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SOME GENERALIZED LACUNARY POWER SERIES WITH ALGEBRAIC COEFFICIENTS FOR MAHLER'S $U-$NUMBER ARGUMENTS

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Abstract. In this work, we show that under certain conditions the values of some generalized lacunary power series with algebraic coefficients for Mahler's U_m-number arguments belong to either a certain algebraic number field or $\bigcup_{i=1}^{t} U_i$ in Mahler's classification of the complex numbers, where t denotes a positive rational integer dependent on the coefficients of the given series and on the argument. Moreover, the obtained results are adapted to the field $\mathbb{Q}_p$ of $p-$adic numbers.

1. Introduction

A power series

$$F(z) = \sum_{h=0}^{\infty} c_h z^h \quad (c_h \in \mathbb{C} \text{ for } h = 0, 1, 2, \ldots \text{ or } c_h \in \mathbb{Q}_p \text{ for } h = 0, 1, 2, \ldots)$$

with a positive radius of convergence, satisfying the following conditions

$$\begin{cases} c_h = 0, & r_n < h < s_n \quad (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = r_n \qquad\qquad (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = s_n \qquad\qquad (n = 0, 1, 2, \ldots), \end{cases}$$

where $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are two infinite sequences of non-negative rational integers with

$$0 \leq s_0 \leq r_1 < s_1 \leq r_2 < s_2 \leq r_3 < s_3 \leq \ldots \quad \text{and} \quad \lim_{n\to\infty} \frac{s_n}{r_n} = \infty,$$

is called a generalized lacunary power series.

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In [7] and [8], Kekeç shows that under certain conditions the values of some generalized lacunary power series with algebraic coefficients from a certain algebraic number field K of degree m for Liouville number arguments belong to either the algebraic number field K or $\bigcup_{i=1}^{m} U_i$ in Mahler's classification of the complex numbers. In the present work, the results given in [7] and [8] are extended to Mahler's U_m-number arguments. Namely, Theorem 3.1 and Theorem 3.2 are the extensions of Theorem 3.1 in [8] to Mahler's U_m-number arguments and Theorem 3.1 in [7] to Mahler's U_m-number arguments, respectively. Then the obtained results are adapted to the field $\mathbb{Q}_p$ of $p-$adic numbers on the light of the idea given in the paper Zeren [20]. Namely, Theorem 3.3 and Theorem 3.4 are the $p-$adic versions of Theorem 3.1 and Theorem 3.2, respectively. The readers are also recommended to refer to Mahler [14], Braune [1], Zeren [21], Yilmaz [18, 19], Gürses [4], Bugeaud [2], Çalşkan [3], and Kekeç [6] for a literature survey.

The main purpose of this work is to give new methods for obtaining $U-$numbers in Mahler's classification of the complex transcendental numbers and $p-$adic $U-$ numbers in Mahler's classification of the $p-$adic transcendental numbers. Our new results are stated and proved in Section 3, and the basic concepts concerning the literature and the lemmas we need to prove the new results of this work are given in Section 2.

2. Preliminaries

The following subsections Subsection 2.1 and Subsection 2.2 are the summary of the well-known basic concepts in the literature and are also available in the author's previous papers Kekeç [8] and Kekeç [9], respectively.

2.1. Mahler's classification of the complex numbers

Mahler [12], in 1932, divided the complex numbers into four classes and called numbers in these classes $A-$numbers, $S-$numbers, $T-$numbers, and $U-$numbers as follows.

We shall be concerned with polynomials $P(x) = a_n x^n + \cdots + a_0$ with rational integral coefficients. The height $H(P)$ of P is defined by $H(P) = \max(|a_n|, \ldots, |a_0|)$, and we shall denote the degree of P by $deg(P)$.

Given a complex number ξ and natural numbers n and H (recall that a natural number means a positive rational integer), Mahler [12] sets

$$w_n(H, \xi) = \min\{|P(\xi)| : P(x) \in \mathbb{Z}[x], deg(P) \leq n, H(P) \leq H, \text{ and } P(\xi) \neq 0\}.$$

The polynomial $P(x) \equiv 1$ is one of the polynomials which lie in the minimum, and so we have $0 < w_n(H, \xi) \leq 1$. We see that $w_n(H, \xi)$ is a non-increasing function of both n and H. Next, Mahler [12] sets

$$w_n(\xi) = \limsup_{H\to\infty} \frac{-\log w_n(H, \xi)}{\log H} \qquad \text{and} \qquad w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n}.$$

Obviously, $w_n(\xi)$ is a non-decreasing function of n. Furthermore, the inequalities $0 \leq w_n(\xi) \leq \infty$ and $0 \leq w(\xi) \leq \infty$ hold. If $w_n(\xi) = \infty$ for some integers n, let $\mu(\xi)$ be the smallest such integer. In this case, we have $w_n(\xi) < \infty$ for $n < \mu(\xi)$ and $w_n(\xi) = \infty$ for $n \geq \mu(\xi)$. If $w_n(\xi) < \infty$ for every n, set $\mu(\xi) = \infty$. Hence, $\mu(\xi)$ and $w(\xi)$ are uniquely determined and are never finite simultaneously. Therefore, there are the following four possibilities for ξ, and ξ is called

- an $A-$number if $w(\xi) = 0$ and $\mu(\xi) = \infty$,
- an $S-$number if $0 < w(\xi) < \infty$ and $\mu(\xi) = \infty$,
- a $T-$number if $w(\xi) = \infty$ and $\mu(\xi) = \infty$,
- a $U-$number if $w(\xi) = \infty$ and $\mu(\xi) < \infty$.

Every complex number ξ is of precisely one of these four types. $A-$numbers are precisely the algebraic numbers (see Schneider [16, pp. 68-69]). Let S, T, and U denote the set of $S-$numbers, the set of $T-$numbers, and the set of $U-$numbers, respectively. Then the transcendental numbers are distributed into the three disjoint classes S, T, and U. Let ξ be a $U-$number such that $\mu(\xi) = m$, and let U_m denote the set of all such numbers, that is, $U_m = \{\xi \in U : \mu(\xi) = m\}$. Obviously, the set U_m $(m = 1, 2, 3, \ldots)$ is a subclass of U, and U is the union of all the disjoint sets U_m. An element of U_m is called a U_m-number. Furthermore, LeVeque [11] showed that U_m is not empty for any natural number m.

Koksma [10], in 1939, set up another classification of the complex numbers. He divided the complex numbers into four classes A^*, S^*, T^*, and U^* as follows.

Suppose that α is an algebraic number. Let $P(x)$ be the minimal defining polynomial of α such that its coefficients are rational integers and relatively prime, and its highest coefficient is positive. Then the height $H(\alpha)$ of α is defined by $H(\alpha) = H(P)$, and the degree $deg(\alpha)$ of α is defined as the degree of P.

Given a complex number ξ and natural numbers n and H, Koksma [10] sets

$$w_n^*(H, \xi) = \min\{|\xi - \alpha| : \alpha \text{ is algebraic}, deg(\alpha) \leq n, H(\alpha) \leq H, \text{and } \alpha \neq \xi\},$$

$$w_n^*(\xi) = \limsup_{H \to \infty} \frac{-\log(H w_n^*(H, \xi))}{\log H}, \qquad \text{and} \qquad w^*(\xi) = \limsup_{n \to \infty} \frac{w_n^*(\xi)}{n}.$$

Obviously, $w_n^*(H, \xi)$ is a non-increasing function of both n and H, and so $w_n^*(\xi)$ is a non-decreasing function of n. Furthermore, the inequalities $0 \leq w_n^*(\xi) \leq \infty$ and $0 \leq w^*(\xi) \leq \infty$ hold. If $w_n^*(\xi) = \infty$ for some integers n, let $\mu^*(\xi)$ be the smallest such integer. In this case, we have $w_n^*(\xi) < \infty$ for $n < \mu^*(\xi)$ and $w_n^*(\xi) = \infty$ for $n \geq \mu^*(\xi)$. If $w_n^*(\xi) < \infty$ for every n, set $\mu^*(\xi) = \infty$. Hence, $\mu^*(\xi)$ and $w^*(\xi)$ are uniquely determined and are never finite simultaneously. Therefore, there are the following four possibilities for ξ, and ξ is called

- an A^*-number if $w^*(\xi) = 0$ and $\mu^*(\xi) = \infty$,
- an S^*-number if $0 < w^*(\xi) < \infty$ and $\mu^*(\xi) = \infty$,

- a T^*-number if $w^*(\xi) = \infty$ and $\mu^*(\xi) = \infty$,
- a U^*-number if $w^*(\xi) = \infty$ and $\mu^*(\xi) < \infty$.

Every complex number ξ is of precisely one of these four types. Let A^*, S^*, T^*, and U^* denote the set of A^*-numbers, the set of S^*-numbers, the set of T^*-numbers, and the set of U^*-numbers, respectively. Then the complex numbers are distributed into the four disjoint classes A^*, S^*, T^*, and U^*. Let ξ be a U^*-number such that $\mu^*(\xi) = m$, and let U_m^* denote the set of all such numbers, that is, $U_m^* = \{\xi \in U^* : \mu^*(\xi) = m\}$. Obviously, the set U_m^* $(m = 1, 2, 3, \ldots)$ is a subclass of U^*, and U^* is the union of all the disjoint sets U_m^*. An element of U_m^* is called a U_m^*-number.

Koksma's classification of the complex numbers is equivalent to Mahler's, that is, the classes A^*, S^*, T^*, and U^* are the same as the classes A, S, T, and U, respectively. Moreover, $U_m = U_m^*$ $(m = 1, 2, 3, \ldots)$ holds (see Schneider [16] and Wirsing [17]).

