

## ENUMERATION PROBLEMS FOR A LINEAR CONGRUENCE EQUATION

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Dedicated to Prof. Peter Bundschuh for the occasion of his 70th birthday

**Abstract.** Let  $m \geq 2$  and  $r \geq 1$  be integers and let  $c \in Z_m = \{0, 1, \dots, m-1\}$ . In this paper, we give an upper bound and a lower bound for the number of unordered solutions  $x_1, \dots, x_n \in Z_m$  of the congruence  $x_1 + x_2 + \dots + x_r \equiv c \pmod{m}$ . Exact formulae are also given when  $m$  or  $r$  is prime. This solution number involves the Catalan number or generalized Catalan number in some special cases. Moreover, the enumeration problem has relationship with the restricted integer partition.

### 1. INTRODUCTION

Consider the congruence equation  $x_1 + x_2 + \dots + x_n \equiv 0 \pmod{n+1}$ , where  $n$  is a positive integer. It is well-known that the number of unordered solutions  $x_1, x_2, \dots, x_n$  in  $Z_{n+1} = \{0, 1, \dots, n\}$  with repetition allowed is  $\frac{1}{n+1} \binom{2n}{n} = C_n$ , the  $n$ th Catalan number (see Guy [3] and Stanley [7]). In this paper, we consider the generalized congruence

$$(1) \quad x_1 + x_2 + \dots + x_r \equiv c \pmod{m}$$

where  $m \geq 2$  and  $r \geq 1$  are integers and  $c \in Z_m = \{0, 1, \dots, m-1\}$ .

If  $x_1 = a_1, \dots, x_r = a_r$  is a solution of (1), the multiset  $\{a_1, \dots, a_r\}$  is called an *unordered solution*. If the multiset  $\{a_1, \dots, a_r\}$  is actually a set, we call it an

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*unordered solution without repetition.* Note that each unordered solution represents several solutions because the coefficients of (1) are all the same. Moreover, different unordered solutions represent different solutions.

In Section 2, we deal with enumeration problems of both unordered solutions and unordered solutions without repetition. One of our results (Theorem 4) involves the Catalan numbers and generalized Catalan numbers (see Sands [6]). We give an relationship with the restricted integer partitions for our problems in Section 3. We give explicit closed forms for numbers of unordered solutions of (1) with either prime  $m$  or prime  $r$  in Section 4.

## 2. UNORDERED SOLUTIONS AND CATALAN NUMBERS

Let  $M_{m,r}$  be the set of all unordered solutions of (1) with  $c$  ranging over all numbers in  $Z_m$ . Every element of  $M_{m,r}$  is a multiset containing  $r$  numbers in  $Z_m$  and so, can be uniquely represented by a non-decreasing sequence of  $r$  elements in  $Z_m$ , i.e.,  $M_{m,r} = \{\{a_1 \leq a_2 \leq \dots \leq a_r\} | a_1, \dots, a_r \in Z_m\}$ .

**Lemma 1.** *Let  $m \geq 2$  and  $r \geq 1$  be integers. Then*

$$|M_{m,r}| = \binom{m+r-1}{r},$$

where the number on the right hand side is the usual binomial coefficient.

*Proof.* This is trivial because  $|M_{m,r}|$  is in fact the number of choices with repetition from the set  $Z_m$  within  $r$  times. ■

Let  $M_{m,r}(c)$  be the set of all unordered solutions of (1). Trivially,  $|M_{m,r}(c)| = 1$  for any  $0 \leq c < m$ . If  $c_1 \neq c_2$ , then  $M_{m,r}(c_1) \cap M_{m,r}(c_2) = \emptyset$ . So,  $M_{m,r} = M_{m,r}(0) \cup \dots \cup M_{m,r}(m-1)$  is a disjoint union.

**Lemma 2.** *Let  $m \geq 2$  and  $r \geq 1$  be integers. For  $0 \leq c_1, c_2 \leq m-1$ , if  $\gcd(m, r, c_1) = \gcd(m, r, c_2)$ , then  $|M_{m,r}(c_1)| = |M_{m,r}(c_2)|$ .*

*Proof.* Let  $c \in Z_m$  and write  $e = \gcd(m, r, c)$ . For proving this lemma, it is enough to show  $|M_{m,r}(c)| = |M_{m,r}(e)|$ .

