TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 3, pp. 731-751, June 2014
DOI: 10.11650/tjm.18.2014.1906
This paper is available online at http://journal.taiwanmathsoc.org.tw

ANNIHILATORS IN IDEALS OF COEFFICIENTS OF ZERO-DIVIDING POLYNOMIALS

Yasuyuki Hirano, Chan Yong Hong, Hong Kee Kim,
Nam Kyun Kim and Yang Lee*

Abstract. We continue the study of the McCoy ring property through examining constant annihilators in the ideals of coefficients of zero-dividing polynomials. In the process we introduce the ideal-McCoy property which is between strongly McCoy and McCoy properties, showing that none of implications can be replaced by an equivalence. We give an example of a right ideal-McCoy ring that is not left ideal-McCoy. We also investigate relations between the ideal-McCoy property and other standard ring theoretic properties. For example, we find possible basic forms of finite right ideal-McCoy rings of minimal order.

1. Right Ideal-mccoy Rings

Throughout this note every ring is associative with identity unless otherwise stated. Let R be a ring and we use $R[x]$ to denote the polynomial ring with an indeterminate x over R. Denote the n by n full matrix ring over R by $Mat_n(R)$ and the n by n upper (resp. lower) triangular matrix ring over R by $U_n(R)$ (resp. $L_n(R)$). Use e_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0. $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the set of integers and the ring of integers modulo n, respectively. Note $Mat_n(R)[x] \cong Mat_n(R[x])$ and $U_n(R)[x] \cong U_n(R[x])$, $L_n(R)[x] \cong L_n(R[x])$. We will apply these isomorphisms freely.

McCoy [20, Theorem 2] showed the following fact in 1942:

$$f(x)g(x) = 0 \text{ implies } f(x)r = 0 \text{ for some nonzero } r \in R,$$

where $f(x)$ and $0 \neq g(x)$ are polynomials over a commutative ring R. Many generalizations have been studied based on this result. Nielsen [21] in 2006 called a ring

Received April 11, 2013, accepted October 21, 2013.
Communicated by Bernd Ulrich.
2010 *Mathematics Subject Classification*: 16D25, 16S36.
Key words and phrases: Right ideal-McCoy ring, (Strongly) Right McCoy ring, Polynomial ring, Matrix ring, Dorroh extension.
*Corresponding author.

R (possibly without identity) *right McCoy* when the equation $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero $r \in R$, where $f(x), 0 \neq g(x)$ are polynomials in $R[x]$. Left McCoy rings are defined symmetrically. Nielsen [21, Section 3 and Section 4] showed that the McCoy condition is not left-right symmetric. Hong et al. [9] called a ring R (possibly without identity) *strongly right McCoy* if $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero r in the right ideal of R generated by the coefficients of $g(x)$, where $f(x)$ and $0 \neq g(x)$ are polynomials in $R[x]$. Strongly left McCoy rings are defined symmetrically. This strong McCoy condition is not left-right symmetric by [12, Remark 2.6(3)]. A ring is called *reduecd* if it has no nonzero nilpotent elements. Due to Cohn [3], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Reduced rings are reversible through a simple computation. Reversible rings are strongly left and right McCoy by [9, Theorem 1.6] or the proof of [21, Theorem 2]. A ring is called *right* (resp. *left*) *duo* if each right (resp. left) ideal is two-sided. Right (resp. left) duo rings are strongly right (resp. left) McCoy by [9, Theorem 1.11] or the proof of [2, Theorem 8.2]. A ring is called *Abelian* if every idempotent is central. The class of Abelain rings contains reversible rings and one-sided duo rings. But one-sided strongly McCoy rings need not be Abelian by [9, Example 1.10].

Now we will study a natural generalization of the strongly McCoy property, considering annihilators in two-sided ideals of coefficients. So a ring R (possibly without identity) will be called *right ideal-McCoy* if $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero r in the ideal of R generated by the coefficients of $g(x)$, where $f(x)$ and $0 \neq g(x)$ are polynomials in $R[x]$. Left ideal-McCoy rings are defined symmetrically. In the following we see that the ideal-McCoy property is not left-right symmetric.

Let R be an algebra (with or without identity) over a commutative ring S. Following Dorroh [4], the *Dorroh extension* of R by S is the Abelian group $R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in R$ and $s_i \in S$.

Proposition 1.1. (1) *Let A be an algebra generated by a, b over a commutative domain K, satisfying the relations*

$$a^2 = a, b^2 = 0, \text{ and } ba = 0.$$

Let R be the subalgebra of A which contains all elements with zero constant term. Then the Dorroh extension of R by K is right ideal-McCoy but not left ideal-McCoy.

(2) *If the algebra A satisfies the relations $a^2 = a, b^2 = 0,$ and $ab = 0$ then the Dorroh extension of R by K is left ideal-McCoy but not right ideal-McCoy.*

Proof. (1) Let D be the Dorroh extension of R by K. Every element in A is expressed by

$$k_0 + k_1a + k_2b + k_3ab$$

where $k_i \in K$ for $i = 0, 1, 2, 3$, and $R = \{k_1a + k_2b + k_3ab \mid k_i \in K \text{ for all } i\}$. Note that

$$(k_0 + k_1a + k_2b + k_3ab)a = (k_0 + k_1)a, (k_0 + k_1a + k_2b + k_3ab)ab = (k_0 + k_1)ab, (k_0 + k_1a + k_2b + k_3ab)b = k_0b + k_1ab,$$

and

$$a(k_0 + k_1a + k_2b + k_3ab) = (k_0 + k_1)a + (k_2 + k_3)ab, b(k_0 + k_1a + k_2b + k_3ab) = k_0b, ab(k_0 + k_1a + k_2b + k_3ab) = k_0ab.$$

Now suppose that $0 \neq f(x) = \sum_{i=0}^{m}(a_i, b_i)x^i$ and $0 \neq g(x) = \sum_{j=0}^{n}(c_j, d_j)x^j$ in $D[x]$ with $f(x)g(x) = 0$. We can rewrite $f(x) = (f_1(x), f_2(x))$ and $g(x) = (g_1(x), g_2(x))$, where $f_1(x) = \sum_{i=0}^{m} a_ix^i$, $f_2(x) = \sum_{i=0}^{m} b_ix^i$, $g_1(x) = \sum_{j=0}^{n} c_jx^j$ and $g_2(x) = \sum_{j=0}^{n} d_jx^j$. We can also express $f_1(x), g_1(x)$ by

$$f_1(x) = h_1(x)a + h_2(x)b + h_3(x)ab \text{ and } g_1(x) = k_1(x)a + k_2(x)b + k_3(x)ab$$

where $h_i(x), k_i(x) \in K[x]$ for $i = 1, 2, 3$. Let $h_0(x) = f_2(x)$ and $k_0(x) = g_2(x)$.

