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IMPULSIVE CONTROLLABILITY OF MULTI-VALUED FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. This paper is concerned with the study of controllability for a class of multi-valued functional differential systems with impulses and delayed control. By making use of fixed point theorem for multi-valued maps, we prove that our system is completely controllable. The result is obtained in the sense of Carathéodory, and without the requirement that the linear part is controllable.

1. INTRODUCTION

Since the impulse phenomenon is a universal one in nature, the differential equations with impulses, including impulsive differential equations and inclusions, have become an important model described the real processes in population dynamics, economics, physics and so on, and have been holding the scholars' interests, see, e.g., [1, 2, 7, 9, 10, 11, 12] and the references contained therein. We remark that the perturbed or impulsive effect can make a noncontrollable system to be a controllable one (see, [11, 12]). Motivated by the papers mentioned above, the present paper will consider a class of impulsive controllable multi-valued functional differential systems with delayed control.

Let us introduce some notations before entering our statements. Let $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ be, respectively, n_1-dimension and n_2-dimension real vector space, and $\mathcal{P}(\mathbb{R}^{n_1})$ the set of all nonempty subset of $\mathbb{R}^{n_1}$. For $a < b$, let $L^1_{n_i}[a,b] := L^1([a,b],\mathbb{R}^{n_i})$, $i = 1,2$, denote the Banach space of Lebesgue integrable functions $v : [a,b] \to \mathbb{R}^{n_i}$ with the norm $||\cdot||_1$ defined by

$$||v||_1 = \int_a^b |v(s)|\mathrm{d}s,$$

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where $|\cdot|$ stands for a norm in $\mathbb{R}^{n_i}$. In particular, by $L^\infty[a,b] := L^\infty([a,b],\mathbb{R}^{n_1\times n_2})$ we denote the Banach space of essentially bounded matrix functions $V : [a,b] \to \mathbb{R}^{n_1\times n_2}$ with the norm $||\cdot||_\infty$ defined by

$$||V||_\infty = \lim_{p\to\infty}\left(\int_a^b |V(s)|^p \mathrm{d}s\right)^{\frac{1}{p}},$$

where $|V(\cdot)|$ indicates the matrix norm derived by the vector norm $|\cdot|$ mentioned above.

For given $r > 0$, by D_1 we denote the set of functions $\phi : [-r,0] \to \mathbb{R}^{n_1}$ with ϕ continuous everywhere except for finite number of points θ at which $\phi(\theta^-)$ and $\phi(\theta^+)$ exist and $\phi(\theta) = \phi(\theta^+)$. Further, we let $\mathrm{D}_2 := L^1([-r,0],\mathbb{R}^{n_2})$.

For a given $\tau > 0$, set $0 = t_0 < t_1 < t_2 < \ldots < t_n < \tau$. Now consider the impulsive control problem described by

$$\text{(1)}\qquad \begin{cases} x'(t) \in F(t,x_t,u_t) + B(t)u(t),\ t\in[0,\tau] \ \text{ and } \ t \neq t_k, \\ x(t_k) = x(t_k^-) + I_k(x(t_k^-)),\ k = 1,2,3,\cdots,n, \end{cases}$$

where $B(\cdot) \in L^\infty[0,\tau]$, $u(\cdot) \in L^1_{n_2}[-r,\tau]$, $F :\ [0,\tau]\times \mathrm{D}_1 \times \mathrm{D}_2 \to \mathcal{P}(\mathbb{R}^{n_1})$ is a multi-valued map, $I_k : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ is continuous and

$$x(t_k) = x(t_k^+) = \lim_{t\to t_k^+} x(t)\ ,\ x(t_k^-) = \lim_{t\to t_k^-} x(t).$$

As usual, the delay function $x_t \in \mathrm{D}_1$ (likewise for $u_t \in \mathrm{D}_2$) is defined by $x_t(\theta) = x(t+\theta)$ for $\theta\in[-r,0]$.