Since  $\gcd(\frac{m}{e}, \frac{r}{e}, \frac{c}{e}) = 1$ , there are integers  $x$  and  $y$  so that  $\frac{c}{e} + \frac{xr}{e} + \frac{ym}{e} = p > m$  is a prime. So  $\gcd(m, p) = 1$ . Let  $0 \leq x_0, p_0 \leq m-1$  satisfy  $x \equiv x_0 \pmod{m}$  and  $p \equiv p_0 \pmod{m}$ . Then  $\gcd(m, p_0) = 1$  and  $c + x_0r \equiv p_0e \pmod{m}$ . Now, define  $f : M_{m,r}(c) \rightarrow M_{m,r}(p_0e)$  by  $f(\{a_1, \dots, a_r\}) = \{a_1 + x_0, \dots, a_r + x_0\}$  for all  $\{a_1, \dots, a_r\} \in M_{m,r}(c)$ . It is trivial that  $f$  is a bijection. So,  $|M_{m,r}(c)| = |M_{m,r}(p_0e)|$ . On the other hand, define a map  $g : M_{m,r}(e) \rightarrow M_{m,r}(p_0e)$  by  $g(\{a_1, \dots, a_r\}) = \{p_0a_1, \dots, p_0a_r\}$  for all  $\{a_1, \dots, a_r\} \in M_{m,r}(e)$ . Since  $\gcd(m, p_0)$

$= 1$ ,  $g$  is a bijection. This implies  $|M_{m,r}(e)| = |M_{m,r}(p_0e)|$ . Therefore,  $|M_{m,r}(c)| = |M_{m,r}(e)|$ . ■

The following result from this lemma is easy and so its proof is omitted.

**Corollary 3.** *Let  $m \geq 2$  and  $r$  be positive integers. If  $\gcd(m, r) = 1$ , then  $|M_{m,r}(0)| = \dots = |M_{m,r}(m - 1)|$ .*

The last result cannot hold if  $m$  and  $r$  are not relatively prime. For instance,  $M_{4,2}(0) = \{\{0, 0\}, \{1, 3\}, \{2, 2\}\}$ ,  $M_{4,2}(1) = \{\{0, 1\}, \{2, 3\}\}$ ,  $M_{4,2}(2) = \{\{0, 2\}, \{1, 1\}, \{3, 3\}\}$ , and  $M_{4,2}(3) = \{\{0, 3\}, \{1, 2\}\}$ , and so their cardinalities are not identical.

The following result is easy to see from Corollary 3 and Lemma 1.

**Theorem 4.** *Let  $m \geq 2$  and  $r$  be positive integers. If  $\gcd(m, r) = 1$ , then for any integer  $0 \leq c < m$ , we have*

$$|M_{m,r}(c)| = \frac{1}{m} \binom{m+r-1}{r}.$$

**Remark.** The anonymous referee points out that Lemma 2 and Theorem 4 can also be proved by the cycle lemma, see Sands [6] for a combinatorial explanation of generalized Catalan numbers by using cycle lemma.

**Example 1.** Take  $m = n + 1$  and  $r = n$ . Then  $|M_{n+1,n}(c)| = \frac{1}{n+1} \binom{2n}{n} = C_n$  is the number of unordered solution of  $x_1 + \dots + x_n \equiv c \pmod{n+1}$ . Similarly, if  $m = n$  and  $r = n + 1$ , we also have  $|M_{n,n+1}(c)| = \frac{1}{n} \binom{2n}{n+1} = \frac{1}{n+1} \binom{2n}{n} = C_n$ . The interested readers can refer to Stanley [7] and Stanley’s website [8] for an extensive list of combinatorial interpretations of Catalan numbers. Furthermore, G. Birkhoff [2] asked in 1934 whether or not  $B_n = \frac{1}{2n-1} \binom{2n-1}{n}$  is an integer. Indeed,  $B_n = C_{n-1}$  is an integer for all positive integer  $n$  because we have  $\frac{1}{2n-1} \binom{2n-1}{n} = \frac{1}{n} \binom{2(n-1)}{n-1}$ .

**Example 2.** Take  $m = n(k - 1) + 1$  and  $r = n$ . We have  $|M_{n(k-1)+1,n}(c)| = \frac{1}{n(k-1)+1} \binom{nk}{n} = C_{n,k}$ , a generalized Catalan number. If  $m = n$  and  $r = (k - 1)n + 1$ , then  $|M_{n,(k-1)n+1}(c)| = \frac{1}{n} \binom{nk}{n(k-1)+1} = \frac{1}{n} \binom{nk}{n-1} = \frac{1}{n(k-1)+1} \binom{nk}{n} = C_{n,k}$ .