We will show that D is right ideal-McCoy. From the equality $0 = f(x)g(x) = (f_1(x)g_1(x) + f_1(x)g_2(x) + f_2(x)g_1(x), f_2(x)g_2(x))$, we have that $h_0(x) = 0$ or $k_0(x) = 0$. Let L be the ideal of D generated by the coefficients of $g(x)$.

Case 1. $h_0(x) = 0$ and $k_0(x) = 0$

We have $0 = f_1(x)g_1(x) = h_1(x)k_1(x)a + (h_1(x)k_2(x) + h_1(x)k_3(x))ab$ in this case. So $h_1(x)k_1(x) = 0$ and $h_1(x)(k_2(x) + k_3(x)) = 0$.

Subcase 1-1. $h_1(x) = 0$

We have $f_1(x) = h_2(x)b + h_3(x)ab$, and so $f(x)L = 0$ since $L \subseteq (Ka + Kb + Kab, 0)$.

Subcase 1-2. $h_1(x) \neq 0$

From $h_1(x) \neq 0$, we have $k_1(x) = 0$ and $k_2(x) = -k_3(x) \neq 0$. So $g_1(x) = k_2(x)b - k_2(x)ab = k_2(x)(b-ab)$ and so L contains $(\alpha(b-ab), 0)$ for some $0 \neq \alpha \in K$. Now we get

$$f(x)(\alpha(b - ab), 0) = (\alpha(h_1(x)a + h_2(x)b + h_3(x)ab)(b - ab), 0) = 0.$$

Case 2. $h_0(x) = 0$ and $k_0(x) \neq 0$

In this case we have

$$0 = f(x)g(x) = (h_1(x)(k_0(x) + k_1(x))a + (h_1(x)(k_2(x) + k_3(x)) + h_3(x)k_0(x))ab + h_2(x)k_0(x)b, 0).$$

This yields

$$h_1(x)(k_0(x)+k_1(x)) = h_1(x)(k_2(x)+k_3(x)) + h_3(x)k_0(x) = h_2(x)k_0(x) = 0.$$

Then $h_2(x) = 0$ from $k_0(x) \neq 0$. Here assume $h_1(x) = 0$, i.e., $f_1(x) = h_3(x)ab$. Then $h_3(x)k_0(x)ab = 0$ and this yields $h_3(x) = 0$, entailing $f(x) = 0$. This induces a contradiction, and so $h_1(x) \neq 0$. This yields $k_1(x) = -k_0(x)$ and $g_1(x) = -k_0(x)a + k_2(x)b + k_3(x)ab$. Whence L contains $(\alpha(b-ab), 0)$ for some $0 \neq \alpha \in K$ since $g(x)(b,0) = (g_1(x)b + k_0(x)b, 0) = (-k_0(x)ab + k_0(x)b, 0) = (k_0(x)(b-ab), 0)$ and $k_0(x) \neq 0$. Now we get

$$f(x)(\alpha(b-ab), 0) = (\alpha(h_1(x)a + h_2(x)b + h_3(x)ab)(b-ab), 0) = 0.$$

Case 3. $h_0(x) \neq 0$ and $k_0(x) = 0$

In this case we have

$$\begin{aligned} 0 &= f(x)g(x) = ((h_0(x)k_1(x) + h_1(x)k_1(x))a + (h_0(x)k_3(x) \\ &\quad + h_1(x)k_2(x) + h_1(x)k_3(x))ab + h_0(x)k_2(x)b, 0). \end{aligned}$$

This yields

$$h_0(x)k_1(x) + h_1(x)k_1(x) = h_0(x)k_3(x) + h_1(x)k_2(x) + h_1(x)k_3(x) = h_0(x)k_2(x) = 0.$$

Then $k_2(x) = 0$ from $h_0(x) \neq 0$, entailing $g_1(x) = k_1(x)a + k_3(x)ab \neq 0$. Further, we get

$$(h_0(x) + h_1(x))k_1(x) = 0 \text{ and } (h_0(x) + h_1(x))k_3(x) = 0.$$

Here assume $h_0(x) + h_1(x) \neq 0$. Then $k_1(x) = 0$ and $k_3(x) = 0$; hence $g(x) = 0$, a contradiction. So $h_0(x) + h_1(x) = 0$ and $f_1(x) = -h_0(x)a + h_2(x)b + h_3(x)ab$. Since $k_1(x) \neq 0$ or $k_3(x) \neq 0$, L contains $(\beta ab, 0)$ for some $0 \neq \beta \in K$ from $g(x)(b,0) = (g_1(x)b, 0) = (k_1(x)ab, 0)$. Then

$$\begin{aligned} f(x)(\beta ab, 0) &= (f_1(x), h_0(x))(\beta ab, 0) \\ &= ((-h_0(x)a + h_2(x)b + h_3(x)ab)\beta ab + h_0(x)\beta ab, 0) \\ &= (-\beta h_0(x)ab + \beta h_0(x)ab, 0) = 0. \end{aligned}$$

Now by the computations of Cases 1, 2, 3, we can conclude that D is right ideal-McCoy.