Set $J = [-r,\tau]\setminus\{t_k :\ k = 1,2,3,...,n\}$. By $PC[-r,\tau]$ we denote the set

$$\left\{x : [-r,\tau] \to \mathbb{R}^{n_1} : x\in C(J),\ x(t_k^-) \ \text{ and } \ x(t_k^+) \ \text{ exist and } \ x(t_k) = x(t_k^+)\right\}.$$

Endowed with the norm

$$||x|| = \sup_{t\in[-r,\tau]} |x(t)|,$$

$PC[-r,\tau]$ is a Banach space. The norm of D_1 can be defined accordingly.

For given $\varphi \in \mathrm{D}_1$ and $u \in L^1_{n_2}[-r,\tau]$ with $u_0(\theta) = 0$ for $\theta\in[-r,0)$, by a solution of (1) we mean that, there exists a function $x \in PC[-r,\tau]$ such that

$$\text{(2)}\qquad \begin{cases} x'(t) = v(t) + B(t)u(t) \ \text{ a.e. } \ t\in[0,\tau]\setminus\{t_1,t_2,\ldots,t_n\}, \\ x(t_k) = x(t_k^-) + I_k(x(t_k^-)),\ k = 1,2,3,\cdots,n, \\ x_0(\theta) = \varphi(\theta) \ \text{ for } \ \theta\in[-r,0], \end{cases}$$

where $v \in L^1_{n_1}[0,\tau]$ satisfying $v(t)\in F(t,x_t,u_t)$ on $[0,\tau]$. For clarity, we will denote by $x(t,\varphi,u)$ the solution of (1) corresponding to the initial value $x_0 \equiv \varphi$ and the control u.

System (1) is said to be controllable if, for any given $\varphi \in \mathrm{D}_1$ and $x^1 \in \mathbb{R}^{n_1}$, there exists a control $u \in L^1_{n_2}[-r,\tau]$ with $u_0(\theta)=0$ for $\theta \in [-r,0)$ such that the solution $x(t,\varphi,u)$ of (1) satisfies $x(\tau,\varphi,u)=x^1$.

2. Preliminaries

We now recall some concepts, which will be used in the sequel. For the details we refer to [1, 2, 3, 5].

Let $(X,||\cdot||)$ be a Banach space and $\mathcal{P}(X)$ be the set of all nonempty subset of X. A multi-valued map $G : X \to \mathcal{P}(X)$ is said to be closed if $G(x)$ is a closed subset of X for each $x \in X$. The convex and compact G can be defined accordingly. G is said to be upper semi-continuous (u.s.c. for short) if $G(x)$ is a nonempty and closed subset of X for each $x \in X$ and if, for each open subset $N \subseteq X$ containing $G(x)$, there exists a neighborhood M of x such that $G(M) \subseteq N$.

G is said to be bounded on bounded sets if for each bounded subset Δ of X, $G(\Delta)$ is bounded, i.e.,

$$\sup_{x\in\Delta}\{\sup ||y|| : y \in G(x)\} < \infty.$$

G is said to be relatively compact on bounded sets if $G(\Delta)$ is relatively compact for each bounded subset Δ of X. G is said to be completely continuous if G is upper semi-continuous and G is relatively compact on bounded subsets. G has a closed graph if

$$x_n \to x_0,\ y_n \to y_0,\ y_n \in G(x_n) \text{ imply } \ y_0 \in G(x_0).$$

It is well known that if G is relatively compact on bounded sets with nonempty and compact-values, then G is upper semi-continuous if and only if G has a closed graph.

To meet the needs of discussions in what follows, we give the total assumptions as follows:

(A1) The multi-valued map F in (1) has compact and convex values for each $(t,\varphi_1,\varphi_2) \in [0,\tau]\times \mathrm{D}_1 \times \mathrm{D}_2$.