When considering unordered solutions without repetition instead of unordered solutions, we take  $r \leq m$ . In the case  $r = m$ , there is only one  $c$  (say,  $c = 0$  if  $m$  is odd while  $c = \frac{m}{2}$  if  $m$  is even) so that (1) has unordered solutions without repetition (in fact, there is only one unordered solution  $\{0, 1, \dots, m - 1\}$ ). Let  $N_{m,r}$  be the set of all unordered solutions of (1) without repetition with  $c$  ranging over all numbers in  $Z_m$  and let  $N_{m,r}(c)$  be the set of all unordered solutions without repetition for any  $c$  in  $Z_m$ . It is trivial that  $|N_{m,r}| = \binom{m}{r}$  and  $N_{m,r} = \cup_{c=0}^{m-1} N_{m,r}(c)$ , a disjoint union. Note that

Lemma 2 and Corollary 3 still hold. So, if  $\gcd(m, r) = 1$ , then  $|N_{m,r}(c)| = \frac{1}{m} \binom{m}{r}$ . From  $\frac{1}{m+r} \binom{m+r}{m} = \frac{1}{m} \binom{m+r-1}{r}$ , we have  $|N_{m+r,r}(a)| = |M_{m,r}(b)|$  with  $0 \leq a < m+r$  and  $0 \leq b < m$ , whenever  $\gcd(m, r) = 1$ .

**Example 3.** Take  $m = 2n + 1$  and  $r = n$  or  $n + 1$ . We have  $|N_{m,r}(c)| = \frac{1}{2n+1} \binom{2n+1}{n} = \frac{1}{n+1} \binom{2n}{n} = C_n$ . If  $m = nk + 1$  and  $r = n$ , then  $|N_{m,r}(c)| = \frac{1}{nk+1} \binom{nk+1}{n} = \frac{1}{n(k-1)+1} \binom{nk}{n} = C_{n,k}$ .

Let  $m \geq r$  and let  $M_{m,r}(c; j) = \{ \{a_1 \leq a_2 \leq \dots \leq a_r\} \in M_{m,r}(c) | a_1 = j \}$  for  $0 \leq c, j < m$ . Then  $M_{m,r}(c) = \cup_{j=0}^{m-1} M_{m,r}(c; j)$ , a disjoint union. It is trivial that  $|M_{m,2}(c; j)| = 1$  if either  $c \geq 2j$  or  $m + j > m + c \geq 2j$ , and  $|M_{m,2}(c; j)| = 0$  otherwise. This terminology will be used in the proof of the following proposition, which gives a relation between  $|M_{m,r}(c)|$  and  $|N_{m,r}(c)|$ .

**Proposition 5.** For integers  $m \geq r > 1$  and any integer  $c$  with  $0 \leq c < m$ ,

$$|M_{m,r}(c)| = (-1)^{r-1} |N_{m,r}(c)| + \sum_{k=1}^{r-1} (-1)^{k-1} \sum_{a=0}^{m-1} |N_{m,k}(c-a)| |M_{m,r-k}(a)|.$$

*Proof.* For  $0 \leq c < m$ , we have

$$\begin{aligned} |M_{m,r}(c)| &= \sum_{i_1=0}^{m-1} |M_{m,r}(c; i_1)| \\ &= \sum_{i_1=0}^{m-1} |M_{m,r-1}(c-i_1)| - \sum_{i_1=1}^{m-1} \sum_{i_2=0}^{i_1-1} |M_{m,r-1}(c-i_1; i_2)| \\ &= \sum_{i_1=0}^{m-1} |M_{m,r-1}(c-i_1)| - \sum_{i_1=1}^{m-1} \sum_{i_2=0}^{i_1-1} |M_{m,r-2}(c-i_1-i_2)| \\ &\quad + \sum_{i_1=2}^{m-1} \sum_{i_2=1}^{i_1-1} \sum_{i_3=0}^{i_2-1} |M_{m,r-2}(c-i_1-i_2; i_3)|. \end{aligned}$$

Continuing this process, we finally have

$$\begin{aligned} |M_{m,r}(c)| &= \sum_{i_1=0}^{m-1} |M_{m,r-1}(c-i_1)| - \sum_{i_1=1}^{m-1} \sum_{i_2=0}^{i_1-1} |M_{m,r-2}(c-i_1-i_2)| \\ (2) \quad &+ \dots + (-1)^{r-2} \sum_{i_1=r-2}^{m-1} \dots \sum_{i_{r-1}=0}^{i_{r-2}-1} |M_{m,1}(c-i_1-\dots-i_{r-1})| \\ &+ (-1)^{r-1} \sum_{i_1=r-2}^{m-1} \dots \sum_{i_r=0}^{i_{r-1}-1} |M_{m,1}(c-\sum_{j=1}^{r-1} i_j; i_r)| \end{aligned}$$