Next consider two nonzero polynomials

$$f(x) = (a, 0) + (ab, 0)x \text{ and } g(x) = (-a, 1) - (ab, 0)x$$

in $D[x]$. Then $f(x)g(x) = 0$. Consider the ideal J of D generated by the coefficients of $f(x)$. Then $J = (Ka + Kab, 0) = (Ka, 0) + (Kab, 0)$ by the computation above, so we have

$$\begin{aligned}&((\alpha a, 0) + (\beta ab, 0))g(x)\\ &= ((\alpha a, 0) + (\beta ab, 0))((-a, 1) - (ab, 0)x) = (\beta ab, 0) - (\alpha ab, 0)x \neq 0\end{aligned}$$

for every nonzero $\alpha a + \beta ab \in J$ with $\alpha, \beta \in K$ since $\alpha \neq 0$ or $\beta \neq 0$. This implies that D is not left ideal-McCoy.

The proof of (2) is similar. ∎

Example 1.2. (1) Let K be a commutative domain. Let $a = e_{11} + e_{12}, b = e_{23}$ in $Mat_3(K)$. Then $a^2 = a, b^2 = 0$, and $ba = 0$. Let R be the subring of $Mat_3(K)$ generated by Ka, Kb. Then $R = Ka+Kb+Kab$, and so $E = K+R$ is isomorphic to the Dorroh extension of R by K. Thus E is right ideal-McCoy but not left ideal-McCoy by Proposition 1.1(1).

(2) Let K be a commutative domain. Let $a = e_{33} + e_{23}, b = e_{12}$ in $Mat_3(K)$. Then $a^2 = a, b^2 = 0$, and $ab = 0$. Let R be the subring of $Mat_3(K)$ generated by Ka, Kb. Then $R = Ka + Kb + Kba$, and so $E = K + R$ is isomorphic to the Dorroh extension of R by K. Thus E is left ideal-McCoy but not right ideal-McCoy by Proposition 1.1(2). ∎

In Proposition 1.1(1), consider the right annihilators taken in the ideal generated by the coefficients of $g(x)$. They are also contained in the right ideal generated by the coefficients of $g(x)$, and so the Dorroh extension is also strongly right McCoy. So this example also provides a ring that asserts that the strongly McCoy property is not left-right symmetric. A ring will be called *ideal-McCoy* if it is both left and right ideal-McCoy.

Strongly right McCoy rings are clearly right ideal-McCoy, but the converse need not hold by the following.

Example 1.3. We use the ring in [2, Proposition 3.2]. Let K be a field and $A = K\langle a_i, b_i, c_i, d_i | i \in \mathbb{N}\rangle$ be the free algebra with non-commuting indeterminates a_i, b_i, c_i, d_i over K, where $\mathbb{N}$ denotes the set of nonnegative integers. Set I_0 be the ideal generated by the relations

$$\sum_{i=0}^{n} a_i c_{n-i} = 0, \sum_{i=0}^{n}(a_i d_{n-i} + b_i c_{n-i}) = 0, \sum_{i=0}^{n} b_i d_{n-i} = 0$$

for each $n \in \mathbb{N}$. Let $R_0 = A/I_0$, and equate the indeterminates with their images in R_0. Let F_0 be the set of all finite subsets of indeterminates in R_0. For every set

$S_0 \in F_0$, adjoin two new variables x_{S_0} and y_{S_0} to R_0 and let I_1 be the ideal generated by the relations

$$x_{S_0}a_i = x_{S_0}b_i = c_i y_{S_0} = d_i y_{S_0} = 0, \text{ for all } i \in \mathbb{N} \text{ and } x_{S_0}s = s y_{S_0} = 0, \text{ for all } s \in S_0.$$

Then we obtain an overring

$$R_1 = K\langle a_i, b_i, c_i, d_i, x_{S_0}, y_{S_0} | i \in \mathbb{N}, S_0 \in F_0\rangle / \cup_{i=0}^{1} I_i.$$

Through this construction, we can obtain two ascending chains $R_0 \subset \cdots \subset R_n \subset R_{n+1} \subset \cdots$ and $I_0 \subset \cdots \subset I_n \subset I_{n+1} \subset \cdots$, where I_i is the ideal of R_i. Note

$$R_{n+1} = K\langle a_i, b_i, c_i, d_i, x_{S_j}, y_{S_j} | i \in \mathbb{N}, j = 0, \ldots, n \text{ and } S_j \in F_j\rangle / \cup_{i=0}^{n+1} I_i.$$

Put $R = \cup_1^{\infty} R_i$. Then R is not strongly right McCoy by [9, Example 1.9].

We will show that R is right ideal-McCoy. Consider nonzero polynomials $f(x), g(x)$ in $R[x]$ with $f(x)g(x) = 0$. Then there exists $k \geq 1$ such that $f(x), g(x) \in R_k[x]$. Let T be the set of all indeterminates in R_k which occur lastly in sum-factors of coefficients of $f(x)$. Then $f(x)y_T = 0$. But $y_T g(x) \neq 0$ and so $y_T\beta \neq 0$ for some coefficient β of $g(x)$. Now we get $f(x)y_T\beta = 0$, entailing that R is right ideal-McCoy. In fact every R_i ($i \geq 1$) is right ideal-McCoy by the same method as just above, and so R is also shown to be right ideal-McCoy by Proposition 2.9(1). ■

The preceding construction is excellent but somewhat complicated to handle. So we will find a simpler constructing method which provides a right ideal-McCoy ring but not strongly right McCoy over given any strongly right McCoy ring. In the following we see a typical kind of ring extension of right ideal-McCoy rings. For any ring A and $n \geq 2$, let

$$D_n(A) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in A \right\}$$

and

$$\begin{aligned} V_n(A) &= \{m = (m_{ij}) \in D_n(A) \mid m_{st} \\ &= m_{(s+1)(t+1)} \text{ for } s = 1, \ldots, n-2 \text{ and } t = 2, \ldots, n-1\}. \end{aligned}$$

Theorem 1.4. *For a ring R and $n \geq 2$, the following conditions are equivalent:*

