$$

$$S_2(x,\lambda_n) = -\cos\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{\mu(x)}{\mu(\pi)}\right] + z_n(x),$$

where

$$\begin{aligned} v_n(x) &= \sin\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{\mu(x)}{\mu(\pi)}\right]\{\cos\epsilon_n\mu(x) - 1\} \\ &+ \cos\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{\mu(x)}{\mu(\pi)}\right]\sin\epsilon_n\mu(x) \\ &+ \int_0^x A_{11}(x,t)\sin\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{1}{\mu(\pi)} + \epsilon_n\right]tdt \\ &- \int_0^x A_{12}(x,t)\cos\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{1}{\mu(\pi)} + \epsilon_n\right]tdt, \end{aligned}$$

$$\begin{aligned} z_n(x) &= -\cos\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{\mu(x)}{\mu(\pi)}\right]\{\cos\epsilon_n\mu(x) - 1\} \\ &+ \sin\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{\mu(x)}{\mu(\pi)}\right]\sin\epsilon_n\mu(x) \\ &+ \int_0^x A_{21}(x,t)sin\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{1}{\mu(\pi)} + \epsilon_n\right]tdt \\ &- \int_0^x A_{22}(x,t)cos\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{1}{\mu(\pi)} + \epsilon_n\right]tdt. \end{aligned}$$

Since $\epsilon_n \in l_2$, then v_n and $z_n \in l_2$ are obtained. ■

Lemma 6. *Normalizing numbers of problem (1.1),(1.2) are as follows:*

$$\alpha_n = \mu(\pi) + \tau_n, \qquad \tau_n \in l_2. \tag{2.5}$$

Proof. For normalizing number of (1.1), (1.2), we have

$$\alpha_n = \int_0^\pi \left\{|S_1(x,\lambda_n)|^2 + |S_2(x,\lambda_n)|^2\right\}\rho(x)dx + \frac{1}{k}\left|b_1S_2(\pi,\lambda_n) + b_2S_1(\pi,\lambda_n)\right|^2.$$

From this equality,

$$\begin{aligned} \alpha_n &= \int_0^\pi \sin^2\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{\mu(x)}{\mu(\pi)}\right]\rho(x)dx \\ &+ \int_0^\pi \cos^2\left[\left(n\pi + \arctan\frac{b_1}{b_2}\right)\frac{\mu(x)}{\mu(\pi)}\right]\rho(x)dx + \tau_n \\ &= \mu(\pi) + \tau_n, \end{aligned}$$

where

$$\begin{aligned}\tau_n &= 2\int_0^{\pi}\left\{\sin\left[\left(n\pi+\arctan\frac{b_1}{b_2}\right)\frac{\mu(x)}{\mu(\pi)}\right]v_n(x)\right\}\rho(x)dx\\ &\quad -2\int_0^{\pi}\left\{\cos\left[\left(n\pi+\arctan\frac{b_1}{b_2}\right)\frac{\mu(x)}{\mu(\pi)}\right]z_n(x)\right\}\rho(x)dx\\ &\quad +\int_0^{\pi}\left[v_n^2(x)+z_n^2(x)\right]\rho(x)dx+\frac{1}{k}\left|b_1 z_n(\pi)+b_2 v_n(\pi)\right|^2.\end{aligned}$$

Furthermore, using $v_n(x)\in l_2$ and $z_n(x)\in l_2$, $\tau_n\in l_2$ is found. ∎

Lemma 7. *The eigenvalues of boundary value problem* (*1.1*), (*1.2*) *are simple.*

Proof. Since $S(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of this problem,

$$\begin{aligned}\psi_2'(x,\lambda)+p(x)\psi_1(x,\lambda)+q(x)\psi_2(x,\lambda)&=\lambda\rho(x)\psi_1(x,\lambda),\\ -\psi_1'(x,\lambda)+q(x)\psi_1(x,\lambda)-p(x)\psi_2(x,\lambda)&=\lambda\rho(x)\psi_2(x,\lambda),\\ S_2'(x,\lambda_n)+p(x)S_1(x,\lambda_n)+q(x)S_2(x,\lambda_n)&=\lambda_n\rho(x)S_1(x,\lambda_n),\\ -S_1'(x,\lambda_n)+q(x)S_1(x,\lambda_n)-p(x)S_2(x,\lambda_n)&=\lambda_n\rho(x)S_2(x,\lambda_n)\end{aligned}$$

are valid. Multiplying the equations by $S_1'(x,\lambda_n), S_2'(x,\lambda_n), -\psi_1'(x,\lambda), -\psi_2'(x,\lambda)$ and adding them together, we get

$$\begin{aligned}&\frac{d}{dx}\left\{S_1(x,\lambda_n)\psi_2(x,\lambda)-\psi_1(x,\lambda)S_2(x,\lambda_n)\right\}\\ &=(\lambda-\lambda_n)\,\rho(x)\left\{S_1(x,\lambda_n)\psi_1(x,\lambda)+S_2(x,\lambda_n)\psi_2(x,\lambda)\right\}.\end{aligned}$$