Let  $0 < k < r$ . For  $a \in \mathbb{Z}_m$ , the total number of terms  $|M_{m,r-k}(a)|$  in the summation  $\sum_{i_1=k-1}^{m-1} \cdots \sum_{i_k=0}^{i_{k-1}-1} |M_{m,r-k}(c - i_1 - \cdots - i_k)|$  of (2), with  $a \equiv c - i_1 - \cdots - i_k \pmod m$ , equals  $|N_{m,k}(c-a)|$ , the number of unordered solutions without repetition of  $x_1 + x_2 + \cdots + x_k \equiv c-a \pmod m$ . So,  $\sum_{i_1=k-1}^{m-1} \cdots \sum_{i_k=0}^{i_{k-1}-1} |M_{m,r-k}(c - i_1 - \cdots - i_k)| = \sum_{a=0}^{m-1} |N_{m,k}(c-a)| |M_{m,r-k}(a)|$ . Note that  $|M_{m,1}(c - \sum_{j=1}^{r-1} i_j; i_r)| = 1$  if and only if  $c - i_1 - \cdots - i_{r-1} \equiv i_r \pmod m$ . So, the last term of (2) becomes  $\sum_{i_1=r-2}^{m-1} \cdots \sum_{i_r=0}^{i_{r-1}-1} |M_{m,1}(c - \sum_{j=1}^{r-1} i_j; i_r)| = |N_{m,r}(c)|$ . Combining together, we get the result. ■

Using this proposition, one can prove that  $\sum_{k=0}^r (-1)^k \binom{m}{k} \binom{m+r-k-1}{r-k} = 0$  for integers  $m \geq r > 0$ .

It seems that the expression for  $|M_{m,r}(c)|$  is quite complicated in general. The only cases which we can give exact formulae for  $|M_{m,r}(c)|$  are either  $m$  or  $r$  is prime. Those formulae will be given in Section 4. Here we present a lower bound and an upper bound of  $|M_{m,r}(c)|$ .

**Theorem 6.** *Let  $m, r \geq 4$  and  $0 \leq c < m$  be integers. Then*

$$\left\lceil \frac{1}{m} \binom{m+r-2}{r-1} \right\rceil \leq |M_{m,r}(c)| \leq \sum_{i=1}^{\lfloor \frac{r-3}{2} \rfloor} \binom{m+r-2i-1}{r-2i+1} + \left\lceil \frac{1}{m} \binom{m+2}{3} \right\rceil,$$

where  $\lceil a \rceil$  denotes the least integer greater than or equal to  $a$  and  $\lfloor a \rfloor$  denotes the greatest integer less than or equal to  $a$ .

*Proof.* Note that  $|M_{m,r}(c; 0)| = |M_{m,r-1}(c)|$  and  $|M_{m,r}(c; i)| = |M_{m,r-1}(c - i) - \sum_{j=0}^{i-1} |M_{m,r-1}(c - i; j)|$  for  $r \geq 2$  and  $1 \leq i < m$ . So,

$$\begin{aligned} |M_{m,r}(c)| &= \sum_{i=0}^{m-1} |M_{m,r-1}(c - i)| - \sum_{i=1}^{m-1} \sum_{j=0}^{i-1} |M_{m,r-1}(c - i; j)| \\ &= \binom{m+r-2}{r-1} - \sum_{i=1}^{m-1} \sum_{j=0}^{i-1} |M_{m,r-1}(c - i; j)|. \end{aligned}$$

This implies

$$\begin{aligned} |M_{m,r}(c)| &\leq \binom{m+r-2}{r-1} - \sum_{i=0}^{m-1} |M_{m,r-1}(c - i; 0)| + |M_{m,r-1}(c; 0)| \\ &= \binom{m+r-2}{r-1} - \binom{m+r-3}{r-2} + |M_{m,r-2}(c)| \\ &= \binom{m+r-3}{r-1} + |M_{m,r-2}(c)|. \end{aligned}$$

The last inequality holds for all  $r > 3$ . Moreover,  $|M_{m,i}(c)| \leq |M_{m,3}(c)| \leq \lceil \frac{1}{m} \binom{m+2}{3} \rceil$  (see Theorem 8 in Section 4) for  $i = 1, 2, 3$ . Iteratedly, we have an upper bound  $|M_{m,r}(c)| \leq \sum_{i=1}^{\lfloor \frac{r-3}{2} \rfloor} \binom{m+r-2i-1}{r-2i+1} + \lceil \frac{1}{m} \binom{m+2}{3} \rceil$ .

Let  $|M_{m,r-1}(c_0)|$  be a largest one among all values  $|M_{m,r-1}(c)|$ ,  $0 \leq c < m$ . So,  $|M_{m,r-1}(c_0)| \geq \lceil \frac{1}{m} \binom{m+r-2}{r-1} \rceil$ . For  $0 \leq c < m$ , define  $f : M_{m,r-1}(c_0) \rightarrow M_{m,r}(c)$  by  $f(\{y_1, \dots, y_{r-1}\}) = \{c - c_0, y_1, \dots, y_{r-1}\}$ . Then  $f$  is injective and so we get a lower bound  $|M_{m,r}(c)| \geq |M_{m,r-1}(c_0)| \geq \lceil \frac{1}{m} \binom{m+r-2}{r-1} \rceil$ . ■

### 3. RELATION WITH THE RESTRICTED INTEGER PARTITIONS

In this section, we are going to give two relations with the restricted integer partitions for  $|M_{m,r}(c)|$ .