(1) *R is right ideal-McCoy;*

(2) *$D_n(R)$ is right ideal-McCoy for any n;*

(3) *$V_n(R)$ is right ideal-McCoy for any n.*

Proof. (1)$\Rightarrow$ (2): Let R be right ideal-McCoy. We will use the ring isomorphism $(D_n(R))[x] \cong D_n(R[x])$ freely. Let

$$f(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ 0 & f_{11}(x) & \cdots & f_{2n}(x) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & f_{11}(x) \end{pmatrix} = A_0 + A_1 x + \cdots + A_m x^m$$

and

$$g(x) = \begin{pmatrix} g_{11}(x) & g_{12}(x) & \cdots & g_{1n}(x) \\ 0 & g_{11}(x) & \cdots & g_{2n}(x) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & g_{11}(x) \end{pmatrix} = B_0 + B_1 x + \cdots + B_l x^l$$

be nonzero polynomials in $D_n(R)[x]$ such that $f(x)g(x) = 0$, where $A_h = (a(h)_{st})$, $B_k = (b(k)_{uv}) \in D_n(R)$ for $h = 0, \ldots, m, k = 0, \ldots, l$ and $f_{st}(x) = \sum_{h=0}^m a(h)_{st} x^h$, $g_{uv}(x) = \sum_{k=0}^l b(k)_{uv} x^k \in R[x]$. Note $f_{11}(x)g_{11}(x) = 0$. Since $g(x) \neq 0$, we can take a nonzero $g_{ij}(x)$ such that $f_{11}(x)g_{ij}(x) = 0$ as follows. If $g_{11}(x) \neq 0$ then $f_{11}(x)g_{11}(x) = 0$. Assume $g_{11}(x) = 0$. Then we can find i, j such that i, j are both largest with respect to the property of $g_{ij}(x) \neq 0$. Note that $i < j$ and the (i, j)-entry of $f(x)g(x)$ is $f_{11}(x)g_{ij}(x) = f_{ii}(x)g_{ij}(x) = 0$. Recall $g_{ij}(x) = \sum_{k=0}^l b(k)_{ij} x^k$. Since R is right ideal-McCoy, there exists nonzero α in $\sum_{k=0}^l Rb(k)_{ij}R$, say $\alpha = \sum_{c=1}^d r_c \beta_c s_c$ with $r_c, s_c \in R$ and $\beta_c \in \{b(0)_{ij}, \ldots, b(l)_{ij}\}$ for all c, such that $f_{11}(x)\alpha = 0$. Let $\Omega = \sum_{c=1}^d (r_c I_n) B_c (s_c I_n) \in D_n(R)$, where $B_c \in \{B_0, \ldots, B_l\}$ and I_n is the n by n identity matrix. Then the (i, j)-entry of Ω is α. Now consider $\Omega' = e_{1i} \Omega e_{jn} = \alpha e_{1n}$. Then Ω' is contained in the ideal of $D_n(R)$ generated by B_k's and $f(x)\Omega' = 0$. This implies that $D_n(R)$ is right ideal-McCoy.

(2) $\Rightarrow$ (1): Let $D_n(R)$ be right ideal-McCoy for any n, and let $0 \neq f(x) = \sum_{i=0}^m a_i x^i, 0 \neq g(x) = \sum_{j=0}^n b_j x^j \in R[x]$ with $f(x)g(x) = 0$. Letting

$$a(x) = \sum_{i=0}^m \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} x^i \text{ and } b(x) = \sum_{j=0}^n \begin{pmatrix} b_j & 0 \\ 0 & b_j \end{pmatrix} x^j,$$

we have $a(x) = \begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix}$ and $b(x) = \begin{pmatrix} g(x) & 0 \\ 0 & g(x) \end{pmatrix}$ with $a(x)b(x) = 0$. Since $D_2(R)$ is right ideal-McCoy, there exists nonzero $C \in \sum_{j=0}^n D_2(R) \begin{pmatrix} b_j & 0 \\ 0 & b_j \end{pmatrix} D_2(R)$ such that $a(x)C = 0$. Here say

$$C = \sum_{t=0}^u \begin{pmatrix} c_{1t} & c_{2t} \\ 0 & c_{1t} \end{pmatrix} \begin{pmatrix} b_{s_t} & 0 \\ 0 & b_{s_t} \end{pmatrix} \begin{pmatrix} d_{1t} & d_{2t} \\ 0 & d_{1t} \end{pmatrix} = \sum_{t=0}^u \begin{pmatrix} c_{1t} b_{s_t} d_{1t} & c_{1t} b_{s_t} d_{2t} + c_{2t} b_{s_t} d_{1t} \\ 0 & c_{1t} b_{s_t} d_{1t} \end{pmatrix},$$

where $b_{s_t} \in \{b_0, \ldots, b_n\}$. Since C is nonzero, we have that $\sum_{t=0}^{u} c_{1t} b_{s_t} d_{1t} \neq 0$ or $\sum_{t=0}^{u}(c_{1t} b_{s_t} d_{2t} + c_{2t} b_{s_t} d_{1t}) \neq 0$. Now $a(x)C = 0$ yields that $f(x)(\sum_{t=0}^{u} c_{1t} b_{s_t} d_{1t}) = 0$ or $f(x)(\sum_{t=0}^{u}(c_{1t} b_{s_t} d_{2t} + c_{2t} b_{s_t} d_{1t})) = 0$, entailing that R is right ideal-McCoy.

The proof for (1) $\Leftrightarrow$ (3) is similar to the preceding case. ∎

Corollary 1.5. *A ring R is right ideal-McCoy if and only if so is $R[x]/(x^n)$, where $n \geq 2$ and (x^n) is the ideal of $R[x]$ generated by x^n.*

Proof. We get the proof from Theorem 1.4 and the isomorphism $V_n(R) \cong R[x]/(x^n)$. ∎

Strongly right McCoy rings are clearly right ideal-McCoy, but the converse need not hold also by Theorem 1.4 since $D_n(A)$ (when $n \geq 3$) cannot be strongly right McCoy, over any strongly right McCoy ring A, by Remark after [9, Theorem 2.2]. Thus we can say that given any strongly right McCoy ring we can construct right ideal-McCoy rings but not strongly right McCoy.

Right ideal-McCoy rings are clearly right McCoy, but the converse need not hold by the following.