(A2) There exist $a_i \in L^1([0,\tau],[0,\infty))$ for $i=0,1,2$, and a constant $\alpha \in (0,1]$ such that

$$\begin{aligned} ||F(t,\varphi_1,\varphi_2)|| &= \sup\{|v| : v \in F(t,\varphi_1,\varphi_2)\} \\ &\le a_0(t) + a_1(t)||\varphi_1||^\alpha + a_2(t)||\varphi_2||_1^\alpha \\ &\qquad \text{for all } \ \varphi_1 \in \mathrm{D}_1, \varphi_2 \in \mathrm{D}_2 \ \text{ and a. e. } \ t \in [0,\tau]. \end{aligned} \tag{3}$$

Furthermore, F satisfies that

(i) $t \to F(t,\varphi_1,\varphi_2)$ is measurable for each $\varphi_1 \in \mathrm{D}_1$ and $\varphi_2 \in \mathrm{D}_2$, and

(ii) $(\varphi_1,\varphi_2) \to F(t,\varphi_1,\varphi_2)$ is upper semi-continuous for almost all $t \in [0,\tau]$.

(A3) For the functions I_k in (1), there exist constants $b_i^{(k)} \geq 0$ for $i = 1, 2$, and $\alpha_k \in (0, 1]$ for $k = 1, 2, 3, ..., n$, such that

$$|I_k(x)| \leq b_1^{(k)} + b_2^{(k)} |x|^{\alpha_k} \quad \text{for} \quad x \in \mathbb{R}^{n_1}.$$

For $x \in PC[-r, \tau]$ and $u \in L^1_{n_2}[-r, \tau]$ the set $S_F(x, u)$ of selection functions is defined by

$$S_F(x, u) = \{v \in L^1_{n_1}[0, \tau] : v(t) \in F(t, x_t, u_t) \quad \text{for} \quad t \in [0, \tau]\}.$$

Let $CC(X)$ denote the set of nonempty compact and convex subset of the Banach space X. We remark that F is L^1- Carathéodory under the assumption (A2)(see [2]). Hence, referring to [6] we have the following results.

Lemma 1. *Under the assumptions (A1) and (A2), it follows that*

(i) $S_F(x, u) \neq \emptyset$ *for each* $x \in PC[-r, \tau]$ *and* $u \in L^1_{n_2}[-r, \tau]$*; and*

(ii) *if* $K : L^1_{n_1}[0, \tau] \to C[0, \tau] \times L^1_{n_2}[0, \tau]$ *is a linear continuous mapping, then*

$$K \circ S_F : PC[-r, \tau] \times L^1_{n_2}[-r, \tau] \to CC(C[0, \tau] \times L^1_{n_2}[0, \tau])$$

is a closed graph operator.

Next conclusion is due to Martelli [8].

Lemma 2. *Let* X *be a Banach space and* $\Psi : X \to CC(X)$ *be a completely continuous multi-valued map. If the set*

$$E = \{x \in X : x \in \lambda \Psi(x) \text{ for some } 0 < \lambda < 1\}$$

is bounded, then Ψ *has a fixed point.*

3. Main Results

Now we are in a position to consider our main result. By the fundamental theorem for the Lebesgue integral we know that, (2) is equivalent to the following formula:

$$\text{(4)} \quad \begin{cases} x(t) = \varphi(0) + \displaystyle\int_0^t [v(s) + B(s)u(s)] \, \mathrm{d}s + \sum_{k:\ t_k \in (0,\ t)} I_k(x(t_k^-)), \ t \in [0, \tau], \\ x_0(\theta) = \varphi(\theta) \quad \text{and} \quad u_0(\theta) = 0 \quad \text{for} \quad \theta \in [-r, 0). \end{cases}$$

Next we consider the controllability of (1). To do this, we will require the condition

$$\text{(5)} \qquad B(t)B^T(t) = E_0 \quad \text{on} \quad [0, \tau],$$

where T represents the transpose of matrices and E_0 an identity matrix. Usually one transforms the controllability problem to the existence of fixed point, see, for example, the references [1, 2, 4, 9, 11]. In what follows we will keep on the approach.