Integrating it from 0 to π and using the condition (1.2),

$$\begin{aligned}&\int_0^{\pi}\left\{S_1(x,\lambda_n)\psi_1(x,\lambda)+S_2(x,\lambda_n)\psi_2(x,\lambda)\right\}\rho(x)dx+\\ &+\frac{1}{k}\left[b_1S_2(\pi,\lambda_n)+b_2S_1(\pi,\lambda_n)\right]\left[b_1\psi_2(\pi,\lambda)+b_2\psi_1(\pi,\lambda)\right]=\frac{\Delta(\lambda)}{\lambda-\lambda_n}\end{aligned}$$

is found. From Lemma 3, since $\psi(x,\lambda_n)=\beta_nS(x,\lambda_n)$, as $\lambda\to\lambda_n$, we obtain

$$\beta_n\alpha_n=\dot{\Delta}(\lambda_n),$$

where $\beta_n=-\psi_2(0,\lambda_n)$. Thus, it follows that $\dot{\Delta}(\lambda_n)\neq 0$. ∎

3. Spectral Expansion Formula

If λ is not a spectrum point of operator L, then the resolvent $R_\lambda=(L-\lambda I)^{-1}$ exists. Now we find this expression of the operator R_λ.

Lemma 8. *The resolvent R_λ is the integral operator with the kernel which has the following form*

$$R_\lambda(x,t) = -\frac{1}{\Delta(\lambda)} \begin{cases} \psi(x,\lambda)\widetilde{S}(t,\lambda), & t \le x, \\ S(x,\lambda)\widetilde{\psi}(t,\lambda), & t \ge x, \end{cases} \tag{3.1}$$

here $\widetilde{S}(t,\lambda)$ denotes the transposed vector function of $S(t,\lambda)$.

Proof. Let $F(x) = \begin{pmatrix} f(x) \\ f_3 \end{pmatrix} \in D(L)$ and $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$. To construct the resolvent operator of L, we need to solve the initial value problem

$$BY' + \Omega(x)Y = \lambda\rho(x)Y + \rho(x)f(x), \tag{3.2}$$

$$\begin{gathered} y_1(0) = 0, \\ \lambda(b_1 y_2(\pi) + b_2 y_1(\pi)) + a_1 y_1(\pi) + a_2 y_2(\pi) = -f_3. \end{gathered} \tag{3.3}$$

By applying the method of variation of parameters, we want to find the solution of problem (3.2), (3.3) which has a form

$$Y(x,\lambda) = c_1(x,\lambda)S(x,\lambda) + c_2(x,\lambda)\psi(x,\lambda), \tag{3.4}$$

where $S(x,\lambda)$, $\psi(x,\lambda)$ are solutions of homogeneous problem. Then we get the equations system

$$\begin{gathered} c_1'(x,\lambda)\widetilde{\psi}(x,\lambda)BS(x,\lambda) = \widetilde{\psi}(x,\lambda)f(x)\rho(x), \\ c_2'(x,\lambda)\widetilde{S}(x,\lambda)B\psi(x,\lambda) = \widetilde{S}(x,\lambda)f(x)\rho(x). \end{gathered} \tag{3.5}$$

Using the system (3.5), we get

$$c_1(x,\lambda) = c_1(\pi,\lambda) - \frac{1}{\Delta(\lambda)}\int_x^\pi \widetilde{\psi}(t,\lambda)f(t)\rho(t)dt, \tag{3.6}$$

$$c_2(x,\lambda) = c_2(0,\lambda) - \frac{1}{\Delta(\lambda)}\int_0^x \widetilde{S}(t,\lambda)f(t)\rho(t)dt. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.4), we obtain

$$Y(x,\lambda) = \int_0^\pi R_\lambda(x,t)f(t)\rho(t)dt + c_2(0,\lambda)\psi(x,\lambda) + c_1(\pi,\lambda)S(x,\lambda),$$

where

$$R_\lambda(x,t) = -\frac{1}{\Delta(\lambda)} \begin{cases} \psi(x,\lambda)\widetilde{S}(t,\lambda), & t \le x, \\ S(x,\lambda)\widetilde{\psi}(t,\lambda), & t \ge x. \end{cases}$$