Example 1.6. Let K be a field and $A = K\langle a, b, c, d, e\rangle$ be the free algebra generated by the noncommuting indeterminates a, b, c, d, e over K. Let I be the ideal of A generated by

$$ab, ad + cb, cd, es, se$$

where $s \in \{a, b, c, d, e\}$. Set $R = A/I$ and identify a, b, c, d, e with their images in R for simplicity. Then $(a + cx)(b + dx) = 0$. Let J be the ideal of R generated by b, d. Since $ab = eb = cd = ed = 0$, every element of J is of the form

$$r = a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4$$

where α_i is a polynomial in A generated by a, b, c, d for $i = 1, 2, 3, 4$. So

$$\begin{aligned} 0 &= (a + cx)r = (a + cx)(a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4) \\ &= (a^2\alpha_1 + ac\alpha_3 + ad\alpha_4) + (ca\alpha_1 + cb\alpha_2 + c^2\alpha_3)x \end{aligned}$$

yields

$$a^2\alpha_1 + ac\alpha_3 + ad\alpha_4 = 0 \text{ and } ca\alpha_1 + cb\alpha_2 + c^2\alpha_3 = 0.$$

Consider $ca\alpha_1 + cb\alpha_2 + c^2\alpha_3 = 0$. Then $c(a\alpha_1 + b\alpha_2 + c\alpha_3) = 0$ and so $a\alpha_1 + b\alpha_2 + c\alpha_3 = 0$. This yields $a\alpha_1 + c\alpha_3 = -b\alpha_2$, and so we must get $b\alpha_2 = 0$, entailing $a\alpha_1 + c\alpha_3 = 0$. Recall $ad + cb = 0$, and so we must have that either $a\alpha_1 = c\alpha_3 = 0$ or $\alpha_1 = d\beta, \alpha_3 = b\beta$ for some $\beta \in A$. Consequently we now have $r = d\alpha_4$ and $0 = (a + cx)d\alpha_4 = ad\alpha_4$. This also implies $d\alpha_4 = 0$, entailing $r = 0$. These conclude that R is not right ideal-McCoy.

Next we will show that R is McCoy. Let $f(x), g(x)$ be nonzero polynomials in $R[x]$ such that $f(x)g(x) = 0$. We can write

$$f(x) = s(x) + h_1(x)a + h_2(x)b + h_3(x)c + h_4(x)d + h_5(x)e \quad \text{and}$$
$$g(x) = t(x) + k_1(x)a + k_2(x)b + k_3(x)c + k_4(x)d + k_5(x)e$$

where $s(x), t(x) \in K[x]$ and $h_i(x), k_i(x) \in R[x]$ for $i = 1, 2, 3, 4, 5$. Assume $s(x) \neq 0$. Then $t(x) = 0$ clearly and so $g(x) = k_1(x)a + k_2(x)b + k_3(x)c + k_4(x)d + k_5(x)e$. Next we can obtain $g(x) = 0$ through a similar computation to the preceding one, a contradiction. Thus we must have $f(x) = h_1(x)a+h_2(x)b+h_3(x)c+h_4(x)d+h_5(x)e$, and so $f(x)e = 0$. This implies that R is right McCoy. The left McCoy property of R can be proved symmetrically. ■

2. Properties and Examples of Right Ideal-mccoy Rings

In this section we observe various kinds of properties of right ideal-McCoy rings, examining ordinary ring extensions of right ideal-McCoy rings. We also investigate the basic forms of finite right ideal-McCoy rings.

A ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$. Due to Feller [6], a ring is called *right* (resp. *left*) *duo* if every right (resp. left) ideal is two-sided. Right or left duo rings are clearly Abelian via a simple computation. Right duo rings are strongly right McCoy by [9, Theorem 1.11].

Proposition 2.1. *Given a regular ring R the following conditions are equivalent:* (1) *R is reduced;* (2) *R is reversible;* (3) *R is right duo;* (4) *R is Abelian;* (5) *R is strongly right McCoy;* (6) *R is right ideal-McCoy;* (7) *R is right McCoy.*

Proof. It suffices to prove (7)⇒(1) by [7, Theorem 3.2]. Let R be right McCoy and assume on the contrary that there exists nonzero $a \in R$ with $a^2 = 0$. Since R is regular, there exists $b \in R$ with $aba = a$. Note $baba = ba$. Consider two nonzero polynomials

$$f(x) = (1 - ba) + ax \text{ and } g(x) = ba - ax$$

in $R[x]$. Then $f(x)g(x) = 0$. But since R is right McCoy, there exists nonzero c in R such that $f(x)c = 0$. This yields $(1 - ba)c = 0$ and $ac = 0$, entailing $c = c - bac = (1 - ba)c = 0$. This induces a contradiction. ■

A ring R is called π-*regular* if for each $a \in R$ there exist a positive integer n, depending on a, and $b \in R$ such that $a^n = a^n ba^n$. Regular rings are clearly π-regular. So one may conjecture that right (ideal-)McCoy π-regular ring may be reduced. However the following argument answers negatively. Note that $D_n(A)$ ($n \geq 2$) is π-regular over a division ring A. Further, it is right (ideal-)McCoy by Theorem 1.4, but not reduced.

Remark 2.2. (1) $Mat_n(A)$ cannot be one-sided McCoy for any ring A and $n \geq 2$.
(2) $U_n(A)$ cannot be one-sided McCoy for any ring A and $n \geq 2$.
(3) The class of right (ideal-)McCoy rings is not closed under subrings.
(4) The class of right ideal-McCoy rings is not closed under homomorphic images.

Proof. (1) and (2) are shown by [11, Proposition 1.6] and [11, Example 1.3] respectively.