2.2. Mahler's classification of the $p-$adic numbers

Let p be a fixed prime number, and let $|\cdot|_p$ denote the $p-$adic absolute value function on the field $\mathbb{Q}$ of rational numbers. We shall denote the unique extension of $|\cdot|_p$ to the field $\mathbb{Q}_p$ of $p-$adic numbers, the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$, by the same notation $|\cdot|_p$.

By analogy with his classification of the complex numbers, Mahler [13], in 1934, proposed a classification of the $p-$adic numbers. Given a $p-$adic number ξ and natural numbers n and H, define the quantity (see Bugeaud [2])

$$w_n(H, \xi) = \min\{|P(\xi)|_p : P(x) \in \mathbb{Z}[x], deg(P) \leq n, H(P) \leq H, \text{ and } P(\xi) \neq 0\}$$

and set

$$w_n(\xi) = \limsup_{H\to\infty} \frac{-\log(Hw_n(H, \xi))}{\log H} \qquad \text{and} \qquad w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n}.$$

The inequalities $0 \leq w_n(\xi) \leq \infty$ and $0 \leq w(\xi) \leq \infty$ hold. If $w_n(\xi) = \infty$ for some integers n, then $\mu(\xi)$ is defined as the smallest such integer. If $w_n(\xi) < \infty$ for every n, set $\mu(\xi) = \infty$. Hence, $\mu(\xi)$ and $w(\xi)$ are uniquely determined and are never finite simultaneously. Therefore, there are the following four possibilities for the $p-$adic number ξ, and ξ is called

- a $p-$adic $A-$number if $w(\xi) = 0$ and $\mu(\xi) = \infty$,
- a $p-$adic $S-$number if $0 < w(\xi) < \infty$ and $\mu(\xi) = \infty$,
- a $p-$adic $T-$number if $w(\xi) = \infty$ and $\mu(\xi) = \infty$,
- a $p-$adic $U-$number if $w(\xi) = \infty$ and $\mu(\xi) < \infty$.

Every $p-$adic number ξ is of precisely one of these four types. The $p-$adic $A-$numbers are precisely the $p-$adic algebraic numbers. Let S, T, and U denote the set of $p-$adic

$S-$numbers, the set of $p-$adic $T-$numbers, and the set of $p-$adic $U-$numbers, respectively. Then the $p-$adic transcendental numbers are distributed into the three disjoint classes S, T, and U. Let ξ be a $p-$adic $U-$number such that $\mu(\xi) = m$, and let U_m denote the set of all such numbers, that is, $U_m = \{\xi \in U : \mu(\xi) = m\}$. Obviously, the set U_m $(m = 1, 2, 3, \ldots)$ is a subclass of U, and U is the union of all the disjoint sets U_m. An element of U_m is called a $p-$adic U_m-number (see Bugeaud [2] for more information about Mahler's classification in $\mathbb{Q}_p$).

Given a $p-$adic number ξ and natural numbers n and H, by analogy with Koksma's classification of the complex numbers, define the quantity (see Bugeaud [2] and Schlickewei [15])

$$w_n^*(H, \xi) = \min\{|\xi - \alpha|_p : \alpha \text{ is algebraic in } \mathbb{Q}_p, deg(\alpha) \leq n, H(\alpha) \leq H, \text{ and } \alpha \neq \xi\}$$

and set

$$w_n^*(\xi) = \limsup_{H\to\infty} \frac{-\log(Hw_n^*(H, \xi))}{\log H} \qquad \text{and} \qquad w^*(\xi) = \limsup_{n\to\infty} \frac{w_n^*(\xi)}{n}.$$

The inequalities $0 \leq w_n^*(\xi) \leq \infty$ and $0 \leq w^*(\xi) \leq \infty$ hold. If $w_n^*(\xi) = \infty$ for some integers n, then $\mu^*(\xi)$ is defined as the smallest such integer. If $w_n^*(\xi) < \infty$ for every n, set $\mu^*(\xi) = \infty$. Hence, $\mu^*(\xi)$ and $w^*(\xi)$ are uniquely determined and are never finite simultaneously. Therefore, there are the following four possibilities for the $p-$adic number ξ, and ξ is called

- a $p-$adic A^*-number if $w^*(\xi) = 0$ and $\mu^*(\xi) = \infty$,
- a $p-$adic S^*-number if $0 < w^*(\xi) < \infty$ and $\mu^*(\xi) = \infty$,
- a $p-$adic T^*-number if $w^*(\xi) = \infty$ and $\mu^*(\xi) = \infty$,
- a $p-$adic U^*-number if $w^*(\xi) = \infty$ and $\mu^*(\xi) < \infty$.

Every $p-$adic number ξ is of precisely one of these four types. Let A^*, S^*, T^*, and U^* denote the set of $p-$adic A^*-numbers, the set of $p-$adic S^*-numbers, the set of $p-$adic T^*-numbers, and the set of $p-$adic U^*-numbers, respectively. Then the $p-$adic numbers are distributed into the four disjoint classes A^*, S^*, T^*, and U^*. Let ξ be a $p-$adic U^*-number such that $\mu^*(\xi) = m$, and let U_m^* denote the set of all such numbers, that is, $U_m^* = \{\xi \in U^* : \mu^*(\xi) = m\}$. Obviously, the set U_m^* $(m = 1, 2, 3, \ldots)$ is a subclass of U^*, and U^* is the union of all the disjoint sets U_m^*. An element of U_m^* is called a $p-$adic U_m^*-number.

Both classifications in $\mathbb{Q}_p$ are equivalent, that is, the classes A, S, T, and U are the same as the classes A^*, S^*, T^*, and U^*, respectively. Moreover, $U_m = U_m^*$ $(m = 1, 2, 3, \ldots)$ holds (see Bugeaud [2] for all references and Schlickewei [15]).

2.3. Lemmas

We need the following two lemmas to prove the new results of this work.

Lemma 2.1. (İçen [5]). *Let* $\alpha_1, \ldots, \alpha_k \quad (k \geq 1)$ *be algebraic numbers which belong to an algebraic number field* K *of degree* m, *and let* $F(y, x_1, \ldots, x_k)$ *be a polynomial with rational integral coefficients and with degree at least* 1 *in* y. *If* η *is any algebraic number such that* $F(\eta, \alpha_1, \ldots, \alpha_k) = 0$, *then*

$$deg(\eta) \leq dm$$

and

$$H(\eta) \leq 3^{2dm+(l_1+\cdots+l_k)m} H^m H(\alpha_1)^{l_1 m} \ldots H(\alpha_k)^{l_k m},$$

where H *is the height of the polynomial* F, d *is the degree of* F *in* y, *and* $l_i \quad (i = 1, \ldots, k)$ *is the degree of* F *in* $x_i \quad (i = 1, \ldots, k)$.

Lemma 2.2. (LeVeque [11]). *Let* α *be an algebraic number of degree* m, *and let* $\alpha^{\{1\}} = \alpha, \ldots, \alpha^{\{m\}}$ *be its conjugates. Then*

$$|\overline{\alpha}| \leq 2H(\alpha),$$

where $|\overline{\alpha}| = \max\left(|\alpha^{\{1\}}|, \ldots, |\alpha^{\{m\}}|\right)$.

3. New Results

3.1. Generalized lacunary power series in the field $\mathbb{C}$ of complex numbers

Theorem 3.1. *Let* $K = \mathbb{Q}(\theta)$ *be an algebraic number field of degree* g, *and let* $F(z) = \sum_{h=0}^{\infty} c_h z^h \quad (c_h \in K,\ h = 0, 1, 2, \ldots)$ *be a power series which satisfies the following conditions*

$$\left\{\begin{array}{lll} c_h = 0, & r_n < h < s_n & (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = r_n & (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = s_n & (n = 0, 1, 2, \ldots), \end{array}\right. \tag{3.1}$$

where $\{s_n\}_{n=0}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *are two infinite sequences of non-negative rational integers with*

$$0 = s_0 < r_1 < s_1 \leq r_2 < s_2 \leq r_3 < s_3 \leq r_4 < s_4 \leq \ldots \quad \textit{and} \quad \lim_{n\to\infty} \frac{s_n}{r_n} = \infty. \tag{3.2}$$

Suppose that the radius of convergence R *of the series* $\sum_{h=0}^{\infty} |\overline{c_h}|\, z^h$ [1] *is positive (*R *may be finite or infinite) and*

$$\limsup_{h\to\infty} \frac{\log A_h}{h} < \infty \qquad (A_h = [a_0, a_1, \ldots, a_h],\ \ h = 1, 2, 3, \ldots)\,, \text{[2]} \tag{3.3}$$

[1] $|\overline{c_h}|$ denotes the maximum of the absolute values of the conjugates of the algebraic number c_h over $\mathbb{Q}$.
[2] $[a_0, a_1, \ldots, a_h]$ denotes the least common multiple of the rational integers $a_0, a_1, \ldots, a_h$.

where a_h $(h = 0, 1, 2, \ldots)$ is a suitable natural number such that $a_h c_h$ $(h = 0, 1, 2, \ldots)$ is an algebraic integer. Let $L = \mathbb{Q}(\beta)$ be an algebraic number field of degree m and α_n $(n = 1, 2, 3, \ldots)$ be algebraic numbers in L, and let $deg(\alpha_n) = m$ $(n = 1, 2, 3, \ldots)$. Moreover, assume that ξ is a U_m-number such that

$$|\xi - \alpha_n| \leq \frac{1}{H(\alpha_n)^{r_n \omega_n}} \qquad (n = 1, 2, 3, \ldots), \tag{3.4}$$

where $H(\alpha_n) > 1$ $(n = 1, 2, 3, \ldots)$ and $\omega_n = \frac{s_n}{r_n \log H(\alpha_n)}$ $(n = 1, 2, 3, \ldots)$ with $\lim_{n\to\infty} \omega_n = \infty$, and

$$|\xi| < R. \tag{3.5}$$

Then either $F(\xi)$ is an algebraic number in the algebraic number field $\mathbb{Q}(\theta, \beta)$, or $F(\xi) \in \bigcup_{i=1}^{t} U_i$, where t is the degree of $\mathbb{Q}(\theta, \beta)$ over $\mathbb{Q}$.