#### 3.1. First Relation

For  $0 \leq c < m$ ,  $|M_{m,r}(c)|$  is the total number of integral solutions of equations

$$(3) \quad \begin{aligned} x_1 + x_2 + \dots + x_r &= km + c, 0 \leq k < r, \\ \text{with } 0 \leq x_1 \leq \dots \leq x_r &\leq m - 1. \end{aligned}$$

For  $0 \leq k < r$ , each unordered solution of (3) represents a restricted partition of  $km + c$  into at most  $r$  parts with each part  $\leq m - 1$ . For each non-negative integer  $n$ , let  $p_{m-1,r}(n)$  be the number of restricted partitions of  $n$  into at most  $r$  parts, each  $\leq m - 1$ , with the convention  $p_{m-1,r}(0) = 1$ . Let  $G_{m-1,r}(x) = \sum_{n=0}^{\infty} p_{m-1,r}(n)x^n$  be the generating function of restricted partitions of non-negative integers into at most  $r$  parts with each part  $\leq m - 1$ . According to Theorem 3.1 in [1],

$$(4) \quad G_{m-1,r}(x) = \frac{(1 - x^{m+r-1}) \dots (1 - x^{r+1})}{(1 - x^{m-1}) \dots (1 - x)}.$$

Note that for  $0 \leq k < r$ , the number of solutions of  $x_1 + x_2 + \dots + x_r = km + c$  with  $0 \leq x_1 \leq \dots \leq x_r \leq m - 1$  equals  $p_{m-1,r}(km + c)$ . From (3),

$$(5) \quad |M_{m,r}(c)| = \sum_{k=0}^{r-1} p_{m-1,r}(km + c) = \sum_{k=0}^{r-1} [x^{km+c}] G_{m-1,r}(x),$$

where  $[x^\ell] G_{m-1,r}(x)$  is the coefficient of the term  $x^\ell$  in  $G_{m-1,r}(x)$ .

As an example, consider the case  $m = 6$  and  $r = 3$ . Using (4), we have  $G_{5,3}(x) = \frac{((1-x^8)(1-x^7)(1-x^6)(1-x^5)(1-x^4))}{((1-x^5)(1-x^4)(1-x^3)(1-x^2)(1-x))} = x^{15} + x^{14} + 2x^{13} + 3x^{12} + 4x^{11} + 5x^{10} + 6x^9 + 6x^8 + 6x^7 + 6x^6 + 5x^5 + 4x^4 + 3x^3 + 2x^2 + x + 1$ . By (5), we get  $|M_{6,3}(0)| = \sum_{k=0}^2 [x^{6k}] G_{5,3}(x) = 1 + 6 + 3 = 10 = \lceil \frac{1}{6} \binom{6+2}{3} \rceil$ ,  $|M_{6,3}(1)| = \sum_{k=0}^2 [x^{6k+1}] G_{5,3}(x) = 9 = \lfloor \frac{1}{6} \binom{6+2}{3} \rfloor$ ,  $|M_{6,3}(2)| =$

$\sum_{k=0}^2 [x^{6k+2}]G_{5,3}(x) = 9 = \left\lfloor \frac{1}{6} \binom{6+2}{3} \right\rfloor$ ,  $|M_{6,3}(3)| = \sum_{k=0}^2 [x^{6k+3}]G_{5,3}(x) = 10 = \left\lfloor \frac{1}{6} \binom{6+2}{3} \right\rfloor$ ,  $|M_{6,3}(4)| = \sum_{k=0}^1 [x^{6k+4}]G_{5,3}(x) = 9 = \left\lfloor \frac{1}{6} \binom{6+2}{3} \right\rfloor$ , and  $|M_{6,3}(5)| = \sum_{k=0}^1 [x^{6k+5}]G_{5,3}(x) = 9 = \left\lfloor \frac{1}{6} \binom{6+2}{3} \right\rfloor$ . These are consistent to the results computed by Theorem 8 in next section.

Write  $y_r = x_1, y_{r-1} = x_2 - x_1, \dots, y_1 = x_r - x_{r-1}$ . Then (3) becomes

$$(6) \quad \begin{aligned} & y_1 + 2y_2 + \dots + ry_r = km + c, 0 \leq k < r, \\ & \text{with } y_1, \dots, y_r \geq 0 \text{ and } y_1 + \dots + y_r \leq m - 1. \end{aligned}$$