Taking the condition (3.3), we get

$$c_2(0,\lambda)=0, \qquad c_1(\pi,\lambda)=-\frac{f_3}{\Delta(\lambda)}.$$

Thus

$$Y(x,\lambda)=\int_0^\pi R_\lambda(x,t)f(t)\rho(t)dt-\frac{f_3}{\Delta(\lambda)}S(x,\lambda). \tag{3.8}$$ ■

Theorem 9. *The eigenvector functions* $\{S(x,\lambda_n)\}_{n=-\infty}^{\infty}$ *of boundary value problem (1.1), (1.2) form a complete system in* $L_{2,\rho}(0,\pi;\mathbb{C}^2)\oplus\mathbb{C}$.

Proof. According to Lemma 3

$$\psi(x,\lambda_n)=\frac{\dot{\Delta}(\lambda_n)}{\alpha_n}S(x,\lambda_n), \tag{3.9}$$

using (3.1) and (3.8) we get

$$\underset{\lambda=\lambda_n}{Res}\,Y(x,\lambda)=-\frac{1}{\alpha_n}S(x,\lambda_n)\int_0^\pi \widetilde{S}(t,\lambda_n)f(t)\rho(t)dt-\frac{f_3}{\dot{\Delta}(\lambda_n)}S(x,\lambda_n). \tag{3.10}$$

We assume that

$$\begin{aligned}&\langle F(x),S(x,\lambda_n)\rangle\\ &=\int_0^\pi \widetilde{S}(t,\lambda_n)f(t)\rho(t)dt+\frac{f_3}{k}\left(b_1S_2\left(\pi,\lambda\right)+b_2S_1\left(\pi,\lambda\right)\right)=0,\end{aligned} \tag{3.11}$$

where $F(x)=\begin{pmatrix} f(x)\\ f_3\end{pmatrix}\in L_{2,\rho}(0,\pi;\mathbb{C}^2)\oplus\mathbb{C}$. From (3.9), we have

$$S_1(\pi,\lambda_n)=\frac{\lambda b_1+a_2}{\beta_n} \quad and \quad S_2(\pi,\lambda_n)=\frac{-\lambda b_2-a_1}{\beta_n}. \tag{3.12}$$

The formula (3.10) can be rewritten as follows

$$\underset{\lambda=\lambda_n}{Res}\,Y(x,\lambda)=-\frac{1}{\alpha_n}S(x,\lambda_n)\left\{\int_0^\pi \widetilde{S}(t,\lambda_n)f(t)\rho(t)dt+\frac{f_3}{\beta_n}\right\}.$$

Using (3.11) and (3.12), we obtain $\underset{\lambda=\lambda_n}{Res}\,Y(x,\lambda)=0$. Consequently $Y(x,\lambda)$ is entire function with respect to λ for each fixed $x\in[0,\pi]$. Taking into account

$$\Delta(\lambda)\geq|\lambda|\,C_\delta\exp(|Im\lambda\mu(\pi)|)$$

and the following equalities being valid according to [14, Lemma 1.3.1]

$$\lim_{|\lambda|\to\infty}\max_{0\leq x\leq\pi}\exp(-|Im\lambda\mu(x)|)\left|\int_0^x \widetilde{S}(t,\lambda)f(t)\rho(t)dt\right|=0, \tag{3.13}$$

$$\lim_{|\lambda|\to\infty}\max_{0\le x\le\pi}\frac{1}{|\lambda|}\exp(-|Im\lambda\,(\mu(\pi)-\mu(x))|)\left|\int_x^\pi \widetilde{\psi}(t,\lambda)f(t)\rho(t)dt\right|=0, \tag{3.14}$$

we find

$$\lim_{|\lambda|\to\infty}\max_{0\le x\le\pi}|Y(x,\lambda)|=0.$$

Hence $Y(x,\lambda)\equiv 0$ is obtained. It follows from (3.2) and (3.3) that $F(x)=0$ a.e. on $(0,\pi)$. ■

Theorem 10. *Let $f(x)$ be an absolutely continuous vector function on $[0,\pi]$ and $f_1(0)=0,\ f(\pi)=\begin{pmatrix}\lambda b_1+a_2\\ -\lambda b_2-a_1\end{pmatrix}$. Then the expansion formula*