Proof. By (3.1), the series $F(z)$ can be written, for the complex numbers z at which $F(z)$ converges, as

$$F(z) = \sum_{h=0}^{\infty} c_h z^h = \sum_{k=0}^{\infty} P_k(z), \tag{3.6}$$

where $P_k(z) = \sum_{h=s_k}^{r_{k+1}} c_h z^h$ $\quad (k = 0, 1, 2, \ldots)$. We shall prove the theorem in four steps.

(1) The radius of convergence of the series $F(z) = \sum_{h=0}^{\infty} c_h z^h$ is greater than or equal to R. For since $|c_h| \leq |\overline{c_h}|$ $(h = 0, 1, 2, \ldots)$, $F(z)$ converges for all the complex numbers z for which the series $\sum_{h=0}^{\infty} |\overline{c_h}|\, z^h$ converges. Then $F(z)$ converges for $z = \xi$.

(2) We shall consider the polynomials

$$F_n(z) = \sum_{k=0}^{n-1} P_k(z) \qquad (n = 1, 2, 3, \ldots). \tag{3.7}$$

Define the algebraic numbers

$$\eta_n = F_n(\alpha_n) = \sum_{h=s_0}^{r_n} c_h \alpha_n^h \in \mathbb{Q}(\theta, \beta) \qquad (n = 1, 2, 3, \ldots). \tag{3.8}$$

Since $\eta_n \in \mathbb{Q}(\theta, \beta)$ $\quad (n = 1, 2, 3, \ldots)$, we have $deg(\eta_n) \leq t$ $\quad (n = 1, 2, 3, \ldots)$, where t is the degree of $\mathbb{Q}(\theta, \beta)$ over $\mathbb{Q}$. By multiplying both sides of the equality

$$\eta_n = \sum_{h=s_0}^{r_n} c_h \alpha_n^h \qquad (n = 1, 2, 3, \ldots)$$

by A_{r_n}, we obtain

$$A_{r_n}\eta_n - \sum_{h=s_0}^{r_n} A_{r_n}c_h\alpha_n^h = 0. \tag{3.9}$$

$A_{r_n}c_h \ \ (h = s_0, s_0+1, \ldots, r_n)$ is an algebraic integer in the algebraic number field $K = \mathbb{Q}(\theta)$. Moreover, we can assume that the algebraic number $\theta \in K$ given in the hypothesis of the theorem is an algebraic integer and shall do so. Then we have

$$A_{r_n}c_h = \frac{\zeta_0^{(h)}}{D} + \frac{\zeta_1^{(h)}}{D}\theta + \cdots + \frac{\zeta_{g-1}^{(h)}}{D}\theta^{g-1} \quad (h = s_0, s_0+1, \ldots, r_n), \tag{3.10}$$

where $\zeta_0^{(h)}, \zeta_1^{(h)}, \ldots, \zeta_{g-1}^{(h)}$, and $D = \left|\Delta^2(1, \theta, \ldots, \theta^{g-1})\right| > 0$ are rational integers. Here,

$$\Delta = \Delta(1, \theta, \ldots, \theta^{g-1}) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \theta^{\{1\}} & \theta^{\{2\}} & \ldots & \theta^{\{g\}} \\ \vdots & \vdots & \vdots & \vdots \\ (\theta^{g-1})^{\{1\}} & (\theta^{g-1})^{\{2\}} & \ldots & (\theta^{g-1})^{\{g\}} \end{vmatrix},$$

where $(\theta^i)^{\{1\}}, \ldots, (\theta^i)^{\{g\}}$ $(i = 1, 2, \ldots, g-1)$ denote the field conjugates of θ^i $(i = 1, 2, \ldots, g-1)$ for $K = \mathbb{Q}(\theta)$. Obviously, Δ and D depend only on θ and the conjugates of θ. We obtain from (3.9) and (3.10)

$$DA_{r_n}\eta_n - \sum_{h=s_0}^{r_n}\sum_{\mu=0}^{g-1} \zeta_\mu^{(h)}\theta^\mu\alpha_n^h = 0. \tag{3.11}$$

Then we have

$$T(\eta_n, \theta, \alpha_n) = 0, \tag{3.12}$$

where

$$T(y, x_1, x_2) = DA_{r_n}y - \sum_{h=s_0}^{r_n}\sum_{\mu=0}^{g-1} \zeta_\mu^{(h)} x_1^\mu x_2^h \tag{3.13}$$

is a polynomial in y, x_1, x_2 with rational integral coefficients. Since $DA_{r_n} \neq 0$, the polynomial $T(y, x_1, x_2)$ is of degree 1 in y. The degree of $T(y, x_1, x_2)$ in x_1 is less than or equal to $g-1$, and the degree of $T(y, x_1, x_2)$ in x_2 is r_n. Denote the height of the polynomial $T(y, x_1, x_2)$ by H. Then, by Lemma 2.1, we get

$$H(\eta_n) \leq 3^{2t+(g-1+r_n)t} H^t H(\theta)^{(g-1)t} H(\alpha_n)^{r_n t} \qquad (n = 1, 2, 3, \ldots). \tag{3.14}$$

Now let us determine an upper bound for the height H of the polynomial $T(y, x_1, x_2)$. It follows from (3.13) that

$$H = \max_{\substack{h=s_0,\ldots,r_n \\ \mu=0,\ldots,g-1}} \left(DA_{r_n}, |\zeta_\mu^{(h)}|\right). \tag{3.15}$$

Now we shall determine an upper bound for $|\zeta_\mu^{(h)}|$ $(\mu = 0, 1, \ldots, g-1;\ h = s_0, s_0 + 1, \ldots, r_n)$. Put

$$\delta = DA_{r_n} c_h. \tag{3.16}$$

Since $A_{r_n} c_h$ is an algebraic integer in K and D is a natural number, δ is an algebraic integer in K. By (3.10) and (3.16), we have

$$\delta = \zeta_0^{(h)} + \zeta_1^{(h)}\theta + \cdots + \zeta_{g-1}^{(h)}\theta^{g-1} \quad (h = s_0, s_0+1, \ldots, r_n)\,. \tag{3.17}$$

By using the field conjugates of θ for K in (3.17), we obtain the system of linear equations

$$\begin{cases} \delta^{\{1\}} = \zeta_0^{(h)} + \zeta_1^{(h)}\theta^{\{1\}} + \cdots + \zeta_{g-1}^{(h)}(\theta^{g-1})^{\{1\}} \\ \delta^{\{2\}} = \zeta_0^{(h)} + \zeta_1^{(h)}\theta^{\{2\}} + \cdots + \zeta_{g-1}^{(h)}(\theta^{g-1})^{\{2\}} \\ \quad\vdots \\ \delta^{\{g\}} = \zeta_0^{(h)} + \zeta_1^{(h)}\theta^{\{g\}} + \cdots + \zeta_{g-1}^{(h)}(\theta^{g-1})^{\{g\}} \end{cases} \tag{3.18}$$

in the unknowns $\zeta_0^{(h)}, \zeta_1^{(h)}, \ldots, \zeta_{g-1}^{(h)}$. Since $\Delta^2(1, \theta, \ldots, \theta^{g-1}) \neq 0$, the coefficient matrix of (3.18) is different from zero. Thus, the system of linear equations (3.18) has a unique solution which is

$$\zeta_\mu^{(h)} = \sum_{j=1}^{g} \frac{\Delta_{\mu j}}{\Delta}\delta^{\{j\}} \quad (\mu = 0, 1, \ldots, g-1), \tag{3.19}$$

where $\Delta_{\mu j}$ $(\mu = 0, 1, \ldots, g-1;\ j = 1, 2, \ldots, g)$ are complex constants which depend only on θ and the conjugates of θ. It follows from (3.19) that

$$|\zeta_\mu^{(h)}| \leq \sum_{j=1}^{g} \frac{|\Delta_{\mu j}|}{|\Delta|}|\delta^{\{j\}}| \leq \sum_{j=1}^{g} \frac{|\Delta_{\mu j}|}{|\Delta|}|\overline{\delta}| \leq |\overline{\delta}| \sum_{\mu=0}^{g-1}\sum_{j=1}^{g} \frac{|\Delta_{\mu j}|}{|\Delta|}. \tag{3.20}$$