Since the changing variable  $y_r = x_1, y_{r-1} = x_2 - x_1, \dots, y_1 = x_r - x_{r-1}$  is linear and non-singular, the total number of solutions  $(y_1, \dots, y_r)$  of (6) is equal to  $|M_{m,r}(c)|$ . Moreover, for  $0 \leq k < r$ , the equation  $y_1 + 2y_2 + \dots + ry_r = km + c$  with  $y_1, \dots, y_r \geq 0$  and  $y_1 + \dots + y_r \leq m - 1$  represents a restricted partition of  $km + c$  into at most  $m - 1$  parts with each part  $\leq r$ . As in the last paragraph, let  $p_{r,m-1}(n)$  be the number of restricted partitions of  $n$  into at most  $m - 1$  parts, each part  $\leq r$ , with the convention  $p_{r,m-1}(0) = 1$ , and let  $G_{r,m-1}(x) = \sum_{n=0}^{\infty} p_{r,m-1}(n)x^n$  be the generating function. According to Theorem 3.1 in [1] again, we have

$$(7) \quad G_{r,m-1}(x) = \frac{(1 - x^{m+r-1}) \dots (1 - x^m)}{(1 - x^r) \dots (1 - x)}.$$

It is easy to see that  $G_{r,m-1}(x) = G_{m-1,r}(x)$ . So, using (6) to compute  $|M_{m,r}(c)|$  is the same as using (3) to compute  $|M_{m,r}(c)|$ .

### 3.2. Second Relation

Denote  $x_{r+1} = m - 1$  and write  $y_{r+1} = x_1, y_r = x_2 - x_1, \dots, y_1 = x_{r+1} - x_r$ . Then (3) becomes

$$(8) \quad \begin{aligned} & y_1 + 2y_2 + \dots + (r + 1)y_{r+1} = km + c - 1, 1 \leq k < r + 1, \\ & \text{with } y_1, \dots, y_{r+1} \geq 0 \text{ and } y_1 + \dots + y_{r+1} = m - 1. \end{aligned}$$

For integers  $\ell, n \geq 0$ , let  $q_{r+1}(\ell, n)$  be the number of restricted partitions of  $n$  into  $\ell$  parts, each part  $\leq r + 1$ , with the convention  $q_{r+1}(\ell, 0) = 1 = q_{r+1}(0, n)$ . Let  $H_{r+1}(y, z) = \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} q_{r+1}(\ell, n)y^\ell z^n$ . Then

$$(9) \quad H_{r+1}(y, z) = \prod_{i=1}^{r+1} \frac{1}{1 - yz^i}$$

by formula (2.1.1) in [1]. From (8), we have

$$(10) \quad |M_{m,r}(c)| = \sum_{k=1}^r q_{r+1}(m - 1, km + c - 1) = \sum_{k=1}^r \left[ y^{m-1} z^{km+c-1} \right] H_{r+1}(y, z).$$

As an example, considering the case  $m = 2$  and  $r = 2\ell$ , we have

$$|M_{2,2\ell}(c)| = \sum_{k=1}^{2\ell} \left[ yz^{2k+c-1} \right] \prod_{i=1}^{2\ell+1} \frac{1}{1-yz^i} = \begin{cases} \ell + 1, & c = 0 \\ \ell, & c = 1. \end{cases}$$

These results are consistent with the results obtained by Theorem 7 in next section.

Note that even though we know the exact forms (7) and (9), it seems that there is no computationally closed form for either  $p_{r,m-1}(km+c)$  or  $q_{r+1}(m-1, km+c-1)$ . So, it is not easy to get an exact form for  $|M_{m,r}(c)|$  using either (5) or (10). However, there are algorithms to compute both  $p_{r,m-1}(n)$  and  $q_{r+1}(m-1, km+c-1)$  (see [4] and [5] for instance). One might employ those algorithms and either (5) or (10) to compute the number  $|M_{m,r}(c)|$ .

#### 4. SOME SPECIAL CASES FOR COMPUTING $|M_{m,r}(c)|$

In this section, we compute  $M_{m,r}(c)$  whenever either  $m$  or  $r$  is prime.

**Theorem 7.** *Let  $m$  be a prime number. Then, for any positive integer  $r$ ,*

$$|M_{m,r}(c)| = \begin{cases} \left\lfloor \frac{1}{m} \binom{m+r-1}{r} \right\rfloor & \text{if } c \equiv 0 \pmod{m}, \\ \left\lfloor \frac{1}{m} \binom{m+r-1}{r} \right\rfloor & \text{if } c \not\equiv 0 \pmod{m}. \end{cases}$$

*Proof.* If  $\gcd(m, r) = 1$ , the theorem holds by Theorem 4. So, we consider only that  $m$  and  $r$  are not relatively prime. Then,  $\gcd(m, r) = m$  and  $r \geq m$  because  $m$  is prime. For any  $0 \leq c < m - 1$ ,

$$\begin{aligned} |M_{m,r}(c)| &= \sum_{i_1=0}^{m-1} |M_{m,r}(c; i_1)| \\ &= \sum_{i_1=0}^{m-1} |M_{m,r-1}(c - i_1)| - \sum_{i_1=1}^{m-1} \sum_{i_2=0}^{i_1-1} |M_{m,r-1}(c - i_1; i_2)| \\ &= \sum_{i_1=0}^{m-1} |M_{m,r-1}(c - i_1)| - \sum_{i_1=1}^{m-1} \sum_{i_2=0}^{i_1-1} |M_{m,r-2}(c - i_1 - i_2)| \\ &\quad + \sum_{i_1=2}^{m-1} \sum_{i_2=1}^{i_1-1} \sum_{i_3=0}^{i_2-1} |M_{m,r-2}(c - i_1 - i_2; i_3)|. \end{aligned}$$