Theorem 1. *Suppose that the assumptions (A1)–(A3) and (5) are fulfilled. Then system (1) is controllable if one of the following conditions holds:*

(i) $\max_k\{\alpha, \alpha_k\} < 1$;

(ii) $\alpha = 1,\ \max_k\{\alpha_k\} < 1$ *and* $(2 + ||B^T||_\infty) \int_0^\tau [a_1(s) + a_2(s)]\mathrm{d}s < 1$;

(iii) $\alpha < 1,\ \max_k\{\alpha_k\} = 1$ *and* $(2 + ||B^T||_\infty) \sum_{k:\ \alpha_k=1} b_k^{(2)} < 1$;

(iv) $\alpha = 1,\ \max_k\{\alpha_k\} = 1$ *and* $(2 + ||B^T||_\infty) \int_0^\tau [a_1(s) + a_2(s)]\mathrm{d}s$
$+ (2 + ||B^T||_\infty) \sum_{k:\ \alpha_k=1} b_k^{(2)} < 1.$

Proof. For any given $\varphi \in \mathrm{D}_1$ and $x^1 \in \mathbb{R}^n$, let the multi-valued map $\Psi : PC[-r,\tau] \times L^1_{n_2}[-r,\tau] \to \mathcal{P}(PC[-r,\tau] \times L^1_{n_2}[-r,\tau])$ be defined by

(6) $$\Psi(x,u) = \left\{(\psi_1, \psi_2) : \psi_1 \in PC[-r,\tau] \ \text{ and } \ \psi_2 \in L^1_{n_2}[-r,\tau]\right\},$$

where, for $v \in S_F(x,u)$, ψ_1 and ψ_2 are defined, respectively, by

(7) $$\begin{cases} \psi_1(x,u)(t) = \varphi(0) + \displaystyle\int_0^t [v(s) + B(s)\psi_2(x,u)(s)]\,\mathrm{d}s \\ \qquad\qquad\qquad + \displaystyle\sum_{k:\ t_k \in (0,\ t)} I_k(x(t_k^-)),\ t \in [0,\tau], \\ \psi_1(x,u)(t) = \varphi(t),\ t \in [-r,0), \end{cases}$$

(8) $$\begin{cases} \psi_2(x,u)(t) = \dfrac{1}{\tau}B^T(t)\left(x^1 - \varphi(0) - \displaystyle\int_0^\tau v(s)\mathrm{d}s\right) \\ \qquad\qquad\qquad -\dfrac{1}{\tau}B^T(t)\displaystyle\sum_{k=1}^n I_k(x(t_k^-)),\ t \in [0,\tau], \\ \psi_2(x,u)(t) = 0,\ t \in [-r,0). \end{cases}$$

Note that, equipped with the norm $||(x,u)|| = ||x|| + ||u||_1$ for $(x,u) \in PC[-r,\tau] \times L^1_{n_2}[-r,\tau]$, then $PC[-r,\tau] \times L^1_{n_2}[-r,\tau]$ becomes a Banach space. Note further that ψ_i defined by (7) and (8) make sense by Lemma 1(i), so $\Psi(x,u) \neq \emptyset$. Now if we can find a fixed point $(x,u) \in \Psi(x,u)$, then, from (4) we learn that $x(t) = [\psi_1(x,u)](t)$ is a solution of (1) with the initial condition $x_0 \equiv \varphi$ and, by the straightforward verification, have $x(\tau) = x^1$. For the proof, we proceed in steps.

