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# CLOSEDNESS OF SET OF EFFICIENT SOLUTIONS FOR GENERALIZED KY FAN INEQUALITY PROBLEMS

Xun-Hua Gong* and Xin-Min Yang

**Abstract.** In this paper, we discuss the closedness of set of efficient solutions for generalized Ky Fan inequality problems in topological vector spaces. We introduce a concept of section mapping of a bifunction. By using the lower semicontinuity of the section mapping, we present sufficient conditions for the closedness of set of efficient solutions to the generalized Ky Fan inequality problems. We give conditions to guarantee the lower semicontinuity of the section mapping. We give also an example to illustrate that the condition of the lower semicontinuity of the section mapping is essential for the closedness of set of efficient solutions for generalized Ky Fan inequality problems. As an application, we give results of closedness of set of efficient solutions for vector optimization problems and for Lipschitz vector variational inequalities.

## 1. Introduction

Generalized Ky Fan inequality problems is an extension of a well known Ky Fan Inequality. Generalized Ky Fan inequality problems have been intensively explored recently since they include many other problems as special cases, including vector variational inequality problems, vector optimization problems, vector Nash equilibrium problems, and vector complementary problems [1, 2, 3].

Many studies have focused on weakly efficient solutions to the generalized Ky Fan inequality problems [1]. In general, the approach relies on an assumption that the ordering cone has a nonempty interior, taking the advantage of the openness of the interior of the ordering cone. In many cases, the ordering cone has an empty interior. For example, for each $1 < p < +\infty$, the positive cone of the normed linear spaces $l^p$

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*Corresponding author.

and $L^p(\Omega)$ has an empty interior. The efficient solution is an important solution to the generalized Ky Fan inequality problems as well as to the vector optimization problems. The concept of the efficient solution does not require the condition that the ordering cone has an nonempty interior. Thus, to study the properties of the efficient solutions of the generalized Ky Fan inequality problems is interesting. The difficulty of such study mainly lies in the fact that we can't longer take the advantage of the openness of the interior of the ordering cone. So far, only a few papers have dealt with this problems. Fang and Huang [4] studied the existence of efficient solution for vector variational inequalities in Banach spaces. Gong and Yue [5] studied the existence of efficient solution for generalized Ky Fan inequality problems. Gong and Yao [6,7] studied the connectedness of the set of efficient solutions for generalized Ky Fan inequality problems and the lower semicontinuity of the efficient solution mapping for generalized Ky Fan inequality problems. By using the generalization of Ljusternik theorem, the open mapping theorem of convex process, and the convex sets separation theorem, Gong [8] gave the necessary conditions for the efficient solution to the constrained vector equilibrium problems without requiring that the ordering cone in the objective space has a nonempty interior and without requiring that the the convexity conditions.

One of the important problems of generalized Ky Fan inequality problems is to investigate the topological structure of the set of efficient solutions for generalized Ky Fan inequality problems. Algorithmically, some approaches for generating all or part of set of efficient solutions of the vector optimization problems require the set of efficient solutions to be closed (see [9]). Hence it is also important to study the closedness of set of efficient solutions of the generalized Ky Fan inequality problems.

Benson and Sun [9] have studied the closedness of the set of efficient solutions for the vector optimization problems in finite-dimensional space; Dong, Gong, Wang and Coladas [10] have studied the closedness of the set of efficient solutions for the vector optimization problems in infinite-dimensional space. Until now, there has been no study on the closedness of set of efficient solutions for generalized Ky Fan inequality problems.

In this paper, we study the closedness of set of efficient solutions for generalized Ky Fan inequality problems in topological vector space. In Section 3, we introduce a concept of section mapping of a bifunction. By using the lower semicontinuity of the section mapping, we then present sufficient conditions for the closedness of set of efficient solutions for generalized Ky Fan inequality problems. Then, we give conditions to guarantee the lower semicontinuity of section mapping, and an example to illustrate the condition that the section mapping is lower semicontinuous is essential for the closedness of set of efficient solutions of the generalized Ky Fan inequality problems. In Section 4 and Section 5, we use the methods of Section 3 to study the closedness of set of efficient solutions to the vector optimization problems and Lipschitz vector variational inequalities, respectively.

## 2. Preliminaries

Throughout this paper, let $X$ and $Y$ be a topological vector spaces, $C$ be a closed, convex and pointed cone in $Y$. Let $A$ be a nonempty subset of $X$ and $F : A \times A \to Y$ be a bifunction. We consider the generalized Ky Fan inequality problems(for short, GFIP): find $x \in A$, such that

$$F(x, y) \notin -P\backslash\{0\} \text{ for all } y \in A,$$

where $P$ is a cone in $Y$.

If we replace above $P$ by int$C \cup \{0\}$, or by $C$, then we can give the following two definitions.

**Definition 2.1.** If int$C \neq \emptyset$, a vector $x \in A$, satisfying

$$F(x, y) \notin -\text{int}C \ \text{ for all } y \in A,$$

is called a weakly efficient solution to the GFIP. The set of weakly efficient solutions to the GFIP is denoted by $V_W(A, F)$.