(3) We use the ring in [2, Theorem 7.1] and arguments in [9, Examples 1.10 and 1.12]. Let K be a field and $K\{e, x, y, z\}$ be the free algebra with noncommuting indeterminates e, x, y, z over K. Due to [2, Theorem 7.1], set R be the factor ring of $K\{e, x, y, z\}$ with the relations $e^2 = e, ex = x, xe = 0, ey = ye = 0, ez = ze = z, x^2 = y^2 = z^2 = xy = xz = yx = yz = zx = zy = 0$. Then R is strongly right McCoy (hence right ideal-McCoy) by the computation in [9, Examples 1.10]. Next consider the subring of R generated by $\{\alpha, e, x \mid \alpha \in K\}$, according to [9, Examples 1.12]. Then this subring is not right McCoy by the argument in [9, Examples 1.12], recalling that the overring R is right ideal-McCoy.

(4) Let R be the ring of quaternions with integer coefficients. Then R is a domain, so ideal-McCoy. However for any odd prime integer q, the ring R/qR is isomorphic to $Mat_2(\mathbb{Z}_q)$ by the argument in [8, Exercise 2A]. Thus R/qR is not one-sided ideal-McCoy by (1). ∎

One may conjecture that a ring R may be right ideal-McCoy when R/I and I are both right ideal-McCoy rings for any nonzero proper ideal I of R, where I is considered as a ring without identity. However the answer is negative by the following.

Example 2.3. Let F be a field and consider $R = U_2(F)$. Then R is not right McCoy (hence not right ideal-McCoy) by Remark 2.2(2). Note that all nonzero proper ideals of R are

$$\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

We will show that R/I and I are both right ideal-McCoy for any nonzero ideal I of R. First, let $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then $R/I \cong F$ is right ideal-McCoy. Let $f(x)g(x) = 0$ for $0 \neq f(x) = A_0 + A_1x + \cdots + A_mx^m$ and $0 \neq g(x) = B_0 + B_1x + \cdots + B_nx^n$ in $I[x]$, where $A_i = \begin{pmatrix} a_i & b_i \\ 0 & 0 \end{pmatrix}$ and $B_j = \begin{pmatrix} c_j & d_j \\ 0 & 0 \end{pmatrix}$ for $1 \leq i \leq m$, $1 \leq j \leq n$. We can write $f(x) = \begin{pmatrix} f_1(x) & f_2(x) \\ 0 & 0 \end{pmatrix}$ and $g(x) = \begin{pmatrix} g_1(x) & g_2(x) \\ 0 & 0 \end{pmatrix}$ where $f_1(x) = \sum_{i=0}^{m} a_ix^i$, $f_2(x) = \sum_{i=0}^{m} b_ix^i$ and $g_1(x) = \sum_{j=0}^{n} c_jx^j, g_2(x) = \sum_{j=0}^{n} d_jx^j$. From $f(x)g(x) = 0$ we have $f_1(x)g_1(x) = 0$ and $f_1(x)g_2(x) = 0$. If $f_1(x) \neq 0$, then

$g_1(x) = 0 = g_2(x)$ and so $g(x) = 0$, a contradiction. Thus $f_1(x) = 0$ and $f_2(x) \neq 0$, i.e., $f(x) = \begin{pmatrix} 0 & f_2(x) \\ 0 & 0 \end{pmatrix}$. This yields that $f(x)C = 0$ for every nonzero C in the ideal of I generated by B_j's. Next let $I = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Then $R/I \cong F$ is right ideal-McCoy. Let $f(x)g(x) = 0$ for $0 \neq f(x) = A_0 + A_1x + \cdots + A_mx^m = \begin{pmatrix} 0 & f_1(x) \\ 0 & f_2(x) \end{pmatrix}$ and $0 \neq g(x) = B_0 + B_1x + \cdots + B_nx^n = \begin{pmatrix} 0 & g_1(x) \\ 0 & g_2(x) \end{pmatrix} \in I[x]$, where $A_i = \begin{pmatrix} 0 & a_i \\ 0 & b_i \end{pmatrix}$ and $B_j = \begin{pmatrix} 0 & c_j \\ 0 & d_j \end{pmatrix}$ for $1 \leq i \leq m$, $1 \leq j \leq n$. Since $f_1(x) \neq 0$ or $f_2(x) \neq 0$, we get $g_2(x) = 0$ from $f(x)g(x) = 0$, entailing $g(x) = \begin{pmatrix} 0 & g_1(x) \\ 0 & 0 \end{pmatrix}$. This yields that $f(x)D = 0$ for every nonzero D in the ideal of I generated by B_j's. Finally let $I = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Then $R/I = F \oplus F$ is right ideal-McCoy. I is clearly right ideal-McCoy since $I^2 = 0$. ■

In ring theoretical process, it is also natural to observe the McCoy and ideal-McCoy properties to be hereditary to ideals. According to Ramamurthi [22], a ring R (possibly without identity) is *right* (resp. *left*) *weakly regular* if $I^2 = I$ for every right (resp. left) ideal I of R. It is shown, by [22, Proposition 1], that a ring R is right (resp. left) weakly regular if and only if $a \in (aR)^2$ (resp. $a \in (Ra)^2$) for every $a \in R$.

Remark 2.4. Let R be a ring and I be a proper ideal of R.

(1) If R is right ideal-McCoy then I is right McCoy as a ring without identity.

(2) The class of right McCoy rings is not closed under ideals.

(3) Suppose that if $Iv \neq 0$ for $v \in I$ then $vI \neq 0$. If R is strongly right McCoy then I is strongly right McCoy as a ring without identity.

(4) Suppose that R is right weakly regular. If R is strongly right McCoy then I is strongly right McCoy as a ring without identity.

(5) Suppose that R is right weakly regular. If R is right ideal-McCoy then I is right ideal-McCoy as a ring without identity.

Proof. (1) Consider nonzero polynomials $f(x), g(x)$ in $I[x]$ with $f(x)g(x) = 0$. Say $g(x) = \sum_{j=0}^{n} b_jx^j$. Since R is right ideal-McCoy, there exists nonzero $c \in \sum_{j=0}^{n} Rb_jR$ such that $f(x)c = 0$. But $c \in I$ and so I is right McCoy.

(2) Consider the ring R in Example 1.6 and let K be the ideal of R generated by $\{a, b, c, d\}$. Consider two polynomials $f(x) = a + cx, g(x) = b + dx$ in $K[x]$. Every element in K is of the form $a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4$ where α_i is a polynomial in A

generated by a, b, c, d for $i = 1, 2, 3, 4$. So the right annihilator of $f(x)$ in K is only zero, entailing that K is not right McCoy.