Assertion 1: $\Psi(\Delta)$ is bounded in $PC[-r,\tau] \times L^1_{n_2}[-r,\tau]$ for any bounded subset Δ of $PC[-r,\tau] \times L^1_{n_2}[-r,\tau]$. It is enough to show that $\psi_i(\Delta)$ are bounded for $i = 1, 2$. Indeed, we may assume that

$$\max_{a.e.\ t \in [0,\tau]} \left\{1, \frac{1}{\tau}|B^T(t)|\right\} \int_0^\tau a_i(s)\mathrm{d}s \leq M \quad \text{for} \quad i = 0, 1, 2$$

and

$$\max_{a.e.\ t\in[0,\tau]}\left\{1,\frac{1}{\tau}|B^T(t)|\right\}\sum_{k=1}^{n}|I_k(x(t_k^-))|\le M,$$

where $(x,u)\in\Delta$. By the assumptions above we have from (7) and (8) that

$$\begin{cases} |\psi_1(x,u)(t)| \le 2|\varphi(0)|+|x^1|+4M+2M(||x||^\alpha+||u||_1^\alpha),\ t\in[0,\tau], \\ |\psi_1(x,u)(t)| = |\varphi(t)|,\ t\in[-r,0) \end{cases}$$

as well as

$$\begin{cases} |\psi_2(x,u)(t)| \le \dfrac{1}{\tau}|B^T(t)(x^1-\varphi(0))|+2M+M(||x||^\alpha+||u||_1^\alpha),\ a.a.\ t\in[0,\tau], \\ |\psi_2(x,u)(t)| = 0,\ t\in[-r,0), \end{cases}$$

which produce that $\psi_1(x,u)(t)$ and $\psi_2(x,u)(t)$ are bounded on Δ.

Assertion 2: Ψ defined as (6) is relatively compact for each bounded subset Δ of $PC[-r,\tau]\times L^1_{n_2}[-r,\tau]$. As a matter of fact, it is sufficient to show that ψ_i maps Δ into a relatively compact subset of $PC[-r,\tau]$ for $i=1$ and of $L^1_{n_2}[-r,\tau]$ for $i=2$. To achieve our objective, we first note that $\psi_i(\Delta)$ is bounded for $i=1,2$. Now for any given $\varepsilon>0$ we take $\delta=\delta(\varepsilon)$ such that when $|t^{(2)}-t^{(1)}|<\delta$,

$$\left|\int_{t^{(1)}}^{t^{(2)}}[v(s)+B(s)\psi_2(x,u)(s)]\,\mathrm{d}s\right|<\varepsilon.$$

On the other hand, it is clear that

$$\sum_{k:\ t_k\in(0,\ t^{(2)})}I_k(x(t_k^-))-\sum_{k:\ t_k\in(0,\ t^{(1)})}I_k(x(t_k^-))=0$$

provided $t^{(1)},t^{(2)}\in[t_{k-1},t_k)$, $k=1,2,\ldots,n$, or $t^{(1)},t^{(2)}\in[t_n,\tau]$.

Hence from (7) we have when $t^{(1)},t^{(2)}\in[t_{k-1},t_k)$, $k=1,2,\ldots,n$, or $t^{(1)},t^{(2)}\in[t_n,\tau]$,

$$|\psi_1(x,u)(t^{(2)})-\psi_1(x,u)(t^{(1)})|<\varepsilon \quad\text{for}\quad |t^{(2)}-t^{(1)}|<\delta \quad\text{and}\quad (x,u)\in\Delta,$$

which, together with Ascoli-Azela theorem, means that $\psi_1(\Delta)$ is a relatively compact subset of $PC[-r,\tau]$.