We infer from (3.16) that

$$|\overline{\delta}| \leq DA_{r_n}|\overline{c_h}|. \tag{3.21}$$

By (3.20) and (3.21), we get

$$|\zeta_\mu^{(h)}| \leq \overline{C}(K)A_{r_n}|\overline{c_h}| \quad (\mu = 0, 1, \ldots, g-1;\ h = s_0, \ldots, r_n), \tag{3.22}$$

where $\overline{C}(K) = D\sum_{\mu=0}^{g-1}\sum_{j=1}^{g}\frac{|\Delta_{\mu j}|}{|\Delta|}$ is a positive real number which depends only on θ and the conjugates of θ. It follows from (3.15) and (3.22) that

$$H \leq \max_{h=s_0,\ldots,r_n}\left(DA_{r_n}, \overline{C}(K)A_{r_n}|\overline{c_h}|\right) \leq C(K)A_{r_n}\max_{h=s_0,\ldots,r_n}(1, |\overline{c_h}|)\,, \tag{3.23}$$

where $C(K)=\max\left(D,\overline{C}(K)\right)\geq 1$ is a real constant which depends only on θ and the conjugates of θ. Let us choose a real number ρ satisfying the inequality

$$0<|\xi|<\rho<R \tag{3.24}$$

(If $R=\infty$, then ρ is chosen as $\rho>|\xi|$). By (3.24), the series $\sum_{h=0}^{\infty}|\overline{c_h}|\,\rho^h$ is convergent. Thus, we have $\lim_{h\to\infty}\left(|\overline{c_h}|\,\rho^h\right)=0$, so the sequence $\left\{|\overline{c_h}|\,\rho^h\right\}_{h=0}^{\infty}$ is bounded, and therefore there is a real number $M>0$ such that

$$|\overline{c_h}|\leq\frac{M}{\rho^h}\quad(h=0,1,2,\ldots). \tag{3.25}$$

By (3.25), we have

$$\max_{h=s_0,\ldots,r_n}(1,|\overline{c_h}|)\leq\max_{h=s_0,\ldots,r_n}\left(1,\frac{M}{\rho^h}\right)\leq M_1\left(\max\left(1,\frac{1}{\rho}\right)\right)^{r_n}, \tag{3.26}$$

where $M_1=\max(1,M)\geq 1$. We deduce from (3.3) that the sequence $\left\{\frac{\log A_h}{h}\right\}_{h=1}^{\infty}$ is bounded above. So there exists a real number $\sigma>0$ such that

$$\frac{\log A_h}{h}\leq\sigma\quad(h=1,2,3,\ldots). \tag{3.27}$$

We obtain from (3.27)

$$A_{r_n}\leq e^{\sigma r_n}\quad(n=1,2,3,\ldots). \tag{3.28}$$

By (3.14), (3.23), (3.26), and (3.28), we get

$$H(\eta_n)\leq e_1^{r_nt}H(\alpha_n)^{r_nt}\quad(n=1,2,3,\ldots), \tag{3.29}$$

where $e_1=3^{g+2}C(K)e^{\sigma}M_1\max\left(1,\frac{1}{\rho}\right)H(\theta)^{g-1}>1$ is a real constant independent of $n,r_n,s_n,\eta_n,\alpha_n$, and $H(\alpha_n)$. On the other hand, by (3.4) and the fact that ξ is a U_m-number, we can assume that $\lim_{n\to\infty}H(\alpha_n)=\infty$ and shall do so. Thus, $e_1\leq H(\alpha_n)$ holds for sufficiently large n. Hence, it follows from (3.29) that

$$H(\eta_n)\leq H(\alpha_n)^{2r_nt} \tag{3.30}$$

for sufficiently large n.

(3) We have

$$|F(\xi)-\eta_n|\leq|F(\xi)-F_n(\xi)|+|F_n(\xi)-\eta_n|\qquad(n=1,2,3,\ldots). \tag{3.31}$$

Now we shall determine an upper bound for $|F(\xi)-F_n(\xi)|$ and $|F_n(\xi)-\eta_n|$. By (3.6), (3.7), (3.24), and (3.25), we get

$$|F(\xi)-F_n(\xi)|\leq\sum_{h=s_n}^{\infty}|\overline{c_h}|\,|\xi|^h\leq M\left(\frac{|\xi|}{\rho}\right)^{s_n}\left(1+\frac{|\xi|}{\rho}+\left(\frac{|\xi|}{\rho}\right)^2+\cdots\right).$$

Thus, we obtain

$$|F(\xi) - F_n(\xi)| \leq \frac{e_2}{e_3^{s_n}} \qquad (n = 1, 2, 3, \ldots), \tag{3.32}$$

where $e_2 = \frac{M}{1-\frac{|\xi|}{\rho}} > 0$ and $e_3 = \frac{\rho}{|\xi|} > 1$ are real constants independent of $n, r_n, s_n, \eta_n, \alpha_n$, and $H(\alpha_n)$. By (3.25), we have

$$|\overline{c_h}| \leq \frac{M}{\rho^h} \leq MB^h \leq M_1 B^h \qquad (h = 0, 1, 2, \ldots), \tag{3.33}$$

where $B = \max\left(1, \frac{1}{\rho}\right) \geq 1$. It follows from (3.4) that

$$|\alpha_n| < |\xi| + 1 \qquad (n = 1, 2, 3, \ldots). \tag{3.34}$$

From (3.4), (3.7), (3.8), (3.33), (3.34), and the fact that $|\xi| < |\xi| + 1$, we obtain

$$|F_n(\xi) - \eta_n| \leq \frac{1}{H(\alpha_n)^{r_n \omega_n}} (r_n + 1)^2 M_1^{r_n} B^{r_n} (|\xi| + 1)^{r_n}. \tag{3.35}$$

Since $\{r_n\}_{n=1}^{\infty}$ is a strictly increasing subsequence of natural numbers, it follows that $\lim_{n\to\infty} \sqrt[r_n]{(r_n + 1)^2} = 1$. Hence, there is a real number $e_4 > 1$ such that

$$(r_n + 1)^2 \leq e_4^{r_n} \tag{3.36}$$

for sufficiently large n. By (3.35) and (3.36), we have for sufficiently large n

$$|F_n(\xi) - \eta_n| \leq \frac{e_5^{r_n}}{H(\alpha_n)^{r_n \omega_n}}, \tag{3.37}$$

where $e_5 = e_4 M_1 B (|\xi| + 1) > 1$. From (3.37) and the fact $e_5 \leq H(\alpha_n)$ for sufficiently large n, we get

$$|F_n(\xi) - \eta_n| \leq \frac{1}{H(\alpha_n)^{r_n(\omega_n - 1)}} \tag{3.38}$$

for sufficiently large n. Let λ be a real number such that $0 < \lambda < \min(1, \log e_3)$. Then the inequalities

$$\frac{e_2}{e_3^{s_n}} \leq \frac{1}{H(\alpha_n)^{r_n(\omega_n - 1)\lambda}} \tag{3.39}$$

and

$$\frac{1}{H(\alpha_n)^{r_n(\omega_n - 1)}} \leq \frac{1}{H(\alpha_n)^{r_n(\omega_n - 1)\lambda}} \tag{3.40}$$

hold for sufficiently large n. It follows from (3.31), (3.32), (3.38), (3.39), and (3.40) that

$$|F(\xi)-\eta_n| \leq \frac{2}{H(\alpha_n)^{r_n(\omega_n-1)\lambda}} \leq \frac{1}{H(\alpha_n)^{r_n(\omega_n-2)\lambda}} \tag{3.41}$$

for sufficiently large n. We deduce from (3.41) that $\lim_{n\to\infty}|F(\xi)-\eta_n|=0$. Hence, we get $\lim_{n\to\infty}\eta_n=F(\xi)$. We infer from (3.30) and (3.41) that

$$|F(\xi)-\eta_n| \leq \frac{1}{H(\eta_n)^{\gamma_n}} \qquad (\lim_{n\to\infty}\gamma_n=\infty) \tag{3.42}$$

for sufficiently large n, where $\gamma_n=\frac{(\omega_n-2)\lambda}{2t}\quad(n=1,2,3,\ldots)$.

(4) There exist the following two cases for the sequence $\{|F(\xi)-\eta_n|\}$:

(a) $|F(\xi)-\eta_n|=0\quad$ from some n onward:

In this case, $\eta_n=F(\xi)$ from some n onward, that is, $\{\eta_n\}$ is a constant sequence. Since $\eta_n\in\mathbb{Q}(\theta,\beta)\quad(n=1,2,3,\ldots)$, in case a), we see that $F(\xi)$ is an algebraic number in $\mathbb{Q}(\theta,\beta)$.

(b) $|F(\xi)-\eta_n|\neq 0\quad$ for infinitely many n:

In this case, $F(\xi)$ is a U^*-number with $\mu^*(F(\xi))\leq t$, and therefore we have $F(\xi)\in\bigcup_{i=1}^{t}U_i^*$. Hence, in case b), we see that $F(\xi)\in\bigcup_{i=1}^{t}U_i$ since U_i^* is identical with U_i for any natural number i. This completes the proof of Theorem 3.1.