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n S(x,\lambda_n), \\ f_3 &= \sum_{n=-\infty}^{\infty} c_n\left(b_1 S_2(\pi,\lambda_n)+b_2 S_1(\pi,\lambda_n)\right) \end{aligned} \tag{3.15}$$

is valid, where

$$c_n=\frac{1}{\alpha_n}\left\langle f(x),S(x,\lambda_n)\right\rangle.$$

The series converges uniformly with respect to $x\in[0,\pi]$. Moreover, the Parseval equality holds:

$$\|f\|^2=\sum_{n=-\infty}^{\infty}\alpha_n\left|c_n\right|^2. \tag{3.16}$$

Proof. Since $S(x,\lambda)$ and $\psi(x,\lambda)$ are solution of the problem (1.1), (1.2),

$$\begin{aligned} Y(x,\lambda) = &-\frac{1}{\lambda\Delta(\lambda)}\psi(x,\lambda)\int_0^x\left\{-\frac{\partial}{\partial t}\widetilde{S}(t,\lambda)B+\widetilde{S}(t,\lambda)\Omega(t)\right\}f(t)dt \\ &-\frac{1}{\lambda\Delta(\lambda)}S(x,\lambda)\int_x^\pi\left\{-\frac{\partial}{\partial t}\widetilde{\psi}(t,\lambda)B+\widetilde{\psi}(t,\lambda)\Omega(t)\right\}f(t)dt \\ &-\frac{f_3}{\Delta(\lambda)}S(x,\lambda) \end{aligned}$$

can be written. Integrating by parts and using the expression Wronskian

$$Y(x,\lambda)=-\frac{1}{\lambda}f(x)-\frac{1}{\lambda}Z(x,\lambda)-\frac{f_3}{\Delta(\lambda)}S(x,\lambda) \tag{3.17}$$

is obtained, where

$$Z(x,\lambda)=\frac{1}{\Delta(\lambda)}\left\{\psi(x,\lambda)\int_0^x\widetilde{S}(t,\lambda)Bf'(t)dt+S(x,\lambda)\int_x^\pi\widetilde{\psi}(t,\lambda)Bf'(t)dt\right\}.$$

By applying (3.13) and (3.14), we have

$$\lim_{|\lambda|\to\infty} \max_{0\le x\le\pi} |Z(x,\lambda)| = 0, \qquad \lambda \in G_\delta. \tag{3.18}$$

Now, we integrate of $Y(x,\lambda)$ with respect to λ over the contour Γ_N with oriented counter clockwise as follows:

$$I_n(x) = \frac{1}{2\pi i}\oint_{\Gamma_N} Y(x,\lambda)d\lambda,$$

where

$$\Gamma_N = \left\{\lambda : |\lambda| = \left(N\pi + \arctan\frac{b_1}{b_2}\right)\frac{1}{\mu(\pi)} + \frac{\pi}{2\mu(\pi)}\right\},$$

N is suffeciently large naturel number. Using Residue theorem, we get

$$\begin{aligned} I_n(x) &= \sum_{n=-N}^{N} \underset{\lambda=\lambda_n}{Res}\, Y(x,\lambda) \\ &= -\sum_{n=-N}^{N}\frac{1}{\alpha_n}S(x,\lambda_n)\int_0^\pi \widetilde{S}(t,\lambda_n)f(t)\rho(t)dt - \sum_{n=-N}^{N}\frac{f_3}{\dot\Delta(\lambda_n)}S(x,\lambda_n). \end{aligned}$$

On the other hand, considering the equation (3.17)

$$f(x) = \sum_{n=-N}^{N} c_n S(x,\lambda_n) + \epsilon_N(x) \tag{3.19}$$

is found, where

$$\epsilon_N(x) = -\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{1}{\lambda}Z(x,\lambda)d\lambda$$

and

$$c_n = \frac{1}{\alpha_n}\int_0^\pi \widetilde{S}(t,\lambda_n)f(t)\rho(t)dt.$$

It follows from (3.18) that,

$$\lim_{|\lambda|\to\infty}\max_{0\le x\le\pi}|\epsilon_N(x)| = 0.$$

Thus, by going over in (3.19) to the limit as $N\to\infty$ the expansion formula (3.15) with respect to eigenfunction is obtained. Since the system of $\{S(x,\lambda_n)\}_{n=-\infty}^{\infty}$ is complete and orthogonal in $L_{2,\rho}(0,\pi;\mathbb{C}^2)\oplus\mathbb{C}$, Parseval equality (3.16) is valid. The extending the Parseval equation to arbitrary vector valued functions of class $L_{2,\rho}(0,\pi;\mathbb{C}^2)$ is done by the usual methods. ■