**Definition 2.2.** A vector $x \in A$, satisfying

$$F(x, y) \notin -C\backslash\{0\} \ \text{ for all } y \in A,$$

is called an efficient solution to the GFIP. The set of efficient solutions to the GFIP is denoted by $V(A, F)$.

A special case of GFIP is a vector optimization problem (for short, VOP) involving

$$F(x, y) = f(y) - f(x),\ x, y \in A,$$

where $f : A \to Y$ is a mapping.

**Definition 2.3.** If

$$F(x, y) = f(y) - f(x),\ x, y \in A,$$

and if $x \in A$ is an efficient solution to the GFIP, then $x \in A$ is called an efficient solution to the VOP. The set of efficient solutions to the VOP is denoted by $E(A, f)$.

**Definition 2.4.** ([11]) Let $G$ be a set-valued map from a topological space $W$ to another topological space $Q$. We say that $G : W \rightrightarrows Q$ is lower semicontinuous at $w_0 \in W$ if, for any $y_0 \in G(w_0)$ and any neighborhood $U(y_0)$ of $y_0$, there exists a neighborhood $U(w_0)$ of $w_0$ such that

$$G(w) \cap U(y_0) \neq \emptyset \quad \text{for all } w \in U(w_0).$$

$G$ is said to be lower semicontinuous on $W$ if it is lower semicontinuous at each $w \in W$. Moreover, when $W$ and $Q$ are metric spaces, $G$ is lower semicontinuous at

$w_0 \in W$ if and only if, for any $y_0 \in G(w_0)$ and any sequence $\{w_n\}$ with $w_n \to w_0$, there is a sequence $\{y_n\}$ with $y_n \in G(w_n)$ such that $y_n \to y_0$.

If $A$ is a nonempty subset of a topology space $X$, $M : A \rightrightarrows A$ is a set-valued mapping, we can define the lower semi continuity of $M$ on $A$ with respect to the topology induced on $A$ by the topology of $X$.

**Definition 2.5.** Let $A$ be a nonempty subset of a topology space $X$, $M : A \rightrightarrows A$ be a set-valued mapping. We say that $M$ is lower semicontinuous at $x_0 \in A$ if, for any $y_0 \in M(x_0)$ and any neighborhood $U(y_0)$ of $y_0$, there exists a neighborhood $U(x_0)$ of $x_0$ such that

$$M(x) \cap U(y_0) \cap A = M(x) \cap U(y_0) \neq \emptyset \quad \text{for all } x \in U(x_0) \cap A.$$

$M$ is said to be lower semicontinuous on $A$ if it is lower semicontinuous at each $x \in A$.

**Remark 2.1.** By Definition 2.4 and Definition 2.5, we can see that if $X$ is a metric space, $A$ is a nonempty subset of a $X$, then the set-valued mapping $M : A \rightrightarrows A$ is lower semicontinuous at $x_0 \in A$ if, for any $y_0 \in M(x_0)$ and any sequence $\{x_n\} \subset A$ with $x_n \to x_0$, there is a sequence $\{y_n\}$ with $y_n \in M(x_n)$ such that $y_n \to y_0$.

**Definition 2.6.** Let $X$ be a topological space, $Y$ be a topological vector space, and $A$ be a nonempty subset of $X$, $C$ be a closed, convex and pointed cone in $Y$. A mapping $g : A \to Y$ is called $C$-upper semicontinuous at $x_0 \in A$ if, for any neighborhood $U$ of 0 in $Y$, there is a neighborhood $U(x_0)$ of $x_0$ in $X$ such that

$$g(x) \in g(x_0) + U - C \quad \text{for all } x \in U(x_0) \cap A.$$

$g$ is said to be $C$-upper semicontinuous on $A$ if it is $C$-upper semicontinuous at each $x \in A$.

If $X$ is a topological vector space, we denote the neighborhood system of zero in $X$ by $\mathcal{N}(0)$.

## 3. Closedness of Efficient Solutions Set of GFIP

In this section, we discuss mainly the closedness of efficient solutions set to the generalized Ky Fan inequality problems.

By the openness of the ordering cone $C$, we can easily get the following result, we omit the proof.

**Theorem 3.1.** *Let $X$ be a topological space, $A$ be a nonempty and closed subset of $X$, $Y$ be a topological vector space, $C$ be a closed, convex and pointed cone in $Y$. Let $F : A \times A \to Y$ be a bifunction. If for each $y \in A$, $F(\cdot, y)$ is $C$- upper semicontinuous on $A$, then $V_W(A, F)$ is closed.*

However, the efficient solutions set of GFIP is not necessary closed under the conditions of Theorem 3.1 as we shall see in Example 3.2. We need a new method to prove the closedness of the set of efficient solutions of GFIP. Now we introduce the following concept.

**Definition 3.1.** Let $X$ be a topological space, $A$ be a nonempty subset of $X$, $Y$ be a topological vector space, $C$ be a closed, convex and pointed cone in $Y$. Let $F : A \times A \to Y$ be a bifunction. The set-valued mapping

$$M(x) := \{y \in A : F(x, y) \in -C\}, \quad x \in A$$

is called a section mapping of $F$.