(3) Consider nonzero polynomials $f(x), g(x)$ in $I[x]$ with $f(x)g(x) = 0$. Say $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j$. If $f(x)b_k = 0$ for some $k \in \{0, \ldots, n\}$ then we are done. So assume that $f(x)b_j \neq 0$ for every nonzero b_j, i.e., $\{a_0 b_j, \ldots, a_m b_j\} \neq 0$ for every nonzero b_j. This yields $Ib_j \neq 0$ for every nonzero b_j. Then, by hypothesis, we get $b_j I \neq 0$ for every nonzero b_j, and so $g(x)I \neq 0$. Say $\sum_{j=0}^{n} b_j c x^j = g(x)c \neq 0$ for some $c \in I$. Then $f(x)g(x)c = 0$ clearly. Since R is strongly right ideal-McCoy, there exists nonzero d in the right ideal of R generated by $b_j c$'s such that $f(x)d = 0$. Say $d = \sum_{j=0}^{n} b_j c r_j$ with r_j's in R. Then $d = \sum_{j=0}^{n} b_j (c r_j) \in \sum_{j=0}^{n} b_j I$. This implies that I is strongly right McCoy.

(4) Let $Iv \neq 0$ for $v \in I$. Then $IvR \neq 0$. Since R is right weakly regular, $IvIvR = IvRIvR = IvR \neq 0$ and so vI must be nonzero. Thus I is strongly right McCoy by (3) when R is strongly right McCoy.

(5) Let R be right ideal-McCoy. Consider nonzero polynomials $f(x), g(x)$ in $I[x]$ with $f(x)g(x) = 0$. Say $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j$. Since R is right ideal-McCoy, there exists a nonzero α in the ideal J of R generated by b_j's such that $f(x)\alpha = 0$. Since R is right weakly regular, we have $J = J^2 = J^3 = \cdots$. Note that $J = RBR$ with $B = \{b_0, \ldots, b_n\}$. This yields $\alpha \in J = (RBR)^3 = (RBR)B(RBR) \subseteq IBI$, and so I is right ideal-McCoy. ■

Questions. (1) Is the class of right ideal-McCoy rings closed under ideals?

(2) Is the class of strongly right McCoy rings closed under ideals?

Finite dimensional algebras need not be right (ideal-)McCoy as we see in $U_n(A)$ ($n \geq 2$) over any finite ring A. We next investigate the basic forms of finite right ideal-McCoy rings.

Given a ring R the Jacobson radical is written by $J(R)$. Recall that R is called *local* if $R/J(R)$ is a division ring. Local rings are Abelian through a simple computation. R is called *semilocal* if $R/J(R)$ is semisimple Artinian, and R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are clearly semilocal.

Lemma 2.5. (1) (Eldridge) [5, Theorem]. *Let R be a finite ring of order m with an identity. If m has a cube free factorization, then R is a commutative ring.*

(2) (Eldridge) [5, Proposition]. *If a noncommutative ring with identity is of order p^3, p a prime, then it is isomorphic to $U_2(\mathbb{Z}_p)$.*

(3) *A ring R is Abelian, semiperfect, and right ideal-McCoy if and only if R is a finite direct product of local right ideal-McCoy rings.*

(4) *A ring R is Abelian, semiperfect, and right McCoy if and only if R is a finite direct product of local right McCoy rings.*

Proof. (3) Let R be a semiperfect right ideal-McCoy ring. The proof of [10, Lemma 2.2(3)] is applied. Since R is semiperfect, R has a finite orthogonal set $\{e_1, \ldots, e_n\}$ of local idempotents whose sum is 1 by [16, Corollary 3.7.2]. This implies that $R = \prod_{i=1}^{n} e_i R$ such that each $e_i R e_i$ is a local ring. Since R is Abelian, every $e_i R = e_i R e_i$ is an ideal of R. Moreover each $e_i R$ is a right ideal-McCoy ring by Proposition 2.9(2) to follow. Conversely suppose that R is a finite direct product of local right ideal-McCoy rings. Then R is Abelian and semiperfect since local rings are both Abelian and semiperfect. Next Proposition 2.9(2) implies that R is right ideal-McCoy.

(4) The proof is obtained by [2, Lemma 4.1] and a similar method to (3). ■

Due to Lambek [18], a ring R is called *symmetric* if $rst = 0$ implies $rts = 0$ for all $r, s, t \in R$. Lambek proved that a ring R is symmetric if and only if $r_1 r_2 \cdots r_n = 0$, with n any positive integer, implies $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \ldots, n\}$ and $r_i \in R$, in [18, Proposition 1]. Symmetric rings are strongly left and right McCoy by [9, Proposition 1.7]. Commutative rings are clearly symmetric. A simple computation gives that symmetric rings are Abelian. Reduced rings are symmetric by [1, Theorem I.3], but there are many non-reduced commutative (so symmetric) rings. $GF(p^n)$ denotes the Galois field of order p^n. Xue [23] proved that finite rings are right duo if and only if they are left duo. We will characterize minimal noncommutative right ideal-McCoy rings, analyzing the following examples.

Let $R_1 = \left\{ \begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix} \in U_2(GF(2^2)) \mid a, b \in GF(2^2) \right\}$, according to Xue [24, Example 2]. Then $J(R_1) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in GF(2^2) \right\}$, $R_1/J(R_1) \cong GF(2^2)$; hence R_1 is local. Note that R_1 is symmetric (hence strongly left and right McCoy) by the argument in [10, Example 2.5]. Further, R_1 is both left and right duo through a simple computation.

Let $R_2 = D_3(\mathbb{Z}_2)$. Then $J(R_2) = \{m \in D_3(\mathbb{Z}_2) \mid$ the diagonal entries of m are zero$\}$ and $R_2/J(R_2) \cong \mathbb{Z}_2$; hence R_2 is local. Note that R_2 is ideal-McCoy by Theorem 1.4, moreover R_2 is strongly left and right McCoy by [14, Proposition 2]. But R_2 is neither left nor right duo, considering the left ideal $R_2 e_{12}$ and right ideal $e_{23} R_2$.