4. Weyl Solution, Weyl Function

Let the function $\Phi(x,\lambda)$ be the solutions of the system (1.1), satisfying the conditions

$$\Phi_1(0,\lambda)=1,$$
$$\lambda\left(b_1\Phi_2(\pi,\lambda)+b_2\Phi_1(\pi,\lambda)\right)+a_1\Phi_1(\pi,\lambda)+a_2\Phi_2(\pi,\lambda)=0.$$

The function $\Phi(x,\lambda)$ is called Weyl solution of the problem (1.1),(1.2). Denote by $C(x,\lambda)$ the solution of system (1.1), satisfying the initial condition $C(0,\lambda)=\begin{pmatrix}1\\0\end{pmatrix}$. The solution $\psi(x,\lambda)$ can be written the following form

$$\psi(x,\lambda)=-\psi_2(0,\lambda)S(x,\lambda)-\Delta(\lambda)C(x,\lambda). \tag{4.1}$$

We define that

$$M(\lambda)=\frac{\psi_2(0,\lambda)}{\Delta(\lambda)}. \tag{4.2}$$

It is obvious that

$$\Phi(x,\lambda)=C(x,\lambda)+M(\lambda)S(x,\lambda). \tag{4.3}$$

The function $M(\lambda)=-\Phi_2(0,\lambda)$ is called the Weyl function of the problem (1.1),(1.2). The Weyl solution and Weyl function are meromorphic functions having simple poles at points λ_n eigenvalues of problem (1.1), (1.2). It is obtained from (4.2) and (4.3) that

$$\Phi(x,\lambda)=-\frac{\psi(x,\lambda)}{\Delta(\lambda)}. \tag{4.4}$$

The solution $\psi(x,\lambda)$ has the representation as

$$\begin{aligned}&\psi(x,\lambda)\\&=\psi_0(x,\lambda)+\int_0^{\alpha(\pi-x)}\tilde{A}(x,t)\begin{pmatrix}(\lambda b_1+a_2)\cos\lambda t-(\lambda b_2+a_1)\sin\lambda t\\-(\lambda b_1+a_2)\sin\lambda t-(\lambda b_2+a_1)\cos\lambda t\end{pmatrix}dt,\end{aligned} \tag{4.5}$$

where

$$\psi_0(x,\lambda)=\begin{pmatrix}(\lambda b_1+a_2)\cos\lambda\alpha\,(\pi-x)-(\lambda b_2+a_1)\sin\lambda\alpha\,(\pi-x)\\-(\lambda b_1+a_2)\sin\lambda\alpha\,(\pi-x)-(\lambda b_2+a_1)\cos\lambda\alpha\,(\pi-x)\end{pmatrix}$$

and $\tilde{A}_{i,j}(x,.)\in L_2(0,\pi)$, $i,j=1,2$.

We define (1^0), (2^0) boundary value problem in the case of $\Omega(x) \equiv 0$ in the boundary value problem (1.1),(1.2). Thus the boundary value problem (1^0), (2^0) has the characteristic function

$$\Delta_0(\lambda) = \lambda\left(b_2 \sin\lambda\mu\left(\pi\right) - b_1 \cos\lambda\mu\left(\pi\right)\right) + a_1 \sin\lambda\mu\left(\pi\right) - a_2 \cos\lambda\mu\left(\pi\right),$$

Weyl function

$$M_0(\lambda) = -\frac{\psi_{0,2}(0,\lambda)}{\psi_{0,1}(0,\lambda)} = \frac{(\lambda b_1 + a_2)\sin\lambda\alpha\pi + (\lambda b_2 + a_1)\cos\lambda\alpha\pi}{(\lambda b_1 + a_2)\cos\lambda\alpha\pi - (\lambda b_2 + a_1)\sin\lambda\alpha\pi}, \tag{4.6}$$

normalizing numbers α_n^0 and eigenvalues λ_n^0.

Theorem 11. *The expression*

$$M(\lambda) = -\left[\frac{1}{\alpha_0(\lambda-\lambda_0)} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{1}{\alpha_n^0\lambda_n^0} + \frac{1}{\alpha_n(\lambda-\lambda_n)}\right)\right] \tag{4.7}$$

holds.