Theorem 3.2. *Let K be an algebraic number field, and let $F(z)=\sum_{h=1}^{\infty}c_hz^h$ $(c_h\in K,\ h=1,2,3,\ldots)$ be a power series which satisfies the following conditions*

$$\begin{cases} c_h=0, & r_n<h<s_n \quad (n=1,2,3,\ldots),\\ c_h\neq 0, & h=r_n \quad (n=1,2,3,\ldots),\\ c_h\neq 0, & h=s_n \quad (n=0,1,2,\ldots),\end{cases} \tag{3.43}$$

where $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are two infinite sequences of positive rational integers with

$$1=s_0\leq r_1<s_1\leq r_2<s_2\leq r_3<s_3\leq\ldots \quad \textit{and} \quad \lim_{n\to\infty}\frac{s_n}{r_n}=\infty. \tag{3.44}$$

Suppose that the radius of convergence R of the series $\sum_{h=1}^{\infty}H(c_h)z^h$ is positive,

$$\limsup_{h\to\infty}\frac{\log H(c_h)}{h}<\infty, \quad \textit{and} \quad \limsup_{n\to\infty}(r_{n+1}-s_n)<\infty. \tag{3.45}$$

Let $L=\mathbb{Q}(\beta)$ be an algebraic number field of degree m and α_n $(n=1,2,3,\ldots)$ be algebraic numbers in L, and let $deg(\alpha_n)=m$ $(n=1,2,3,\ldots)$. Moreover, assume that ξ is a U_m-number such that

$$|\xi-\alpha_n|\leq\frac{1}{H(\alpha_n)^{r_n\omega_n}} \qquad (n=1,2,3,\ldots), \tag{3.46}$$

where $H(\alpha_n) > 1$ $(n = 1, 2, 3, \ldots)$ *and* $\omega_n = \frac{s_n}{r_n \log H(\alpha_n)}$ $(n = 1, 2, 3, \ldots)$ *with* $\lim_{n\to\infty} \omega_n = \infty$*, and*

$$|\xi| < R. \tag{3.47}$$

Then either $F(\xi)$ *is an algebraic number in the algebraic number field* $K(\beta)$*, or* $F(\xi) \in \bigcup_{i=1}^{t} U_i$*, where* t *is the degree of* $K(\beta)$ *over* $\mathbb{Q}$*.*

Proof. By (3.43), the series $F(z)$ can be written, for the complex numbers z at which it converges, as

$$F(z) = \sum_{h=1}^{\infty} c_h z^h = \sum_{k=0}^{\infty} P_k(z), \tag{3.48}$$

where $P_k(z) = \sum_{h=s_k}^{r_{k+1}} c_h z^h \quad (k = 0, 1, 2, \ldots)$. We shall prove the theorem in four steps.

(1) By Lemma 2.2, the radius of convergence of $F(z)$ is greater than or equal to R. Then $F(z)$ converges for $z = \xi$.

(2) We shall consider the polynomials

$$F_n(z) = \sum_{k=0}^{n-1} P_k(z) \qquad (n = 1, 2, 3, \ldots). \tag{3.49}$$

Define the algebraic numbers

$$\eta_n = F_n(\alpha_n) = c_{s_0}\alpha_n^{s_0} + c_{s_0+1}\alpha_n^{s_0+1} + \cdots + c_{r_n}\alpha_n^{r_n} \in K(\beta) \quad (n = 1, 2, 3, \ldots). \tag{3.50}$$

Since $\eta_n \in K(\beta)$ $(n = 1, 2, 3, \ldots)$, we have $deg(\eta_n) \leq t$ $(n = 1, 2, 3, \ldots)$, where t is the degree of $K(\beta)$ over $\mathbb{Q}$. Then we get

$$T(\eta_n, \alpha_n, c_{s_0}, c_{s_0+1}, \ldots, c_{r_n}) = 0, \tag{3.51}$$

where

$$T(y, x, x_1, x_2, \ldots, x_{r_n}) = y - x_1 x^{s_0} - x_2 x^{s_0+1} - \cdots - x_{r_n} x^{r_n} \tag{3.52}$$

is a polynomial in $y, x, x_1, x_2, \ldots, x_{r_n}$ with rational integral coefficients. The polynomial $T(y, x, x_1, x_2, \ldots, x_{r_n})$ is of degree 1 in each $y, x_1, x_2, \ldots, x_{r_n}$ and is of degree r_n in x. The height $H(T)$ of the polynomial $T(y, x, x_1, x_2, \ldots, x_{r_n})$ is 1. Then, by Lemma 2.1, we obtain

$$H(\eta_n) \leq 3^{4r_n t} H(\alpha_n)^{r_n t} \left(H(c_{s_0})^t \ldots H(c_{r_1})^t\right) \ldots \left(H(c_{s_{n-1}})^t \ldots H(c_{r_n})^t\right) \tag{3.53}$$

for $n = 1, 2, 3, \ldots$. By the first inequality of (3.45), there exists a real number $M^* > 0$ such that $\frac{\log H(c_h)}{h} \leq M^* \quad (h = 1, 2, 3, \ldots)$, and hence we have

$$H(c_h) \leq A^h \qquad (h = 1, 2, 3, \ldots), \tag{3.54}$$

where $A = e^{M^*} > 1$. By the second inequality of (3.45), there exists a rational integer $\sigma > 0$ such that

$$s_i + (s_i + 1) + \cdots + r_{i+1} \leq \sigma r_{i+1} \qquad (i = 0, 1, 2, \ldots). \tag{3.55}$$

It follows from (3.53), (3.54), and (3.55) that

$$H(\eta_n) \leq 3^{4r_n t} H(\alpha_n)^{r_n t} A^{(r_1 + r_2 + \cdots + r_n)\sigma t} \tag{3.56}$$

for $n = 1, 2, 3, \ldots$. Since $r_n \geq s_{n-1} \quad (n = 1, 2, 3, \ldots)$ and $\lim_{n\to\infty} \frac{s_n}{r_n} = \infty$, we have $\lim_{n\to\infty} \frac{r_n}{r_{n-1}} = \infty$. Thus, there exists a natural number $n^* > 1$ such that $2r_{n-1} < r_n$ for $n \geq n^*$. From this, by induction, we obtain

$$r_{n^*} + r_{n^*+1} + \cdots + r_n < 2r_n \qquad (n \geq n^*). \tag{3.57}$$

We deduce from (3.56) and (3.57) that

$$H(\eta_n) \leq e_0^{r_n t} H(\alpha_n)^{r_n t} \quad (n \geq n^*), \tag{3.58}$$

where $e_0 = 3^4 A^{(r_1 + r_2 + \cdots + r_{n^*-1})\sigma} A^{2\sigma} > 1$ is a real constant independent of n, r_n, s_n, η_n, α_n, and $H(\alpha_n)$. On the other hand, by (3.46) and the fact that ξ is a U_m−number, we can assume that $\lim_{n\to\infty} H(\alpha_n) = \infty$ and shall do so. Thus, $e_0 \leq H(\alpha_n)$ holds for sufficiently large n. Hence, it follows from (3.58) that

$$H(\eta_n) \leq H(\alpha_n)^{2r_n t} \tag{3.59}$$

for sufficiently large n.

(3) We have

$$|F(\xi) - \eta_n| \leq |F(\xi) - F_n(\xi)| + |F_n(\xi) - \eta_n| \qquad (n = 1, 2, 3, \ldots). \tag{3.60}$$

Let us choose a real number ρ satisfying the inequality

$$0 < |\xi| < \rho < R. \tag{3.61}$$

By (3.61), the series $F(\rho) = \sum_{h=1}^{\infty} c_h \rho^h$ is convergent, so the sequence $\{c_h \rho^h\}_{h=1}^{\infty}$ is bounded, and therefore there is a real number $M > 0$ such that

$$|c_h| \leq \frac{M}{\rho^h} \qquad (h = 1, 2, 3, \ldots). \tag{3.62}$$

It follows from (3.48), (3.49), (3.61), and (3.62) that

$$|F(\xi) - F_n(\xi)| \leq \frac{e_1}{e_2^{s_n}} \qquad (n = 1, 2, 3, \ldots), \tag{3.63}$$

where $e_1 = \frac{M}{1-\frac{|\xi|}{\rho}} > 0$ and $e_2 = \frac{\rho}{|\xi|} > 1$. By (3.46), we get

$$|\alpha_n| < |\xi| + 1 \qquad (n = 1, 2, 3, \ldots). \tag{3.64}$$

We deduce from (3.46), (3.49), (3.50), (3.62), (3.64), and the fact $|\xi| < |\xi| + 1$ that

$$|F_n(\xi) - \eta_n| \leq \frac{1}{H(\alpha_n)^{r_n(\omega_n - 1)}} \tag{3.65}$$

for sufficiently large n. Let λ be a real number satisfying the inequality $0 < \lambda < \min(1, \log e_2)$. Then, for sufficiently large n, the inequalities

$$\frac{e_1}{e_2^{s_n}} \leq \frac{1}{H(\alpha_n)^{r_n(\omega_n-1)\lambda}} \quad \text{and} \quad \frac{1}{H(\alpha_n)^{r_n(\omega_n-1)}} \leq \frac{1}{H(\alpha_n)^{r_n(\omega_n-1)\lambda}} \tag{3.66}$$

hold. By (3.60), (3.63), (3.65), and (3.66), we have

$$|F(\xi) - \eta_n| \leq \frac{2}{H(\alpha_n)^{r_n(\omega_n-1)\lambda}} \leq \frac{1}{H(\alpha_n)^{r_n(\omega_n-2)\lambda}} \tag{3.67}$$

for sufficiently large n. It follows from (3.67) that $\lim_{n\to\infty} |F(\xi) - \eta_n| = 0$. Hence, we get $\lim_{n\to\infty} \eta_n = F(\xi)$. We infer from (3.59) and (3.67) that

$$|F(\xi) - \eta_n| \leq \frac{1}{H(\eta_n)^{\gamma_n}} \qquad (\lim_{n\to\infty} \gamma_n = \infty) \tag{3.68}$$

for sufficiently large n, where $\gamma_n = \frac{(\omega_n - 2)\lambda}{2t}$ $(n = 1, 2, 3, \ldots)$.

(4) There exist the following two cases for the sequence $\{|F(\xi) - \eta_n|\}$:

(a) $|F(\xi) - \eta_n| = 0$ from some n onward:

In this case, $\eta_n = F(\xi)$ from some n onward, that is, $\{\eta_n\}$ is a constant sequence. Since $\eta_n \in K(\beta)$ $(n = 1, 2, 3, \ldots)$, in case a), we see that $F(\xi)$ is an algebraic number in $K(\beta)$.