**Theorem 3.2.** *Let $X$ be a Hausdorff topological vector space, $A$ be a nonempty and closed subset of $X$, $Y$ be a topological vector space, $C$ be a closed, convex and pointed cone in $Y$. Let $F : A \times A \to Y$ be a bifunction with $F(x, x) \in C$ for all $x \in A$. Assume that*

(**i**) *the section mapping $M$ of $F$ is lower semicontinuous on $A$;*

(**ii**) *for any $x, y \in A$ with $x \neq y$, implies $F(x, y) \neq 0$.*

*Then $V(A, F)$ is closed.*

*Proof.* If $V(A, F) = \emptyset$, it is clear that $V(A, F)$ is closed. Now we assume that $V(A, F) \neq \emptyset$. We claim that cl$V(A, F) \subset V(A, F)$, where cl$V(A, F)$ denote the closure of $V(A, F)$. Suppose to the contrary that cl$V(A, F) \not\subset V(A, F)$, then there exists $x_0 \in$ cl$V(A, F)$ such that $x_0 \notin V(A, F)$. Since $V(A, F) \subset A$, and $A$ is closed, we have $x_0 \in A$. By $x_0 \notin V(A, F)$, there exists $y_0 \in A$ such that

(1) $$F(x_0, y_0) \in -C \backslash \{0\}.$$

By (1), we have $y_0 \in M(x_0)$. By condition (i), $M$ is lower semicontinuous at $x_0$, for any $U \in \mathcal{N}(0)$, there exists a neighborhood $V_U(x_0)$ of $x_0$ such that

$$M(x) \cap (y_0 + U) \neq \emptyset \quad \text{for all } x \in V_U(x_0) \cap A,$$

thus,

(2) $$M(x) \cap (y_0 + U) \neq \emptyset \quad \text{for all } x \in V_U(x_0) \cap (x_0 + U) \cap A.$$

By $x_0 \in$ cl$V(A, F)$, and noting that $V_U(x_0) \cap (x_0 + U)$ is a neighborhood of $x_0$, we have

$$V_U(x_0) \cap (x_0 + U) \cap V(A, F) \neq \emptyset.$$

Pick $x_U \in V_U(x_0) \cap (x_0 + U) \cap V(A, F)$. By (2), we have

$$M(x_U) \cap (y_0 + U) \neq \emptyset.$$

Pick $y_U \in M(x_U) \cap (y_0 + U)$. Thus, we obtain the nets $\{x_U : U \in \mathcal{N}(0)\}$ and $\{y_U : U \in \mathcal{N}(0)\}$. It is clear that $x_U \to x_0$ and $y_U \to y_0$, $x_U \in V(A, F)$ and $y_U \in M(x_U)$ for all $U \in \mathcal{N}(0)$. We have

(3) $$F(x_U, y_U) \in -C \quad \text{for all } U \in \mathcal{N}(0).$$

By (1) and assumption, and noting that $C$ is a pointed cone, we have $x_0 \neq y_0$. Since $X$ is Hausdorff, there exist a neighborhood $U(x_0)$ of $x_0$ and a neighborhood $U(y_0)$ of $y_0$ such that

(4) $$U(x_0) \cap U(y_0) = \emptyset.$$

Since $x_U \to x_0, y_U \to y_0$, there exists $U_0 \in \mathcal{N}(0)$ such that

$$x_U \in U(x_0),\ y_U \in U(y_0) \quad \text{for all } U \in \{U \in \mathcal{N}(0) : U \subset U_0\}.$$

By (4), $x_U \neq y_U$ for all $U \in \{U \in \mathcal{N}(0) : U \subset U_0\}$, and by condition (ii), we have $F(x_U, y_U) \neq 0$. This together with (3) yields

$$F(x_U, y_U) \in -C \backslash \{0\}.$$

Thus, $x_U \notin V(A, F)$. This contradicts that $x_U \in V(A, F)$. Thus, $\text{cl}V(A, F) \subset V(A, F)$, therefore, $V(A, F)$ is closed. ■

Now, we give an example to show that there exists a mapping which satisfying the conditions of Theorem 3.2.

**Example 3.1.** Let $X = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+ = \{y = (y_1, y_2) : y_1 \geq 0, y_2 \geq 0\}$, and $A = [-1, 1] \subset X$. Let

$$g(x) = \begin{cases} -x, & x \in [-1, 0], \\ 0, & x \in (0, 1]. \end{cases}$$

Define the mapping $F : A \times A \to \mathbb{R}^2$ by

$$F(x, y) = (y - x, g(y) - g(x)), \ \ x, y \in A.$$

It is easy to see that

$$M(x) = \begin{cases} \{x\}, & x \in [-1, 0], \\ [0, x], & x \in (0, 1], \end{cases}$$

and the conditions of Theorem 3.2 are satisfied. By Theorem 3.2, $V(A, F)$ is closed. We can see that $V(A, F) = [-1, 0]$.

The following example illustrates that the condition that section mapping $M$ of $F$ is lower semicontinuous on $A$ in Theorem 3.2 is essential.