Proof. It follows from (4.5) that

$$\psi(0,\lambda) = \begin{pmatrix} \psi_{0,1}(0,\lambda) \\ \psi_{0,2}(0,\lambda) \end{pmatrix} + \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix},$$

where

$$\begin{aligned} f_1(\lambda) \;=\; & \int_0^{\mu(\pi)} \tilde{A_{11}}(0,t)\left[(\lambda b_1 + a_2)\cos\lambda t - (\lambda b_2 + a_1)\sin\lambda t\right]dt - \\ & - \int_0^{\mu(\pi)} \tilde{A_{12}}(0,t)\left[(\lambda b_1 + a_2)\sin\lambda t + (\lambda b_2 + a_1)\cos\lambda t\right]dt, \\ f_2(\lambda) \;=\; & \int_0^{\mu(\pi)} \tilde{A_{21}}(0,t)\left[(\lambda b_1 + a_2)\cos\lambda t - (\lambda b_2 + a_1)\sin\lambda t\right]dt - \\ & - \int_0^{\mu(\pi)} \tilde{A_{22}}(0,t)\left[(\lambda b_1 + a_2)\sin\lambda t + (\lambda b_2 + a_1)\cos\lambda t\right]dt. \end{aligned}$$

From (4.2) and (4.6), we have

$$M(\lambda) - M_0(\lambda) = M_0(\lambda)\frac{f_1}{\Delta(\lambda)} + \frac{f_2}{\Delta(\lambda)}.$$

Moreover, applying Lemma 1.3.1 in [14], we have

$$\lim_{|\lambda|\to\infty} e^{-|Im\lambda|\mu(\pi)}\left|f_i(\lambda)\right| = 0, \quad i = 1, 2, \quad \lambda \in G_\delta.$$

Thus using this expression and the estimate $|\Delta(\lambda)| > C_\delta\, |\lambda| \exp(|Im\lambda|\, \mu(\pi))$

$$\lim_{|\lambda|\to\infty} |M(\lambda) - M_0(\lambda)| = 0 \tag{4.8}$$

is obtained. The vector functions $S(x,\lambda_n)$ $\left(S_0(x,\lambda_n^0)\right)$ and $\psi(x,\lambda_n)$ $\left(\psi_0(x,\lambda_n^0)\right)$ are eigenvector functions of the boundary value problem $(1.1),(1.2),((1.1^0),\ (1.2^0))$. Accordingly there exists constants $\beta_n(\beta_n^0)$ such that

$$\psi(x,\lambda_n) = \beta_n S(x,\lambda_n) \ \ (\psi_0(x,\lambda_n^0) = \beta_n^0 S_0(x,\lambda_n^0)).$$

It follows from these equalities that

$$\beta_n = -\psi_2(0,\lambda_n) \ \ (\beta_n^0 = -\psi_{0,2}(0,\lambda_n)).$$

Moreover

$$\alpha_n\beta_n = \dot{\Delta}(\lambda_n) \ \ (\alpha_n^0\beta_n^0 = \dot{\Delta}_0(\lambda_n^0)).$$

Applying these relations, we find

$$\begin{aligned} \underset{\lambda=\lambda_n}{Res}\, M(\lambda) &= \frac{\psi_2(0,\lambda_n)}{\dot{\Delta}(\lambda_n)} = -\frac{1}{\alpha_n} \\ \underset{\lambda=\lambda_n}{Res}\, M_0(\lambda) &= \frac{\psi_{0,2}(0,\lambda_n^0)}{\dot{\Delta}_0(\lambda_n^0)} = -\frac{1}{\alpha_n^0}. \end{aligned} \tag{4.9}$$

We consider the following contour integral

$$I_N(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_N(\lambda)} \frac{M(\xi) - M_0(\xi)}{\xi - \lambda} d\xi, \quad \xi \in int\Gamma_N,$$

where

$$\Gamma_N = \left\{\lambda : |\lambda| = \left(N\pi + \arctan\frac{b_1}{b_2}\right)\frac{1}{\mu\,(\pi)} + \frac{\pi}{2\mu(\pi)}\right\}.$$

Using (4.8), we have

$$\lim_{N\to\infty} I_N(\lambda) = 0.$$

On the other hand, applying the residue calculus and the residues (4.9), we get

$$I_N(\lambda) = M(\lambda) - M_0(\lambda) + \sum_{\lambda_n \in int\Gamma_N}\left(\frac{1}{\alpha_n^0(\lambda_n^0 - \lambda)} - \frac{1}{\alpha_n(\lambda_n - \lambda)}\right).$$

Thus, as $N \to \infty$

$$M(\lambda) = M_0(\lambda) - \sum_{n=-\infty}^{\infty}\left(\frac{1}{\alpha_n(\lambda - \lambda_n)} - \frac{1}{\alpha_n^0(\lambda - \lambda_n^0)}\right) \tag{4.10}$$

is found. We can write for the function $M_0(\lambda)$ the following expansion

$$M_0(\lambda) = -\frac{1}{\lambda\mu(\pi)} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{\alpha_n^0(\lambda - \lambda_n^0)} + \frac{1}{\alpha_n^0 \lambda_n^0} \right).$$