For the Banach space $((L^1_{n_2}[-r,\tau],||\cdot||_1)$, we have

$$\sum_{k:\ t_k\in(0,\ t^{(2)})}I_k(x(t_k^-))-\sum_{k:\ t_k\in(0,\ t^{(1)})}I_k(x(t_k^-))=0$$

for almost all $t^{(1)}, t^{(2)} \in [-r, \tau]$ with $|t^{(2)} - t^{(1)}| < \delta$. Therefore, by similar arguments we can readily show that $\psi_2(\Delta)$ is a relatively compact subset of $L^1_{n_2}[-r, \tau]$. Thus Ψ transforms any bounded subset $\Delta \subseteq PC[-r, \tau] \times L^1_{n_2}[-r, \tau]$ into a relatively compact subset of $PC[-r, \tau] \times L^1_{n_2}[-r, \tau]$.

Assertion 3: Ψ has a closed graph. For this end, we suppose that

$$(x^{(n)}, u^{(n)}) \to (x^*, u^*), \ (y^{(n)}, z^{(n)}) \to (y^*, z^*) \ \text{ and } \ (y^{(n)}, z^{(n)}) \in \Psi(x^{(n)}, u^{(n)}). \tag{9}$$

Now consider an operator $K : L^1_{n_1}[0, \tau] \to C[0, \tau] \times L^1_{n_2}[0, \tau]$, $v \to Kv = (K_1 v, K_2 v)$, here K_1 and K_2 are defined, respectively, by

$$(K_1 v)(t) = \int_0^t v(s)\mathrm{d}s$$

and

$$(K_2 v)(t) = -\frac{1}{\tau} B^T(t) \int_0^\tau v(s)\mathrm{d}s.$$

Then K is linear and continuous. Furthermore, according to Lemma 1(ii) it follows that $K \circ S_F$ is a closed graph operator.

Now that $(y^{(n)}, z^{(n)}) \in \Psi(x^{(n)}, u^{(n)})$, from (7) and (8) there exists a $v^{(n)}(t) \in S_F(x^{(n)}, u^{(n)})$ such that

$$\begin{aligned} y^{(n)}(t) &= \psi_1(x^{(n)}, u^{(n)})(t) \\ &= \varphi(0) + \int_0^t \left[v^{(n)}(s) + B(s)\psi_2(x^{(n)}, u^{(n)})(s) \right] \mathrm{d}s \\ &\quad + \sum_{k:\ t_k \in (0,\ t)} I_k(x^{(n)}(t_k^-)), \ t \in [0, \tau] \end{aligned} \tag{10}$$

and

$$\begin{aligned} z^{(n)}(t) &= \psi_2(x^{(n)}, u^{(n)})(t) \\ &= \frac{1}{\tau} B^T(t) \left(x^1 - \varphi(0) - \int_0^\tau v^{(n)}(s)\mathrm{d}s \right) \\ &\quad - \frac{1}{\tau} B^T(t) \sum_{k=1}^n I_k(x^{(n)}(t_k^-)), \quad t \in [0, \tau]. \end{aligned} \tag{11}$$

From (10) and (11) we have, respectively, that

$$\begin{aligned} Y^{(n)}(t) &:= y^{(n)}(t) - \varphi(0) - \int_0^t B(s)\psi_2(x^{(n)}, u^{(n)})(s)\mathrm{d}s \\ &\quad - \sum_{k:\ t_k \in (0,\ t)} I_k(x^{(n)}(t_k^-)) \in K_1(S_F(x^{(n)}, u^{(n)})) \end{aligned} \tag{12}$$

and

$$(13)\quad \begin{aligned} Z^{(n)}(t) &:= z^{(n)}(t)+\frac{1}{\tau}\left(-B^T(t)\left(x^1-\varphi(0)\right)+B^T(t)\sum_{k:\ t_k\in(0,\ \tau)} I_k(x^{(n)}(t_k^-))\right)\\ &\in\ K_2(S_F(x^{(n)},u^{(n)})). \end{aligned}$$