**Example 3.2.** Let $X = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, A = [-1, 1] \subset X$. Let

$$g(x) = \begin{cases} x, & x \in [-1, 0), \\ 0, & x \in [0, 1]. \end{cases}$$

Define the mapping $F : A \times A \to \mathbb{R}^2$ by

$$F(x, y) = (g(y) - g(x), x - y), \quad x, y \in A.$$

It is easy to see that

$$M(x) = \begin{cases} \{x\}, & x \in [-1, 0), \\ [x, 1], & x \in [0, 1]. \end{cases}$$

$M$ is not lower semicontinuous at $0 \in [-1, 1]$. In fact, there exist $\frac{1}{2} \in M(0) = [0, 1]$, and a neighborhood $(\frac{1}{4}, \frac{3}{4})$ of $\frac{1}{2}$, for any neighborhood $U(0)$ of 0, there exists $x \in U(0) \cap [-1, 0)$ such that

$$M(x) \cap (\frac{1}{4}, \frac{3}{4}) = \{x\} \cap (\frac{1}{4}, \frac{3}{4}) = \emptyset.$$

We can see that the other conditions of Theorem 3.2 are satisfied. However, $V(A, F) = [-1, 0) \cup \{1\}$ is not closed.

In the following, we give sufficient conditions to guarantee the lower semicontinuous of section mapping $M$ of $F$.

**Theorem 3.3.** *Let $X$ and $Y$ be topological vector spaces, $A$ be a nonempty convex subset of $X$, $C$ be a closed, convex and pointed cone in $Y$ with int $C \neq \emptyset$. Let $F : A \times A \to Y$ be a bifunction. Assume that*

(**i**) *for any $y \in A, F(\cdot, y)$ is $C$-upper semicontinuous on $A$;*

(**ii**) *for any $x \in A$, if $F(x, y_1) \in -C, F(x, y_2) \in -C$ and $y_1 \neq y_2$ with $y_1, y_2 \in A$, then*

$$F(x, ty_1 + (1 - t)y_2) \in -\textit{int}\, C \ \text{ for all } t \in (0, 1);$$

(**iii**) *for any $x \in A, x \in M(x)$.*

*Then, the section mapping $M$ of $F$ is lower semicontinuous on $A$.*

*Proof.* For any $x_0 \in A$, by condition (iii), $M(x_0) \neq \emptyset$. For any $y_0 \in M(x_0)$, and any neighborhood $U(y_0)$ of $y_0$, we consider two case:

**Case 1.** if $y_0 = x_0$, then for above $U(y_0)$, taking the neighborhood $U(x_0)$ of $x_0$ with $U(x_0) = U(y_0)$, by condition (iii), we have

$$x \in M(x) \cap U(y_0) \quad \text{for all } x \in U(x_0) \cap A,$$

that is

$$M(x) \cap U(y_0) \neq \emptyset \ \text{ for all } x \in U(x_0) \cap A. \tag{5}$$

**Case 2.** if $y_0 \neq x_0$. Noting that $F(x_0, y_0) \in -C$, and by condition (iii), $F(x_0, x_0) \in -C$, by assumption, we have

$$F(x_0, ty_0 + (1-t)x_0) \in -\text{int } C \ \text{ for all } t \in (0,1). \tag{6}$$

Since $ty_0 + (1-t)x_0 \to y_0$ as $t \to 1$, and $U(y_0)$ is a neighborhood of $y_0$, there exists $t_0 \in (0,1)$ such that

$$t_0 y_0 + (1-t_0)x_0 \in U(y_0). \tag{7}$$

By (6), there exists a neighborhood $U(0)$ of zero in $Y$, such that

$$F(x_0, t_0 y_0 + (1-t_0)x_0) + U(0) \subset -\text{int } C,$$

Therefore, we have

$$F(x_0, t_0 y_0 + (1-t_0)x_0) + U(0) - C \subset -\text{int } C.$$

By assumption, for above $t_0 y_0 + (1-t_0)x_0 \in A$, $F(\cdot, t_0 y_0 + (1-t_0)x_0)$ is $C$-upper semicontinuous at $x_0$, there exists a neighborhood $U(x_0)$ of $x_0$ such that

$$\begin{aligned} &F(x, t_0 y_0 + (1-t_0)x_0) \in F(x_0, t_0 y_0 + (1-t_0)x_0) + U(0) - C \\ &\subset -\text{int} C \quad \text{for all } x \in U(x_0) \cap A. \end{aligned}$$

This together with (7) implies that

$$t_0 y_0 + (1-t_0)x_0 \in M(x) \cap U(y_0) \quad \text{for all } x \in U(x_0) \cap A. \tag{8}$$

By (5) and (8), we can see that $M$ is lower semicontinuous at $x_0$. ∎

The following example illustrates that the condition (ii) in Theorem 3.3 is not necessary.