Substituting the last formula into (4.10),

$$M(\lambda) = -\left[\frac{1}{\alpha_0(\lambda - \lambda_0)} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (\frac{1}{\alpha_n^0 \lambda_n^0} + \frac{1}{\alpha_n(\lambda - \lambda_n)}) \right]$$

is obtained. ■

Now we seek inverse problem of the reconstruction of the problem (1.1), (1.2) by Weyl function and spectral data $\{\lambda_n, \alpha_n\}_{n=-\infty}^{\infty}$. Using the Weyl function, uniquness theorem for the problem is proved. We assume that the problem $\tilde{L}$ with the potential $\tilde{\Omega}(x)$ has the same form with L.

Theorem 12. *If $M(\lambda) = \tilde{M}(\lambda)$, then $L = \tilde{L}$. Namely, the boundary value problem (1.1), (1.2) is uniquely determined by the Weyl function.*

Proof. We describe the matrix $P(x,\lambda) = [P_{ij}(x,\lambda)]_{i,j=1,2}$ with the formula

$$P(x,\lambda) \begin{pmatrix} \tilde{S}_1 & \tilde{\Phi}_1 \\ \tilde{S}_2 & \tilde{\Phi}_2 \end{pmatrix} = \begin{pmatrix} S_1 & \Phi_1 \\ S_2 & \Phi_2 \end{pmatrix}. \tag{4.11}$$

The Wronskian of the solutions

$$\tilde{S}(x,\lambda) = \begin{pmatrix} \tilde{S}_1(x,\lambda) \\ \tilde{S}_2(x,\lambda) \end{pmatrix}, \qquad \tilde{\Phi}(x,\lambda) = \begin{pmatrix} \tilde{\Phi}_1(x,\lambda) \\ \tilde{\Phi}_2(x,\lambda) \end{pmatrix}$$

is as follows

$$W[\tilde{S}(x,\lambda), \tilde{\Phi}(x,\lambda)] = \tilde{S}_1(x,\lambda)\tilde{\Phi}_2(x,\lambda) - \tilde{S}_2(x,\lambda)\tilde{\Phi}_1(x,\lambda) = 1. \tag{4.12}$$

Multiplying both sides of (4.11) from right by the matrix

$$\begin{pmatrix} \tilde{\Phi}_2(x,\lambda) & -\tilde{\Phi}_1(x,\lambda) \\ -\tilde{S}_2(x,\lambda) & \tilde{S}_1(x,\lambda) \end{pmatrix},$$

we get

$$\begin{aligned} P_{11}(x,\lambda) &= S_1(x,\lambda)\tilde{\Phi}_2(x,\lambda) - \Phi_1(x,\lambda)\tilde{S}_2(x,\lambda), \\ P_{12}(x,\lambda) &= \Phi_1(x,\lambda)\tilde{S}_1(x,\lambda) - S_1(x,\lambda)\tilde{\Phi}_1(x,\lambda), \\ P_{21}(x,\lambda) &= S_2(x,\lambda)\tilde{\Phi}_2(x,\lambda) - \Phi_2(x,\lambda)\tilde{S}_2(x,\lambda), \\ P_{22}(x,\lambda) &= \Phi_2(x,\lambda)\tilde{S}_1(x,\lambda) - S_2(x,\lambda)\tilde{\Phi}_1(x,\lambda) \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} S_1(x,\lambda) &= P_{11}(x,\lambda)\tilde{S}_1(x,\lambda) + P_{12}(x,\lambda)\tilde{S}_2(x,\lambda), \\ S_2(x,\lambda) &= P_{21}(x,\lambda)\tilde{S}_1(x,\lambda) + P_{22}(x,\lambda)\tilde{S}_2(x,\lambda), \\ \Phi_1(x,\lambda) &= P_{11}(x,\lambda)\tilde{\Phi}_1(x,\lambda) + P_{12}(x,\lambda)\tilde{\Phi}_2(x,\lambda), \\ \Phi_2(x,\lambda) &= P_{21}(x,\lambda)\tilde{\Phi}_1(x,\lambda) + P_{22}(x,\lambda)\tilde{\Phi}_2(x,\lambda). \end{aligned} \tag{4.14}$$