Let

$$Y^*(t) := y^*(t)-\varphi(0)-\int_0^t B(s)\psi_2(x^*,u^*)(s)\mathrm{d}s-\sum_{k:\ t_k\in(0,\ t)} I_k(x^*(t_k^-))$$

and

$$Z^*(t) := z^*(t)+\frac{1}{\tau}\left(-B^T(t)\left(x^1-\varphi(0)\right)+B^T(t)\sum_{k:\ t_k\in(0,\ \tau)} I_k(x^*(t_k^-))\right).$$

Then, together with (9), (12) and (13), we attain

$$||Y^{(n)}-Y^*||\to 0 \ \text{ as } \ n\to\infty$$

and

$$||Z^{(n)}-Z^*||_1\to 0 \ \text{ as } \ n\to\infty,$$

which, together with the fact that $K\circ S_F$ is a closed graph operator, result in

$$(Y^*,Z^*)\in K(S_F(x^*,u^*)).$$

That is, there exists a $v^*(t)\in S_F(x^*,u^*)$ such that, when the symbol "(n)" in (10) and (11) is replaced by "$*$", y^* and z^* satisfy, respectively, (10) and (11). Therefore Assertion 3 holds.

Assertion 4: Ψ is completely continuous. In fact, since F is compact, from Assertion 2 we see that Ψ is a multi-valued map with nonempty and compact values, Hence, with the aid of Assertion 3, Ψ is upper semi-continuous. Invoking Assertion 2 again we obtain our desired conclusion.

Assertion 5: Ψ has a fixed point. Note first that F is a multi-valued map with convex values, it is easy to show Ψ defined as (6) is convex. Thus Ψ maps $PC[-r,\tau]\times L^1_{n_2}[-r,\tau]$ into $CC(PC[-r,\tau]\times L^1_{n_2}[-r,\tau])$. Next we show that Ψ satisfies Lemma 2. To do this, we assume that $(x,u)\in PC[-r,\tau]\times L^1_{n_2}[-r,\tau]$ such that

$$(x,u)\in\lambda\Psi(x,u) \ \text{ for some } \ \lambda\in(0,1).$$

Let

$$E=\{(x,u):(x,u)\in\lambda\Psi(x,u) \ \text{ for some } \ \lambda\in(0,1)\}.$$

Then, by (6)–(8) there exists a $v \in S_F(x,u)$ so that

(14) $$x(t) = \lambda\psi_1(x,u)(t) \quad \text{and} \quad u(t) = \lambda\psi_2(x,u)(t) \quad \text{for} \quad t \in [0,\tau].$$

Now from (7) and (8) we have

(15) $$||\psi_1(x,u)|| \le m_1 + 2||x||^{\alpha}\int_0^{\tau} a_1(s)\mathrm{d}s + 2||u||_1^{\alpha}\int_0^{\tau} a_2(s)\mathrm{d}s + 2\sum_{k=1}^{n} b_k^{(2)}||x||^{\alpha_k}$$

and

(16) $$\begin{aligned}&||\psi_2(x,u)||_1\\ &\le m_2 + ||B^T||_\infty\left(||x||^{\alpha}\int_0^{\tau} a_1(s)\mathrm{d}s + ||u||_1^{\alpha}\int_0^{\tau} a_2(s)\mathrm{d}s + \sum_{k=1}^{n} b_k^{(2)}||x||^{\alpha_k}\right),\end{aligned}$$

where m_i are constant for $i \in \{1,2\}$, and independent of (x,u). For simplicity we let

$$M_1 = m_1 + m_2, \ M_2 = \int_0^{\tau}[a_1(s) + a_2(s)]\mathrm{d}s \quad \text{and} \quad M_3 = \sum_{k=1}^{n} b_k^{(2)}.$$

Next we discuss in steps.

(i) in case $\max_k\{\alpha, \alpha_k\} < 1$, without loss of generality we may assume that $||(x,u)|| \ge 1$ and $\alpha \ge \alpha_k$ for all k. Then it follows from (15) and (16) that

(17) $$||(x,u)|| \le M_1 + M_2(2 + ||B^T||_\infty)||(x,u)||^{\alpha} + M_3(2 + ||B^T||_\infty)||(x,u)||^{\alpha}.$$

Note that $\alpha < 1$, from (17) we learn that $||(x,u)||$ have a same bound for all $(x,u) \in \lambda\Psi(x,u)$. In other words, $||(x,u)||$ is bounded on E.