**Example 3.3.** Let $X, Y, C, A$, and $F$ be as in the Example 3.1. There exist $x = \frac{1}{2} \in A, y_1 = \frac{1}{4}, y_2 = \frac{1}{8} \in A$, such that

$$F(\frac{1}{2}, \frac{1}{4}) = (-\frac{1}{4}, 0) \in -\mathbb{R}^2_+, \ \text{and} \ F(\frac{1}{2}, \frac{1}{8}) = (-\frac{3}{8}, 0) \in -\mathbb{R}^2_+,$$

$$F(\frac{1}{2}, ty_1 + (1-t)y_2) = (ty_1 + (1-t)y_2 - \frac{1}{2}, 0) \notin -\text{int}\mathbb{R}^2_+ \text{ for all } t \in (0,1).$$

The condition (ii) of Theorem 3.3 is not satisfied. It is clear that the other conditions of Theorem 3.3 are satisfied. By Example 3.1, $M$ is lower semicontinuous on $[-1,1]$. Thus, the condition (ii) of Theorem 3.3 is not necessary.

**Definition 3.2.** Let $X$ and $Y$ be topological vector spaces, $C$ be a closed, convex and pointed cone in $Y$ with int$C \neq \emptyset$. Let $A$ be a nonempty and convex subset of $X$ and let $F : A \times A \to Y$ be a bifunction. $F$ is called to be $C$-strictly convex in its second variable if for each $x \in A$, and for every pair of distinct points $y_1, y_2 \in A$ and $t \in (0,1)$, the following property holds:

$$tF(x, y_1) + (1-t)F(x, y_2) \in F(x, ty_1 + (1-t)y_2) + \text{int}C.$$

**Remark 3.1.** If $F : A \times A \to Y$ is $C$-strictly convex in its second variable, then the condition (ii) of Theorem 3.3 is satisfied. In fact, for any $x \in A$, if $F(x, y_1) \in -C$, $F(x, y_2) \in -C$ and $y_1 \neq y_2$ with $y_1, y_2 \in A$, since $F$ is $C-$strictly convex in its second variable, we have

$$F(x, ty_1 + (1-t)y_2) \in tF(x, y_1) + (1-t)F(x, y_2) - \text{int}C \subset -\text{int}C \text{ for all } t \in (0,1).$$

Combining Theorem 3.2, Theorem 3.3 and Remark 3.1, we can give a sufficient condition guaranteeing that the closedness of $V(A,F)$ without the concept of section mapping $M$ of $F$.

**Theorem 3.4.** *Let $X$ be a Hausdorff topological vector space, $Y$ be a topological vector space, $C$ be a closed, convex and pointed cone in $Y$ with int$C \neq \emptyset$. Let $A$ be a nonempty and closed convex subset of $X$. Assume that*

(**i**) $F(x,x) = 0$ *for all $x \in A$;*

(**ii**) *for any $y \in A, F(\cdot, y)$ is $C$-upper semicontinuous on $A$;*

(**iii**) *for any $x, y \in A$ with $x \neq y$, implies $F(x,y) \neq 0$.*

(**iv**) *$F$ is a $C$-strictly convex in its second variable.*

*Then $V(A,F)$ is closed.*

## 4. Closedness of Efficient Solutions Set of VOP

In this section, we use the method of Section 3 to study the closedness of set of efficient solutions to the vector optimization problems.

Let $Y$ be a topological vector space, $C$ be a closed, convex and pointed cone in $Y$. $C$ induces a partially ordering in $Y$ defined by

$$x \leq y \text{ if and only if } y - x \in C.$$

**Theorem 4.1.** *Let $X$ and $Y$ be topological vector spaces, $A$ be a nonempty and closed subset of $X$, $C$ be a closed, convex and pointed cone in $Y$. Let $f : A \to Y$ be a continuous mapping. Assume that the set-valued mapping*

$$M(x) = \{y \in A : f(y) \leq f(x)\},\ x \in A$$

*is lower semicontinuous on $A$, then $E(A, f)$ is closed.*

*Proof.* If $E(A, f) = \emptyset$, it is clear that $E(A, f)$ is closed. Now we assume that $E(A, f) \neq \emptyset$. We claim that $\mathrm{cl}E(A, f) \subset E(A, f)$, Suppose to the contrary that $\mathrm{cl}E(A, f) \not\subset E(A, f)$, then there exists $x_0 \in \mathrm{cl}E(A, f)$ such that $x_0 \notin E(A, f)$. Since $A$ is closed, we have $x_0 \in A$. By $x_0 \notin E(A, f)$, there exists $\bar{x} \in A$ such that

$$f(\bar{x}) - f(x_0) \in -C\backslash\{0\},$$

that is

$$f(\bar{x}) \leq f(x_0) \text{ and } f(\bar{x}) \neq f(x_0). \tag{9}$$

By (9), we have

$$\bar{x} \in M(x_0) = \{y \in A : f(y) \leq f(x_0)\}.$$

Since the set-valued mapping $M : A \rightrightarrows A$ is lower semicontinuous at $x_0$, for any $U \in \mathcal{N}(0)$, there exists a neighborhood $V_U(x_0)$ of $x_0$ such that

$$M(x) \cap (\bar{x} + U) \neq \emptyset \quad \text{for all } x \in V_U(x_0) \cap A,$$

thus,

$$M(x) \cap (\bar{x} + U) \neq \emptyset \quad \text{for all } x \in V_U(x_0) \cap (x_0 + U) \cap A. \tag{10}$$