Taking into account (4.4), (4.12) and (4.13),

$$P_{11}(x,\lambda)-1 = \frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)}\left\{\tilde{S}_1(x,\lambda) - S_1(x,\lambda)\right\} + \tilde{S}_2(x,\lambda)\left\{\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\right\},$$

$$P_{12}(x,\lambda) = \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\left\{S_1(x,\lambda) - \tilde{S}_1(x,\lambda)\right\} + S_1(x,\lambda)\left\{\frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\right\},$$

$$P_{21}(x,\lambda) = \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\left\{\tilde{S}_2(x,\lambda) - S_2(x,\lambda)\right\} + S_2(x,\lambda)\left\{\frac{\psi_2(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)}\right\},$$

$$P_{22}(x,\lambda)-1 = \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\left\{S_2(x,\lambda) - \tilde{S}_2(x,\lambda)\right\} + \tilde{S}_1(x,\lambda)\left\{\frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\right\}$$

are found. Using $|\Delta(\lambda)| > C_\delta |\lambda|\, exp(|Im\lambda|\, \mu(\pi))$ and the expressions of solutions $S(x,\lambda)$, $\psi(x,\lambda)$, we obtain

$$\begin{aligned} &\lim_{\substack{|\lambda|\to\infty \\ \lambda\in G_\delta}} \max_{0\le x\le\pi} |P_{11}(x,\lambda)-1| = 0, \qquad \lim_{\substack{|\lambda|\to\infty \\ \lambda\in G_\delta}} \max_{0\le x\le\pi} |P_{12}(x,\lambda)| = 0, \\ &\lim_{\substack{|\lambda|\to\infty \\ \lambda\in G_\delta}} \max_{0\le x\le\pi} |P_{22}(x,\lambda)-1| = 0, \qquad \lim_{\substack{|\lambda|\to\infty \\ \lambda\in G_\delta}} \max_{0\le x\le\pi} |P_{21}(x,\lambda)| = 0. \end{aligned} \tag{4.15}$$

Substituting (4.3) into (4.13), we have

$$P_{11}(x,\lambda) = S_1(x,\lambda)\tilde{C}_2(x,\lambda) - C_1(x,\lambda)\tilde{S}_2(x,\lambda) + S_1(x,\lambda)\tilde{S}_2(x,\lambda)\left[\tilde{M}(\lambda) - M(\lambda)\right],$$

$$P_{12}(x,\lambda) = C_1(x,\lambda)\tilde{S}_1(x,\lambda) - S_1(x,\lambda)\tilde{C}_1(x,\lambda) + S_1(x,\lambda)\tilde{S}_1(x,\lambda)\left[M(\lambda) - \tilde{M}(\lambda)\right],$$

$$P_{21}(x,\lambda) = S_2(x,\lambda)\tilde{C}_2(x,\lambda) - C_2(x,\lambda)\tilde{S}_2(x,\lambda) + S_2(x,\lambda)\tilde{S}_2(x,\lambda)\left[\tilde{M}(\lambda) - M(\lambda)\right],$$

$$P_{22}(x,\lambda) = C_2(x,\lambda)\tilde{S}_1(x,\lambda) - S_2(x,\lambda)\tilde{C}_1(x,\lambda) + S_2(x,\lambda)\tilde{S}_1(x,\lambda)\left[M(\lambda) - \tilde{M}(\lambda)\right].$$

Hence, if $M(\lambda) \equiv \tilde{M}(\lambda)$, $P_{ij}(x,\lambda)_{i,j=1,2}$ are entire functions with respect to λ for every fixed x. Then from (4.15), we find

$$P_{11}(x,\lambda) \equiv 1, \quad P_{12}(x,\lambda) \equiv 0, \quad P_{21}(x,\lambda) \equiv 0, \quad P_{22}(x,\lambda) \equiv 1.$$

Substituting these identities into (4.14),

$$S_1(x,\lambda) \equiv \tilde{S}_1(x,\lambda), \quad S_2(x,\lambda) \equiv \tilde{S}_2(x,\lambda)$$
$$\Phi_1(x,\lambda) \equiv \tilde{\Phi}_1(x,\lambda), \quad \Phi_2(x,\lambda) \equiv \tilde{\Phi}_2(x,\lambda)$$

are obtained for all x and λ. Thus, $L \equiv \tilde{L}$. ∎

Theorem 13. *If $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$ for all $n \in Z$, $L = \tilde{L}$. That is, the problem (1.1), (1.2) is uniquelly determined by spectral data.*

Proof. Since $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$ for all $n \in Z$ and considering the formula (4.7), we have $M(\lambda) = \tilde{M}(\lambda)$. Using the previous theorem, $L = \tilde{L}$ is obtained. ∎

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