(ii) For the case $\alpha = 1$, $\max_k\{\alpha_k\} < 1$ and

$$(2 + ||B^T||_\infty)\int_0^{\tau}[a_1(s) + a_2(s)]\mathrm{d}s < 1,$$

we let $\max_k\{||(x,u)||^{\alpha_k}\} = ||(x,u)||^{\beta}$. Then from (15) and (16) it holds that

$$[1 - (2 + ||B^T||_\infty)M_2]||(x,u)|| \le M_1 + (2 + ||B^T||_\infty)M_3||(x,u)||^{\beta},$$

which shows again that $||(x,u)||$ have a same bound for all $(x,u) \in \lambda\Psi(x,u)$.

(iii) Corresponding to the case $\alpha < 1$, $\max_k\{\alpha_k\} = 1$ and

$$((2 + ||B^T||_\infty))\sum_{k:\ \alpha_k = 1} b_k^{(2)} < 1,$$

for simplicity we may only consider the special case that $\alpha_k = 1$ for all k. Then it follows that

$$[1 - (2 + ||B^T||_\infty)M_3]||(x,u)|| \le M_1 + (2 + ||B^T||_\infty)M_2||(x,u)||^{\alpha}.$$

Analogously, in this case $||(x,u)||$ is bounded on E.

(iv) For the fourth case the boundedness of $||(x, u)||$ on E can be shown similarly.

In summary, as long as one of the conditions (i)–(iv) holds, Lemma 2 will implies that there exists a point $(x^*, u^*) \in PC[-r, \tau] \times L^1_{n_2}[-r, \tau]$ such that $(x^*, u^*) \in \Psi(x^*, u^*)$. Subsequently, there exists a $v^* \in S_F(x^*, u^*)$ so that

$$x^*(t) = \psi_1(x^*, u^*)(t) \quad \text{and} \quad u^*(t) = \psi_2(x^*, u^*)(t) \quad \text{for} \quad t \in [0, \tau],$$

and $x^*(t) = \varphi(t)$ and $u^*(t) = 0$ for $t \in [-r, 0)$. Immediately, by the definitions of ψ_1 and ψ_2 as in (7) and (8), it holds that $x^*(t)$ solves the system (4) and satisfies $x^*(\tau) = x^1$. In other words, we have found a solution $x(t, \varphi, u^*)$ of (1), which satisfies $x(\tau, \varphi, u^*) = x^1$. The proof is complete.

Finally, we give an example to end our discussions.

Example 1. Suppose in (1) that

$$B(t) \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F(t, \varphi_1, \varphi_1) = \Big\{ v(t) : v(t) = \frac{a_1(t)\varphi_1(t-r) + a_2(t)\varphi_2(t-r)}{1 + ||\varphi_1|| + ||\varphi_2||_1} + \begin{pmatrix} kt \\ kt \end{pmatrix}, k \in [0, 1] \Big\},$$

as well as

$$I_1(x) = \frac{x}{6}, \ I_2(x) = \begin{pmatrix} x_1^{\frac{1}{3}} \\ x_2^{\frac{1}{3}} \end{pmatrix} \quad \text{for} \quad x = (x_1, x_2)^T \in \mathbb{R}^2,$$

where a_i are defined as in assumption (A2), $i = 1, 2$. Then F satisfies the assumptions (A1)–(A2) and $||B^T||_\infty = 1$. Hence, if we let

$$\int_0^\tau (a_1(s) + a_2(s))\mathrm{d}s < \frac{1}{6},$$

then Theorem 1 implies that system (1) is controllable.

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