By $x_0 \in \mathrm{cl}E(A, f)$, we have

$$V_U(x_0) \cap (x_0 + U) \cap E(A, f) \neq \emptyset.$$

Pick $x_U \in V_U(x_0) \cap (x_0 + U) \cap E(A, f)$. By (10), we have

$$M(x_U) \cap (\bar{x} + U) \neq \emptyset.$$

Pick $y_U \in M(x_U) \cap (\bar{x} + U)$. Thus, we obtain the nets $\{x_U : U \in \mathcal{N}(0)\}$ and $\{y_U : U \in \mathcal{N}(0)\}$. It is clear that $x_U \to x_0$ and $y_U \to \bar{x}$, $x_U \in E(A, f)$ and $y_U \in M(x_U)$ for all $U \in \mathcal{N}(0)$. We have

$$f(y_U) \leq f(x_U) \quad \text{for all } U \in \mathcal{N}(0). \tag{11}$$

Since $C$ is a closed, convex and pointed cone in $Y$, $Y$ is Hausdorff ( see 3.1.2 of [12] ), and by (9), there exist a neighborhood $U(f(\bar{x}))$ of $f(\bar{x})$ and a neighborhood $U(f(x_0))$ of $f(x_0)$ such that

(12) $$U(f(\bar{x})) \cap U(f(x_0)) = \emptyset.$$

Since $y_U \to \bar{x}$ and $x_U \to x_0$, by the continuity of $f$, we have $f(y_U) \to f(\bar{x})$ and $f(x_U) \to f(x_0)$. Thus, there exists $U_0 \in \mathcal{N}(0)$ such that $f(y_{U_0}) \in U(f(\bar{x}))$ and $f(x_{U_0}) \in U(f(x_0))$. This together with (12) imply that

(13) $$f(y_{U_0}) \neq f(x_{U_0}).$$

By (11) and (13), we get $x_{U_0} \notin E(A, f)$. This contradicts that $x_{U_0} \in E(A, f)$. Hence, cl$E(A, f) \subset E(A, f)$. This means that $E(A, f)$ is closed. ■

**Remark 4.1.** Although we can set

$$F(x, y) = f(y) - f(x), \;\; x, y \in A,$$

but, we can not use Theorem 3.2 to obtain Theorem 4.1. Because in Theorem 3.2, $X$ is Hausdorff; while in Theorem 4.1, $X$ is not a Hausdorff space; the assumption (ii) of Theorem 3.2 is also not satisfied. In fact, if the assumption (ii) of Theorem 3.2 is satisfied, then the condition that " for any $x, y \in A$ with $x \neq y$, implies $F(x, y) = f(y) - f(x) \neq 0$ " holds, that is if $x, y \in A$ with $x \neq y$, implies $f(x) \neq f(y)$. But in Theorem 4.1, this condition is lacking.

**Remark 4.2.** Comparing Theorem 4.1 with Theorem 3.3 of [9] and Theorem 4.1 of [10], we not require that $A$ is a convex set and $f$ is $C$-quasiconvex, and $Y$ is a topological vector lattice with the ordering cone $C$. Thus, we improve the results of Theorem 3.3 of [9] and Theorem 4.1 of [10].

## 5. Closedness of Efficient Solutions Set of LVVI

In this section, we use the methods of Section 3 to study the closedness of efficient solutions set to the Lipschitz vector variational inequality.

Let $X$ and $Y$ be real normed linear spaces, $D$ be a nonempty subset of $X$ with $0 \in D$. Let $G : D \to Y$ be a mapping. We say that $G$ is Lipschitz on $D$, if there exists a constant $L > 0$ such that

$$\|G(x) - G(y)\| \leq L\|x - y\| \quad \text{for all } x, y \in D.$$

If $G$ is Lipschitz on $D$, then, $L$ is an upper bound of $\{\frac{\|G(x)-G(y)\|}{\|x-y\|} : \; x, y \in D, x \neq y\}$. Set

$$\text{Lip}(D, Y) = \{G \mid G : D \to Y \text{ is a Lipschitz mapping and } \; G(0) = 0\}.$$

We can see that Lip$(D, Y)$ is a linear space. For each $G \in$ Lip$(D, Y)$, we define

$$\|G\|_L = \sup\{\frac{\|G(x) - G(y)\|}{\|x - y\|} :\ x, y \in D, x \neq y\}.$$

It is clear that$\| \cdot \|_L$ is a norm on Lip$(D, Y)$(see [13]). Thus, (Lip$(D, Y), \| \cdot \|_L)$ is a normed space.
It is clear that for each $G \in$ Lip$(D, Y)$, we have

$$\|G(y)\| = \|G(y) - G(0)\| \leq \| G \|_L \| y \| \quad \text{for all } y \in D.$$

Let $A \subset D$ be a nonempty set, and let $T : A \to$ Lip$(D, Y)$ be a mapping. We consider the Lipschitz vector variational inequality (for short LVVI) which consists in finding $x \in A$ such that

(14) $$(Tx)(y) - (Tx)(x) \notin -C\backslash\{0\} \ \text{ for all } \ y \in A.$$

If $x \in A$ satisfies (14), then we call $x$ is an efficient solution for LVVI. The set of efficient solutions to the LVVI is denoted by LVI(A, T).