(b) $|F(\xi) - \eta_n| \neq 0$ for infinitely many n:

In this case, $F(\xi)$ is a U^*-number with $\mu^*(F(\xi)) \leq t$, and therefore we have $F(\xi) \in \bigcup_{i=1}^{t} U_i^*$. Hence, in case b), we see that $F(\xi) \in \bigcup_{i=1}^{t} U_i$ since U_i^* is identical with U_i for any natural number i. This completes the proof of Theorem 3.2.

3.2. Generalized lacunary power series in the field $\mathbb{Q}_p$ of $p-$adic numbers

Theorem 3.3. *Let $K = \mathbb{Q}(\theta)$ be a $p-$adic algebraic number field of degree g so that θ is a $p-$adic algebraic integer of degree g, and let $F(z) = \sum_{h=1}^{\infty} c_h z^h$ $(c_h \in K,\ h = 1, 2, 3, \ldots)$ be a power series in $\mathbb{Q}_p$ satisfying the following conditions*

$$\begin{cases} c_h = 0, & r_n < h < s_n \quad (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = r_n \qquad\quad (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = s_n \qquad\quad (n = 0, 1, 2, \ldots), \end{cases} \tag{3.69}$$

where $\{s_n\}_{n=0}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *are two infinite sequences of positive rational integers with*

$$1 = s_0 < r_1 < s_1 \le r_2 < s_2 \le r_3 < s_3 \le r_4 < s_4 \le \dots \quad \text{and} \quad \lim_{n\to\infty} \frac{s_n}{r_n} = \infty. \tag{3.70}$$

Suppose that the radius of convergence R *of the series F(z) is positive (*R *may be finite or infinite),*

$$\limsup_{h\to\infty} \frac{\log H(c_h)}{h} < \infty, \tag{3.71}$$

and

$$\limsup_{h\to\infty} \frac{\log A_h}{h} < \infty \qquad (A_h = [a_1, a_2, \dots, a_h], \;\; h = 2, 3, 4, \dots), \tag{3.72}$$

where a_h $(h = 1, 2, 3, \dots)$ *is a suitable natural number such that* $a_h c_h$ $(h = 1, 2, 3, \dots)$ *is a* $p-$*adic algebraic integer. Let* $L = \mathbb{Q}(\beta)$ *be a* $p-$*adic algebraic number field of degree* m *and* α_n $(n = 1, 2, 3, \dots)$ *be* $p-$*adic algebraic numbers in* L*, and let* $deg(\alpha_n) = m$ $(n = 1, 2, 3, \dots)$*. Moreover, assume that* ξ *is a* $p-$*adic* U_m-*number such that*

$$|\xi - \alpha_n|_p \le \frac{1}{H(\alpha_n)^{r_n \omega_n}} \qquad (n = 1, 2, 3, \dots), \tag{3.73}$$

where $H(\alpha_n) > 1$ $(n = 1, 2, 3, \dots)$ *and* $\omega_n = \frac{s_n}{r_n \log H(\alpha_n)}$ $(n = 1, 2, 3, \dots)$ *with* $\lim_{n\to\infty} \omega_n = \infty$*, and*

$$|\xi|_p < R. \tag{3.74}$$

Then either $F(\xi)$ *is a* $p-$*adic algebraic number in the* $p-$*adic algebraic number field* $\mathbb{Q}(\theta, \beta)$*, or* $F(\xi) \in \bigcup_{i=1}^{t} U_i$*, where* t *is the degree of* $\mathbb{Q}(\theta, \beta)$ *over* $\mathbb{Q}$*.*

Proof. By (3.69), the series $F(z)$ can be written, for the $p-$adic numbers z at which $F(z)$ converges, as

$$F(z) = \sum_{h=1}^{\infty} c_h z^h = \sum_{k=0}^{\infty} P_k(z), \tag{3.75}$$

where $P_k(z) = \sum_{h=s_k}^{r_{k+1}} c_h z^h \quad (k = 0, 1, 2, \dots)$. We shall prove the theorem in three steps.

(1) We shall consider the polynomials

$$F_n(z) = \sum_{k=0}^{n-1} P_k(z) \qquad (n = 1, 2, 3, \dots). \tag{3.76}$$

Define the $p-$adic algebraic numbers

$$\eta_n = F_n(\alpha_n) = \sum_{h=s_0}^{r_n} c_h \alpha_n^h \in \mathbb{Q}(\theta, \beta) \qquad (n = 1, 2, 3, \ldots). \tag{3.77}$$

Since $\eta_n \in \mathbb{Q}(\theta, \beta)$ $(n = 1, 2, 3, \ldots)$, we have $deg(\eta_n) \le t$ $(n = 1, 2, 3, \ldots)$, where t is the degree of $\mathbb{Q}(\theta, \beta)$ over $\mathbb{Q}$. By multiplying both sides of the equality

$$\eta_n = \sum_{h=s_0}^{r_n} c_h \alpha_n^h \qquad (n = 1, 2, 3, \ldots)$$

by A_{r_n}, we obtain

$$A_{r_n} \eta_n - \sum_{h=s_0}^{r_n} A_{r_n} c_h \alpha_n^h = 0. \tag{3.78}$$

$A_{r_n} c_h$ $(h = s_0, s_0 + 1, \ldots, r_n)$ is a $p-$adic algebraic integer in the $p-$adic algebraic number field $K = \mathbb{Q}(\theta)$. Then we have

$$A_{r_n} c_h = \frac{\zeta_0^{(h)}}{D} + \frac{\zeta_1^{(h)}}{D} \theta + \cdots + \frac{\zeta_{g-1}^{(h)}}{D} \theta^{g-1} \quad (h = s_0, s_0 + 1, \ldots, r_n), \tag{3.79}$$

where $\zeta_0^{(h)}, \zeta_1^{(h)}, \ldots, \zeta_{g-1}^{(h)}$, and $D = \left|\Delta^2(1, \theta, \ldots, \theta^{g-1})\right| > 0$ are rational integers. Here,

$$\Delta = \Delta(1, \theta, \ldots, \theta^{g-1}) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \theta^{\{1\}} & \theta^{\{2\}} & \ldots & \theta^{\{g\}} \\ \vdots & \vdots & \vdots & \vdots \\ (\theta^{g-1})^{\{1\}} & (\theta^{g-1})^{\{2\}} & \ldots & (\theta^{g-1})^{\{g\}} \end{vmatrix},$$

where $(\theta^i)^{\{1\}}, \ldots, (\theta^i)^{\{g\}}$ $(i = 1, 2, \ldots, g - 1)$ denote the field conjugates of θ^i $(i = 1, 2, \ldots, g - 1)$ for $K = \mathbb{Q}(\theta)$. Obviously, Δ and D depend only on θ and the conjugates of θ. We obtain from (3.78) and (3.79)

$$D A_{r_n} \eta_n - \sum_{h=s_0}^{r_n} \sum_{\mu=0}^{g-1} \zeta_\mu^{(h)} \theta^\mu \alpha_n^h = 0. \tag{3.80}$$

Then we have

$$T(\eta_n, \theta, \alpha_n) = 0, \tag{3.81}$$

where

$$T(y,x_1,x_2)=DA_{r_n}y-\sum_{h=s_0}^{r_n}\sum_{\mu=0}^{g-1}\zeta_\mu^{(h)}x_1^\mu x_2^h \tag{3.82}$$

is a polynomial in y, x_1, x_2 with rational integral coefficients. Since $DA_{r_n}\neq 0$, the polynomial $T(y,x_1,x_2)$ is of degree 1 in y. The degree of $T(y,x_1,x_2)$ in x_1 is less than or equal to $g-1$, and the degree of $T(y,x_1,x_2)$ in x_2 is r_n. Denote the height of the polynomial $T(y,x_1,x_2)$ by H. Then, by Lemma 2.1, we get

$$H(\eta_n)\leq 3^{2t+(g-1+r_n)t}H^tH(\theta)^{(g-1)t}H(\alpha_n)^{r_nt}\qquad(n=1,2,3,\ldots). \tag{3.83}$$

Now let us determine an upper bound for the height H of the polynomial $T(y,x_1,x_2)$. It follows from (3.82) that

$$H=\max_{\substack{h=s_0,\ldots,r_n\\ \mu=0,\ldots,g-1}}\left(DA_{r_n},|\zeta_\mu^{(h)}|\right). \tag{3.84}$$

Now we shall determine an upper bound for $|\zeta_\mu^{(h)}|\ \ (\mu=0,1,\ldots,g-1;\ h=s_0,s_0+1,\ldots,r_n)$. Put

$$\delta=DA_{r_n}c_h. \tag{3.85}$$

Since $A_{r_n}c_h$ is a $p-$adic algebraic integer in K and D is a natural number, δ is a $p-$adic algebraic integer in K. By (3.79) and (3.85), we have

$$\delta=\zeta_0^{(h)}+\zeta_1^{(h)}\theta+\cdots+\zeta_{g-1}^{(h)}\theta^{g-1}\quad(h=s_0,s_0+1,\ldots,r_n). \tag{3.86}$$