Continuing this process, we finally have



$$\begin{aligned}
 |M_{m,r}(c)| &= \sum_{i_1=0}^{m-1} |M_{m,r-1}(c - i_1)| - \sum_{i_1=1}^{m-1} \sum_{i_2=0}^{i_1-1} |M_{m,r-2}(c - i_1 - i_2)| + \dots \\
 (11) \quad &+ (-1)^{m-2} \sum_{i_1=m-2}^{m-1} \dots \sum_{i_{m-1}=0}^{i_{m-2}-1} |M_{m,r-m+1}(c - i_1 - \dots - i_{m-1})| \\
 &+ (-1)^{m-1} |M_{m,r-m+1}(c - \sum_{j=1}^{m-1} j; 0)|
 \end{aligned}$$

Write  $r = tm$ . We show the result by induction on  $t$ . At first, let  $t = 1$ . If  $m = 2$ , the last term of (11) is  $-|M_{m,1}(c - 1; 0)| = -1$  if  $c = 1$ , and  $-|M_{m,1}(c - 1; 0)| = 0$  if  $c = 0$ . So, the theorem holds. If  $m$  is an odd prime, the last term of (11) becomes  $|M_{m,1}(c - m(m - 1)/2; 0)| = |M_{m,1}(c; 0)| = 1$  if  $c = 0$  and  $|M_{m,1}(c; 0)| = 0$  if  $c \neq 0$ . Hence, the theorem also holds in this case.

Finally, assume  $r = tm$  with  $t > 1$  and that the theorem holds for  $r < tm$ . The last term of (11) becomes  $(-1)^{m-1} |M_{m,r-m}(c - m(m - 1)/2)|$ . By the assumption of induction,  $(-1)^{m-1} |M_{m,r-m}(0 - m(m - 1)/2)| = (-1)^{m-1} |M_{m,r-m}(c - m(m - 1)/2)| + 1$  for any  $c \neq 0$ . By the assumption again, the theorem holds because of Theorem 4. ■

**Theorem 8.** *Let  $r \geq 2$  be a prime number. Then for integers  $m > 1$  and  $0 \leq c < m$ ,*

$$|M_{m,r}(c)| = \begin{cases} \left\lfloor \frac{1}{m} \binom{m+r-1}{r} \right\rfloor & \text{if } c \equiv 0 \pmod{r}, \\ \left\lfloor \frac{1}{m} \binom{m+r-1}{r} \right\rfloor & \text{if } c \not\equiv 0 \pmod{r}. \end{cases}$$

*Proof.* If  $\gcd(m, r) = 1$ , the theorem holds by Theorem 4. So, let  $m = \ell r$  for some positive integer  $\ell$ . If  $\ell = 1$ , the theorem holds by Theorem 7. Hence, let  $\ell > 1$ . Since  $r$  is prime,  $\gcd(m, r, c)$  is either 1 or  $r$  for any integer  $0 \leq c < m$ . From Lemma 2, we only consider either  $c = 0$  or  $c = 1$ .

We are going to show  $|M_{\ell r, r}(0)| = |M_{r, \ell r}(0)|$  and  $|M_{\ell r, r}(1)| = |M_{r, \ell r}(1)|$ . Let  $G_{\ell r-1, r}(x)$  and  $G_{r-1, \ell r}(x)$  be generating functions, respectively, as those defined in the last section. Write  $G_{\ell r-1, r}(x) = \sum_{i=0}^{(\ell r-1)r} a_i x^i$  and  $G_{r-1, \ell r}(x) = \sum_{i=0}^{(r-1)\ell r} c_i x^i$ . Then we have  $|M_{\ell r, r}(0)| = \sum_{k=0}^{r-1} a_{k\ell r}$ ,  $|M_{\ell r, r}(1)| = \sum_{k=0}^{r-1} a_{k\ell r+1}$ ,  $|M_{r, \ell r}(0)| = \sum_{k=0}^{\ell(r-1)} c_{kr}$ , and  $|M_{r, \ell r}(1)| = \sum_{k=0}^{\ell(r-1)-1} c_{kr+1}$  from (5).