**Theorem 5.1.** *Let $X$ and $Y$ be real normed linear spaces, $D$ be a nonempty subset of $X$ with $0 \in D$, $A$ be a nonempty and closed subset of $X$ with $A \subset D$, $C$ be a closed, convex and pointed cone in $Y$. Assume that $T : A \to$ Lip$(D, Y)$ is a continuous mapping and the set-valued mapping*

$$M(x) = \{y \in A : (Tx)(y) - (Tx)(x) \in -C\}, \ \ x \in A$$

*is lower semicontinuous on $A$. Then LVI(A, T) is closed.*

*Proof.* If $LVI(A, T) = \emptyset$, it is clear that $LVI(A, T)$ is closed. Now we assume that $LVI(A, T) \neq \emptyset$. Let a sequence $\{x_n\} \subset LVI(A, T)$ and $x_n \to x_0$. Since $A$ is closed, $x_0 \in A$. Suppose to the contrary that $x_0 \notin LVI(A, T)$, then there exists $y_0 \in A$ such that

$$(Tx_0)(y_0) - (Tx_0)(x_0) \in -C\backslash\{0\}.$$

We have

(15) $$(Tx_0)(y_0) \neq (Tx_0)(x_0).$$

Thus,

$$y_0 \in M(x_0) = \{y \in A : (Tx_0)(y) - (Tx_0)(x_0) \in -C\}.$$

Since $x_n \to x_0$ and $y_0 \in M(x_0)$, by the lower semicontinuity of $M$ on $A$, and by Remark 2.1, there exists a sequence $\{y_n\}$ with $y_n \in M(x_n)$ such that

$$y_n \to y_0.$$

By the definition of $M$, we have

(16) $$(Tx_n)(y_n) - (Tx_n)(x_n) \in -C \quad \text{for all } n.$$

Since $Y$ is Hausdorff, by (15), there exist a neighborhood $U((Tx_0)(y_0))$ of $(Tx_0)(y_0)$ and a neighborhood $U((Tx_0)(x_0))$ of $(Tx_0)(x_0)$ such that

(17) $$U((Tx_0)(y_0)) \cap U((Tx_0)(x_0)) = \emptyset.$$

We have

$$\|(Tx_n)(y_n) - (Tx_0)(y_0)\| = \|(Tx_n)(y_n) - (Tx_0)(y_n) + (Tx_0)(y_n) - (Tx_0)(y_0)\|$$

(18) $$\leq \|(Tx_n - Tx_0)(y_n)\| + \|(Tx_0)(y_n) - (Tx_0)(y_0)\|.$$

Since $Tx_0 \in \text{Lip}(D, Y)$ and $y_n \to y_0$, we have

$$(Tx_0)(y_n) \to (Tx_0)(y_0).$$

We have

(19) $$\|(Tx_n - Tx_0)(y_n)\| \leq \|Tx_n - Tx_0\|_L \|y_n\|.$$

Noting that $T : A \to \text{Lip}(D, Y)$ is continuous, we have $\|Tx_n - Tx_0\|_L \to 0$. Since $\{\|y_n\|\}$ is bounded, by (19), we have

$$(Tx_n - Tx_0)(y_n) \to 0.$$

By (18), we have

(20) $$(Tx_n)(y_n) \to (Tx_0)(y_0).$$

We also have

(21) $$(Tx_n)(x_n) \to (Tx_0)(x_0).$$

By (20) and (21), there exists $n_0$ such that

(22) $$(Tx_n)(y_n) \in U((Tx_0)(y_0)) \text{ and } (Tx_n)(x_n) \in U((Tx_0)(x_0)) \quad \text{for all } n \geq n_0.$$

By (22) and (17), we have

$$(Tx_n)(y_n) \neq (Tx_n)(x_n) \text{ for all } n \geq n_0.$$

This together with (16), we get

$$(Tx_n)(y_n) - (Tx_n)(x_n) \in -C\backslash\{0\} \quad \text{for all } n \geq n_0$$

This contradicts that $x_{n_0} \in \text{LVI}(A,T)$. Thus, $x_0 \in \text{LVI}(A,T)$. Hence, $\text{LVI}(A,T)$ is closed. ■

**Remark 5.1.** Although we can set $F(x,y) = (Tx)(y) - (Tx)(x)\ \ x, y \in A$, but, we can not use Theorem 3.2 to obtain Theorem 5.1, because the assumption (ii) of Theorem 3.2 is not satisfied. In fact, if the assumption (ii) of Theorem 3.2 is satisfied, then the condition " for any $x, y \in A$ with $x \neq y$, implies $F(x,y) = (Tx)(y) - (Tx)(x) \neq 0$" holds, that is if $x, y \in A$ with $x \neq y$, implies $(Tx)(y) \neq (Tx)(x)$, but in Theorem 5.1, this condition is lacking.