By using the field conjugates of θ for K in (3.86), we obtain the system of linear equations

$$\begin{cases}\delta^{\{1\}}=\zeta_0^{(h)}+\zeta_1^{(h)}\theta^{\{1\}}+\cdots+\zeta_{g-1}^{(h)}(\theta^{g-1})^{\{1\}}\\ \delta^{\{2\}}=\zeta_0^{(h)}+\zeta_1^{(h)}\theta^{\{2\}}+\cdots+\zeta_{g-1}^{(h)}(\theta^{g-1})^{\{2\}}\\ \vdots\\ \delta^{\{g\}}=\zeta_0^{(h)}+\zeta_1^{(h)}\theta^{\{g\}}+\cdots+\zeta_{g-1}^{(h)}(\theta^{g-1})^{\{g\}}\end{cases} \tag{3.87}$$

in the unknowns $\zeta_0^{(h)},\zeta_1^{(h)},\ldots,\zeta_{g-1}^{(h)}$. Since $\Delta^2(1,\theta,\ldots,\theta^{g-1})\neq 0$, the coefficient matrix of (3.87) is different from zero. Thus, the system of linear equations (3.87) has a unique solution which is

$$\zeta_\mu^{(h)}=\sum_{j=1}^{g}\frac{\Delta_{\mu j}}{\Delta}\delta^{\{j\}}\quad(\mu=0,1,\ldots,g-1), \tag{3.88}$$

where $\Delta_{\mu j}\quad(\mu=0,1,\ldots,g-1;\ j=1,2,\ldots,g)$ are constants which depend only on θ and the conjugates of θ. It follows from (3.88) that

$$|\zeta_\mu^{(h)}|\leq\sum_{j=1}^{g}\frac{|\Delta_{\mu j}|}{|\Delta|}|\delta^{\{j\}}|\leq\sum_{j=1}^{g}\frac{|\Delta_{\mu j}|}{|\Delta|}|\overline{\delta}|\leq|\overline{\delta}|\sum_{\mu=0}^{g-1}\sum_{j=1}^{g}\frac{|\Delta_{\mu j}|}{|\Delta|}. \tag{3.89}$$

We infer from (3.85) that

$$|\overline{\delta}| \leq DA_{r_n}|\overline{c_h}|. \tag{3.90}$$

By (3.89) and (3.90), we get

$$|\zeta_\mu^{(h)}| \leq \overline{C}(K)A_{r_n}|\overline{c_h}| \quad (\mu = 0, 1, \ldots, g-1;\ h = s_0, \ldots, r_n), \tag{3.91}$$

where $\overline{C}(K) = D\sum_{\mu=0}^{g-1}\sum_{j=1}^{g}\frac{|\Delta_{\mu j}|}{|\Delta|}$ is a positive real number which depends only on θ and the conjugates of θ. It follows from (3.84) and (3.91) that

$$H \leq \max_{h=s_0,\ldots,r_n}\left(DA_{r_n}, \overline{C}(K)A_{r_n}|\overline{c_h}|\right) \leq C(K)A_{r_n}\max_{h=s_0,\ldots,r_n}(1, |\overline{c_h}|), \tag{3.92}$$

where $C(K) = \max\left(D, \overline{C}(K)\right) \geq 1$ is a real constant which depends only on θ and the conjugates of θ. By (3.71), there exists a real number $\sigma_1 > 0$ such that $\frac{\log H(c_h)}{h} \leq \sigma_1 \quad (h = 1, 2, 3, \ldots)$, and hence we have

$$H(c_h) \leq B^h \qquad (h = 1, 2, 3, \ldots), \tag{3.93}$$

where $B = e^{\sigma_1} > 1$. By Lemma 2.2,

$$|\overline{c_h}| \leq 2H(c_h) \quad (h = 1, 2, 3, \ldots) \tag{3.94}$$

holds. We infer from (3.93) and (3.94) that

$$|\overline{c_h}| \leq D^h \quad (h = 1, 2, 3, \ldots), \tag{3.95}$$

where $D = 2B > 1$. It follows from (3.95) that

$$\max_{h=s_0,\ldots,r_n}(1, |\overline{c_h}|) \leq \max_{h=s_0,\ldots,r_n}\left(1, D^h\right) = D^{r_n}. \tag{3.96}$$

We deduce from (3.72) that the sequence $\left\{\frac{\log A_h}{h}\right\}_{h=2}^{\infty}$ is bounded above. So there exists a real number $\sigma_2 > 0$ such that

$$\frac{\log A_h}{h} \leq \sigma_2 \quad (h = 2, 3, 4, \ldots). \tag{3.97}$$

We obtain from (3.97)

$$A_{r_n} \leq e^{\sigma_2 r_n} \quad (n = 1, 2, 3, \ldots). \tag{3.98}$$

By (3.83), (3.92), (3.96), and (3.98), we get

$$H(\eta_n) \leq e_0^{r_n t}H(\alpha_n)^{r_n t} \quad (n = 1, 2, 3, \ldots), \tag{3.99}$$

where $e_0 = 3^{g+2}C(K)e^{\sigma_2}DH(\theta)^{g-1} > 1$ is a real constant independent of n, r_n, s_n, η_n, α_n, and $H(\alpha_n)$. On the other hand, by (3.73) and the fact that ξ is a $p-$adic U_m-number, we can assume that $\lim_{n\to\infty} H(\alpha_n) = \infty$ and shall do so. Thus, $e_0 \leq H(\alpha_n)$ holds for sufficiently large n. Hence, it follows from (3.99) that

$$H(\eta_n) \leq H(\alpha_n)^{2r_n t} \tag{3.100}$$

for sufficiently large n.

(2) We have

$$|F(\xi) - \eta_n|_p \leq \max\left(|F(\xi) - F_n(\xi)|_p, |F_n(\xi) - \eta_n|_p\right) \qquad (n = 1, 2, 3, \ldots). \tag{3.101}$$

Now we shall determine an upper bound for $|F(\xi) - F_n(\xi)|_p$ and $|F_n(\xi) - \eta_n|_p$. By (3.74), there exists a real number ε with $0 < \varepsilon < R$ such that

$$0 < |\xi|_p < R - \varepsilon. \tag{3.102}$$

Let the radius of convergence $R = \frac{1}{\limsup_{h\to\infty} \sqrt[h]{|c_h|_p}}$ of the series $F(z)$ be finite. Then there exists a natural number $h_0 > 1$ such that

$$|c_h|_p < \frac{1}{(R-\varepsilon)^h} \qquad \text{for} \quad h \geq h_0. \tag{3.103}$$

In fact, there is a real number $M_1 \geq 1$ such that

$$|c_h|_p < \frac{M_1}{(R-\varepsilon)^h} \qquad (h = 1, 2, 3, \ldots). \tag{3.104}$$

Let the radius of convergence R of the series $F(z)$ be infinite. Then $F(z)$ converges for every $p-$adic number z. Let us choose a $p-$adic number ρ such that

$$|\rho|_p > |\xi|_p > 0. \tag{3.105}$$

The series $F(\rho) = \sum_{h=1}^{\infty} c_h \rho^h$ is convergent in $\mathbb{Q}_p$. Thus, we have $\lim_{h\to\infty} |c_h\rho^h|_p = 0$, so the sequence $\left\{|c_h\rho^h|_p\right\}_{h=1}^{\infty}$ is bounded, and therefore there is a real number $M_2 > 0$ such that

$$|c_h|_p \leq \frac{M_2}{|\rho|_p^h} \quad (h = 1, 2, 3, \ldots). \tag{3.106}$$

Whether the radius of convergence R of the series $F(z)$ is finite or infinite, we obtain from (3.104) and (3.106)

$$0 \leq |c_h|_p \leq \frac{M}{r^h} \quad (h = 1, 2, 3, \ldots), \tag{3.107}$$

where $M = \max(M_1, M_2) \geq 1$ and $r = \min(R - \varepsilon, |\rho|_p) > 0$, and from (3.102) and (3.105)

$$0 < \frac{|\xi|_p}{r} < 1. \tag{3.108}$$

By (3.75), (3.76), (3.107), and (3.108), we get

$$|F(\xi) - F_n(\xi)|_p \leq M \max\left(\left(\frac{|\xi|_p}{r}\right)^{s_n}, \left(\frac{|\xi|_p}{r}\right)^{s_n+1}, \ldots\right) = M\left(\frac{|\xi|_p}{r}\right)^{s_n}.$$

Thus, we obtain

$$|F(\xi) - F_n(\xi)|_p \leq \frac{M}{e_1^{s_n}} \qquad (n = 1, 2, 3, \ldots), \tag{3.109}$$

where $e_1 = \frac{r}{|\xi|_p} > 1$. By (3.107), we have

$$0 \leq |c_h|_p \leq \frac{M}{r^h} \leq ME^h \qquad (h = 1, 2, 3, \ldots), \tag{3.110}$$

where $E = \max\left(1, \frac{1}{r}\right) \geq 1$. It follows from (3.73) that

$$|\alpha_n|_p < |\xi|_p + 1 \qquad (n = 1, 2, 3, \ldots). \tag{3.111}$$

From (3.73), (3.76), (3.77), (3.110), (3.111), and the fact that $|\xi|_p < |\xi|_p + 1$, we obtain

$$|F_n(\xi) - \eta_n|_p \leq \frac{e_2^{r_n}}{H(\alpha_n)^{r_n\omega_n}} \qquad (n = 1, 2, 3, \ldots), \tag{3.112}$$

where $e_2 = ME(|\xi|_p + 1) > 1$. From (3.112) and the fact $e_2 \leq H(\alpha_n)$ for sufficiently large n, we get

$$|F_n(\xi) - \eta_n|_p \leq \frac{1}{H(\alpha_n)^{r_n(\omega_n - 1)}} \tag{3.113}$$

for sufficiently large n. Let λ be a real number such that $0 < \lambda < \min(1, \log e_1)$. Then the inequalities

$$\frac{M}{e_1^{s_n}} \leq \frac{1}{H(\alpha_n)^{r_n(\omega_n - 1)\lambda}} \tag{3.114}$$

and

$$\frac{1}{H(\alpha_n)^{r_n(\omega_n - 1)}} \leq \frac{1}{H(\alpha_n)^{r_n(\omega_n - 1)\lambda}} \tag{3.115}$$

hold for sufficiently large n. It follows from (3.101), (3.109), (3.113), (3.114), and (3.115) that

$$|F(\xi) - \eta_n|_p \leq \frac{1}{H(\alpha_n)^{r_n(\omega_n - 1)\lambda}} \tag{3.116}$$

for sufficiently large n. We deduce from (3.116) that $\lim_{n\to\infty} |F(\xi) - \eta_n|_p = 0$. Hence, we get $\lim_{n\to\infty} \eta_n = F(\xi)$. We infer from (3.100) and (3.116) that

$$|F(\xi) - \eta_n|_p \leq \frac{1}{H(\eta_n)^{\gamma_n}} \qquad (\lim_{n\to\infty} \gamma_n = \infty) \tag{3.117}$$

for sufficiently large n, where $\gamma_n = \frac{(\omega_n - 1)\lambda}{2t} \quad (n = 1, 2, 3, \ldots)$.