As in (4) and (7),  $G_{\ell r-1, r}(x) = ((1 - x^{\ell r+r-1}) \dots (1 - x^{\ell r})) / ((1 - x^r) \dots (1 - x))$  and  $G_{r-1, \ell r}(x) = G_{\ell r, r-1}(x) = ((1 - x^{\ell r+r-1}) \dots (1 - x^{\ell r+1})) / ((1 - x^{r-1}) \dots (1 - x))$ . These imply  $(1 - x^r) G_{\ell r-1, r}(x) = (1 - x^{\ell r}) G_{r-1, \ell r}(x) = \sum_{i=0}^{\ell r^2} b_i x^i$ . Multiplying out both sides of the first equality, we have

$$\begin{aligned} & \sum_{i=0}^{r-1} a_i x^i + \sum_{i=r}^{(\ell r-1)r} (a_i - a_{i-r}) x^i - \sum_{i=1}^r a_{(\ell r-2)r+i} x^{(\ell r-1)r+i} = \sum_{i=0}^{\ell r^2} b_i x^i \\ & = \sum_{i=0}^{\ell r-1} c_i x^i + \sum_{i=\ell r}^{\ell r(r-1)} (c_i - c_{i-\ell r}) x^i - \sum_{i=1}^{\ell r} c_{\ell r(r-2)+i} x^{\ell r(r-1)+i} \end{aligned}$$

The first equality implies that  $a_{kr+j} = \sum_{i=0}^k b_{ir+j}$  for all  $0 \leq k \leq \ell(r-1)$  and for all  $0 \leq j < r$ . Combining with the second equality, we have  $|M_{\ell r,r}(0)| = \sum_{k=0}^{r-1} a_{k\ell r} = \sum_{k=0}^{r-1} \sum_{i=0}^{k\ell} b_{ir} = rb_0 + (r-1) \sum_{i=1}^{\ell} b_{ir} + \dots + \sum_{i=(r-2)\ell+1}^{(r-1)\ell} b_{ir} = \sum_{i=0}^{(r-1)\ell} c_{ir} = |M_{r,\ell r}(0)|$ . Similarly, we have  $|M_{\ell r,r}(1)| = |M_{r,\ell r}(1)|$ .

Since  $r$  is prime,  $|M_{r,\ell r}(0)| = \lceil \frac{1}{r} \binom{\ell r+r-1}{\ell r} \rceil$  and  $|M_{r,\ell r}(1)| = \lfloor \frac{1}{r} \binom{\ell r+r-1}{\ell r} \rfloor$  by Theorem 7. From  $\frac{1}{\ell r} \binom{\ell r+r-1}{r} = \frac{1}{r} \binom{\ell r+r-1}{\ell r}$ , we have  $|M_{\ell r,r}(0)| = \lceil \frac{1}{\ell r} \binom{\ell r+r-1}{r} \rceil$  and  $|M_{\ell r,r}(1)| = \lfloor \frac{1}{\ell r} \binom{\ell r+r-1}{r} \rfloor$ . This completes the proof. ■

In general, the numbers  $|M_{m,r}(c)|$  may not assume only two values as those in Theorems 7 and 8. For instance,  $|M_{4,4}(0)| = 10$ ,  $|M_{4,4}(1)| = |M_{4,4}(3)| = 8$ , and  $|M_{4,4}(2)| = 9$ , which means that  $|M_{4,4}(c)|$  takes 3 values. On the other hand,  $|M_{m,r}(c)|$  might takes two values even though  $\gcd(m, n) \neq 1$ . For instance,  $|M_{4,2}(c)|$  has only values 3 and 2 as we have seen right after Corollary 3. Unfortunately, we are unable to give simple expressions as those stated in theorems above for arbitrary  $m$  and  $r$ .

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REFERENCES

1. G. E. Andrews, *The Theory of Partitions, Encyclo. Math. Its Appls*, Vol. 2, Addison-Wesley, Reading, Mass.-London-Amsterdam, 1976.
2. G. Birkhoff, Problem 3674, *Amer. Math. Monthly*, **41** (1934), 269.
3. R. K. Guy, Parker’s permutation problem involves the Catalan numbers, *Amer. Math. Monthly*, **100** (1993), 287-289.
4. J. D. Opdyke, A unified approach to algorithms generating unrestricted and restricted integer compositions and interfeer partitions, *J. Math. Model Algor.*, **9** (2010), 53-97.
5. L. A. Sanchis and M. B. Squire, Parallel algorithms for counting and randomly generating integer partitions, *J. Paral. Distrib. Computing*, **34** (1996), 29-35.

6. A. D. Sands, On generalized Catalan numbers, *Discrete Math.*, **21** (1978), 219-221.
7. R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge Stud. Adv. Math. Vol. 62, Cambridge Univ. Press, Cambridge, 1998.
8. R. P. Stanley, Catalan Addendum,  
<http://www.math.mit.edu/rstan/ec/catadd.pdf>.

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