**Theorem 5.2.** *Let $X$ and $Y$ be real normed linear spaces, $D$ be a nonempty subset of $X$ with $0 \in D$, $A$ be a nonempty convex subset of $X$ with $A \subset D$, $C$ be a closed, convex and pointed cone in $Y$ with int $C \neq \emptyset$.*
*Assume that*

(**i**) $T : A \to \mathit{Lip}(D,Y)$ *is a continuous mapping;*

(**ii**) *for any $x \in A$, if $(Tx)(y_1) - (Tx)(x) \in -C$, $(Tx)(y_2) - (Tx)(x) \in -C$, and $y_1 \neq y_2$ with $y_1, y_2 \in A$, then*

$$(Tx)(ty_1 + (1-t)y_2) - (Tx)(x) \in -\mathit{int}\, C \ \text{ for all } \ t \in (0,1).$$

*Then the set-valued mapping*

$$M(x) = \{y \in A : (Tx)(y) - (Tx)(x) \in -C\}, \ \ x \in A$$

*is lower semicontinuous on $A$.*

*Proof.* By Theorem 3.3, we only need to show that the mapping

$$F(x,y) = (Tx)(y) - (Tx)(x)$$

is continuous on $A \times A$. For any $(x_0, y_0) \in A$, let $\{(x_n, y_n)\} \subset A \times A$ be a sequence such that $(x_n, y_n) \to (x_0, y_0)$. Similarly to the proof of Theorem 5.1, we can see that

$$(Tx_n)(y_n) \to (Tx_0)(y_0) \text{ and } (Tx_n)(x_n) \to (Tx_0)(x_0).$$

Thus,

$$F(x_n, y_n) = (Tx_n)(y_n) - (Tx_n)(x_n) \to (Tx_0)(y_0) - (Tx_0)(x_0) = F(x_0, y_0).$$

This means that the mapping $F(x,y) = (Tx)(y) - (Tx)(x)$ is continuous on $A \times A$. This completes the proof. ■

Now we give an example to show that there exists a mapping which satisfying the conditions of Theorem 5.1.

**Example 5.1.** Let $X = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, A = D = [-1, 1] \subset X$. For each fixed $x \in A$, define the mapping $Tx : A \to Y$ by

$$(Tx)(y) = (2 + x)(y^2, y^2),\ y \in A.$$

For any $y_1, y_2 \in A$, we have

$$\|(Tx)(y_1)-(Tx)(y_2)\| = \|(2+x)(y_1^2, y_1^2)-(2+x)(y_2^2, y_2^2)\| = (2+x)\|(y_1^2-y_2^2, y_1^2-y_2^2)\|$$

$$= (2 + x)\sqrt{\mid y_1^2 - y_2^2 \mid^2 + \mid y_1^2 - y_2^2 \mid^2} \leq 2\sqrt{2}(2 + x) \mid y_1 - y_2 \mid .$$

Hence, $Tx$ is a Lipschitz mapping on $A$. We have $(Tx)(0) = (2 + x)(0, 0) = (0, 0)$. Therefore, $Tx \in \text{Lip}(A, Y)$. Hence, we have that $T : A \to \text{Lip}(A, Y)$.

Now we show that $T : A \to \text{Lip}(A, Y)$ is continuous. For any $x_0 \in A$, and $x \in A$, we have

$$\begin{aligned}\|Tx - Tx_0\|_L &= \sup\{\frac{\|(Tx - Tx_0)(u) - (Tx - Tx_0)(v)\|}{\mid u - v \mid} : u, v \in A, u \neq v\} \\ &= \sup\{\frac{\mid x - x_0 \mid \sqrt{(u^2 - v^2)^2 + (u^2 - v^2)^2}}{\mid u - v \mid} : u, v \in A, u \neq v\} \\ &= \mid x - x_0 \mid \sup\{\sqrt{2} \mid u + v \mid : u, v \in A, u \neq v\} \leq 2\sqrt{2} \mid x - x_0 \mid .\end{aligned}$$

Thus, $T : A \to \text{Lip}(A, Y)$ is continuous.

For any $x \in A$, if $(Tx)(y_1) - (Tx)(x) \in -C,\ (Tx)(y_2) - (Tx)(x) \in -C$, and $y_1 \neq y_2$, for any $t \in (0, 1)$, we have

$$(Tx)(ty_1 + (1 - t)y_2) - (Tx)(x) \in -C - (r, r),$$

where $r > 0$ because that $y \mapsto y^2$ is a strictly convex on $A$. Thus, we have

$$(Tx)(ty_1 + (1 - t)y_2) - (Tx)(x) \in -\text{int}C \ \text{ for all } t \in (0, 1).$$

By Theorem 5.2,

$$M(x) = \{y \in A : (Tx)(y) - (Tx)(x) \in -C\},\ \ x \in A$$

is lower semicontinuous on $A$. Then by Theorem 5.1, LVI$(A, T)$ is closed.

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Xun-Hua Gong
Department of Mathematics
Nanchang University
Nanchang 330031
P. R. China
E-mail: xunhuagong@gmail.com

Xin-Min Yang
Department of Mathematics
Chongqing Normal University
Chongqing 400047
P. R. China
E-mail: xmyang@cqnu.edu.cn