(3) There exist the following two cases for the sequence $\{|F(\xi) - \eta_n|_p\}$:

(a) $|F(\xi) - \eta_n|_p = 0$ from some n onward:

In this case, $\eta_n = F(\xi)$ from some n onward, that is, $\{\eta_n\}$ is a constant sequence. Since $\eta_n \in \mathbb{Q}(\theta, \beta) \quad (n = 1, 2, 3, \ldots)$, in case a), we see that $F(\xi)$ is a $p-$adic algebraic number in $\mathbb{Q}(\theta, \beta)$.

(b) $|F(\xi) - \eta_n|_p \neq 0$ for infinitely many n:

In this case, $F(\xi)$ is a $p-$adic U^*-number with $\mu^*(F(\xi)) \leq t$, and therefore we have $F(\xi) \in \bigcup_{i=1}^{t} U_i^*$. Hence, in case b), we see that $F(\xi) \in \bigcup_{i=1}^{t} U_i$ since U_i^* is identical with U_i for any natural number i. This completes the proof of Theorem 3.3.

Theorem 3.4. *Let K be a $p-$adic algebraic number field, and let $F(z) = \sum_{h=1}^{\infty} c_h z^h$ $(c_h \in K,\ h = 1, 2, 3, \ldots)$ be a power series in $\mathbb{Q}_p$ satisfying the following conditions*

$$\begin{cases} c_h = 0, & r_n < h < s_n \quad (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = r_n \qquad\quad\ (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = s_n \qquad\quad\ (n = 0, 1, 2, \ldots), \end{cases} \tag{3.118}$$

where $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are two infinite sequences of positive rational integers with

$$1 = s_0 \leq r_1 < s_1 \leq r_2 < s_2 \leq r_3 < s_3 \leq \ldots \quad \textit{and} \quad \lim_{n\to\infty} \frac{s_n}{r_n} = \infty. \tag{3.119}$$

Suppose that the radius of convergence R of the series $F(z)$ is positive,

$$\limsup_{h\to\infty} \frac{\log H(c_h)}{h} < \infty, \quad \textit{and} \quad \limsup_{n\to\infty} (r_{n+1} - s_n) < \infty. \tag{3.120}$$

Let $L = \mathbb{Q}(\beta)$ be a $p-$adic algebraic number field of degree m and α_n $(n = 1, 2, 3, \ldots)$ be $p-$adic algebraic numbers in L, and let $deg(\alpha_n) = m$ $(n = 1, 2, 3, \ldots)$. Moreover, assume that ξ is a $p-$adic U_m-number such that

$$|\xi - \alpha_n|_p \leq \frac{1}{H(\alpha_n)^{r_n\omega_n}} \qquad (n = 1, 2, 3, \ldots), \tag{3.121}$$

where $H(\alpha_n) > 1$ $(n = 1, 2, 3, \ldots)$ *and* $\omega_n = \frac{s_n}{r_n \log H(\alpha_n)}$ $(n = 1, 2, 3, \ldots)$ *with* $\lim_{n\to\infty} \omega_n = \infty$, *and*

$$|\xi|_p < R. \tag{3.122}$$

Then either $F(\xi)$ *is a* $p-$*adic algebraic number in the* $p-$*adic algebraic number field* $K(\beta)$, *or* $F(\xi) \in \bigcup_{i=1}^{t} U_i$, *where* t *is the degree of* $K(\beta)$ *over* $\mathbb{Q}$.

Proof. It follows the same lines of step 2) of the proof of Theorem 3.2 and of steps 2), 3) of the proof of Theorem 3.3.

3.3. Examples

We will make use of the following lemma, due to Zeren [20, Satz 1], in order to construct some examples for our results.

Lemma 3.1. (Zeren [20]). *Let* $F(z) = \sum_{i=0}^{\infty} c_{n_i} z^{n_i}$ *be a power series, where* $c_{n_i} = \frac{b_{n_i}}{a_{n_i}}$, $b_{n_i} \in \mathbb{Z}\backslash\{0\}$, $a_{n_i} \in \mathbb{N}$ $(i = 0, 1, 2, \ldots)$, *and* $\{n_i\}_{i=0}^{\infty}$ *is an infinite sequence of non-negative rational integers with*

$$0 \le n_0 < n_1 < n_2 < n_3 < \ldots \quad \textit{and} \quad \lim_{i\to\infty} \frac{n_{i+1}}{n_i} = \infty.$$

Suppose that the radius of convergence R *of the series* $F(z)$ *is positive and*

$$\limsup_{i\to\infty} \frac{\log A_{n_i}}{n_i} < \infty \qquad (A_{n_i} = [a_{n_0}, a_{n_1}, \ldots, a_{n_i}],\ \ i = 1, 2, 3, \ldots).$$

Moreover, assume that α *is an algebraic number of degree* m *with* $0 < |\overline{\alpha}| < R$ *such that the absolute values of its conjugates are pairwise different. Then* $F(\alpha) \in U_m$.

We give the following example for our result Theorem 3.1.

Example 3.1. Let K be the algebraic number field $\mathbb{Q}(\sqrt[g]{p})$ of degree g, where p is a prime number and $g \ge 2$ is a rational integer, and let $F(z) = \sum_{h=0}^{\infty} c_h z^h$ $(c_h \in K,\ h = 0, 1, 2, \ldots)$ be a power series with

$$\begin{cases} c_h = 0, & r_n < h < s_n \quad (n = 1, 2, 3, \ldots), \\ c_h = \sqrt[g]{p}, & s_n \le h \le r_{n+1} \quad (n = 0, 1, 2, \ldots), \end{cases}$$

where $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are two infinite sequences of non-negative rational integers, determined by

$$s_0 = 0,\ s_n = ((n+2)!)^{(n+2)!} \quad \text{and} \quad r_n = 2\left((n+1)!\right)^{(n+1)!} \quad (n = 1, 2, 3, \ldots).$$

Suppose that α is an algebraic number of degree m with $0 < |\overline{\alpha}| \le e^{-1} < 1$ such that the absolute values of its conjugates are pairwise different. Then the series $F(z) =$

$\sum_{h=0}^{\infty} c_h z^h$ and the Mahler's U_m-number $\xi = \sum_{\nu=1}^{\infty} \alpha^{r_\nu}$, ξ is a U_m-number by Lemma 3.1, satisfy the conditions of Theorem 3.1. (The algebraic numbers α_n $(n = 1, 2, 3, \ldots)$ in the hypothesis of Theorem 3.1 may be taken as $\alpha_n = \sum_{\nu=1}^{n} \alpha^{r_\nu} \in L := \mathbb{Q}(\alpha)$ $(n = 1, 2, 3, \ldots)$. By the proof of Satz 1 in Zeren [20, pp. 93-101], $deg(\alpha_n) = m$ from some n onward and the sequence $\{H(\alpha_n)\}_{n=1}^{\infty}$ is not bounded above. Hence, we can assume that $deg(\alpha_n) = m$ $(n = 1, 2, 3, \ldots)$ and $H(\alpha_n) > 1$ $(n = 1, 2, 3, \ldots)$ by working with an appropriate subsequence of $\{\alpha_n\}_{n=1}^{\infty}$ if necessary. It is also remarkable to notice that $H(\alpha_n) \leq a^{r_n}$ $(n = 1, 2, 3, \ldots)$, where $a > 1$ is a real constant, by Lemma 2.1.) Then either $F(\xi)$ is an algebraic number in the algebraic number field $K(\alpha) = \mathbb{Q}(\sqrt[g]{p}, \alpha)$, or $F(\xi) \in \bigcup_{i=1}^{t} U_i$, where t is the degree of $\mathbb{Q}(\sqrt[g]{p}, \alpha)$ over $\mathbb{Q}$.

Example 3.2. In Example 3.1, if we take the sequences $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ as $s_0 = 1,\ s_n = ((n+1)!)^{(n+1)!}$ $(n = 1, 2, 3, \ldots)$ and $r_n = (n!)^{n!} + 1$ $(n = 1, 2, 3, \ldots)$, then this yields an example for Theorem 3.2.

We may also try to adapt Example 3.1 and Example 3.2 to the field $\mathbb{Q}_p$ of $p-$adic numbers in order to obtain examples for Theorem 3.3 and Theorem 3.4.

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