According to Xue [24, Example 2], let $R_3 = \mathbb{Z}_4\{x, y\}/I$, where $\mathbb{Z}_4\{x, y\}$ is the free algebra with non-commuting indeterminates x, y over $\mathbb{Z}_4$ and I is the ideal of $\mathbb{Z}_4\{x, y\}$ generated by $x^3, y^3, yx, x^2 - xy, x^2 - 2, y^2 - 2, 2x, 2y$. Then R_3 is duo by the argument in [24, Example 2], and thus R_3 is strongly left and right McCoy by [9, Theorem 1.11]. Note that $J(R_3) = \langle 2, x, y \rangle$ (hence $R_3/J(R_3) \cong \mathbb{Z}_2$) and $J(R_3)^3 = 0$.

$|\ |$ denotes the cardinality.

Lemma 2.6. *If R is a noncommutative right (or left) McCoy ring of order 16, then*

$|J(R)|$ *is* 4 *or* 8.

Proof. Let R be a noncommutative right McCoy ring of order 16. We have four cases of $|J(R)| = 0$, $|J(R)| = 2$, $|J(R)| = 4$, or $|J(R)| = 8$. Assume $|J(R)| = 0$. Since R is noncommutative and $|R| = 16$, $R \cong Mat_2(\mathbb{Z}_2)$ by Wedderburn-Artin theorem. But $Mat_2(\mathbb{Z}_2)$ is not right McCoy by Remark 2.2(1), entailing that this case is impossible. Assume $|J(R)| = 2$. If R is local (i.e., $R/J(R)$ is a field), then $J(R)$ is a vector space over $R/J(R)$. This entails $|J(R)| \geq 8$ since $R/J(R) \cong GF(2^3)$, a contradiction. Thus we must have that $R/J(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $|J(R)| = 2$. Note $J(R)^2 = 0$. We can obtain orthogonal nonzero idempotents e_1, e_2, e_3, such that $e_1 + e_2 + e_3 = 1$, by [16, Corollary 3.7.2], and we have

$$R = \{x + y \mid x \in I, y \in J(R)\}$$

where $I = \{0, 1, e_1, e_2, e_3, 1 - e_1, 1 - e_2, 1 - e_3\}$. Note that I is a commutative ring since e_1, e_2, e_3 are orthogonal each other. Say $J(R) = \{0, a\}$. Assume that $e_i a = 0$ (resp. $ae_i = 0$) for all i. Then $a = 1a = (e_1 + e_2 + e_3)a = 0$ (resp. $a = a1 = a(e_1 + e_2 + e_3) = 0$), a contradiction. So $e_k a \neq 0$ (resp. $ae_k \neq 0$) for some k. Here assume that $e_j a \neq 0$ for $j \neq k$. Then $e_k a = e_j a = a$ since $e_k a, e_j a \in J(R)$, and so this entails $0 = e_k e_j a = e_k a = a$, a contradiction. Thus $e_j a = 0$ for all $j \neq k$ if $e_k a \neq 0$. Similarly, $ae_j = 0$ for all $j \neq i$ if $ae_i \neq 0$. Here assume that $e_i a \neq 0$ and $ae_i \neq 0$ for some i. Then $e_i a = a = ae_i$ and $e_j a = 0 = ae_j$ for all $j \neq i$. This implies that R is commutative, a contradiction. So if $e_i a \neq 0$ for some i then $ae_i = 0$, entailing $ae_j \neq 0$ for some $j \neq i$. Say $e_1 a \neq 0, ae_2 \neq 0$, i.e., $e_1 a = ae_2 = a$. Then $e_2 a = e_3 a = 0$ and $ae_1 = ae_3 = 0$. Now consider two polynomials

$$f(x) = (e_1 + e_3) + ax \text{ and } g(x) = e_2 - ax \in R[x].$$

Then $f(x)g(x) = 0$ but there cannot exist $0 \neq r \in R$ such that $f(x)r = 0$, entailing that R is not right McCoy. The computations for remaining cases are similar. So this case of $|J(R)| = 2$ is also impossible. Therefore $|J(R)|$ is either 4 or 8. The proof for the left case is similar. ∎

As an application of Lemma 2.6, $Mat_2(\mathbb{Z}_2)$ cannot be right (ideal-)McCoy since the Jacobson radical is zero.

In the following we can see all possible basic forms of finite right ideal-McCoy rings of minimal order. We use the term "minimal" in the names of such kinds of rings. The characteristic of a ring R is written by $char(R)$.

Theorem 2.7. *If R is a minimal noncommutative Abelian right McCoy ring, then R is of order 16 and is isomorphic to R_i for some $i \in \{1, 2, 3\}$, where R_i's are the rings above.*

Proof. Suppose that R is a minimal noncommutative right McCoy ring. Then $|R|$ has a cube factor by Lemma 2.5(1) since R is noncommutative. $U_2(A)$ is not right McCoy by Remark 2.2(2) for any ring A. So Lemma 2.5(2) implies that $|R|$ is not of the form p^3 for some prime p since R is right McCoy. These yield that $|R|$ is equal to or larger than 2^4 since R is minimal such a ring. But the rings R_i's above are right McCoy rings of order 16, and so R must be of order 16. Note that R is semiperfect.

Now since R is a noncommutative right McCoy ring of order 16, we have two cases of $|J(R)| = 4$ and $|J(R)| = 8$, by Lemma 2.6.

Case 1. $|J(R)| = 4$.

R is local by Lemma 2.5(4) since R is a minimal noncommutative Abelian right McCoy ring. Then R is a noncommutative duo ring of order 16 by the proof of [10, Theorem 2.6], and thus R is isomorphic to the ring R_1 above by [24, Theorem 3].

Case 2. $|J(R)| = 8$.

Since R is local with $|R/J(R)| = 2$, we have $R = \{x + y \mid x \in I, y \in J(R)\}$ where $I = \{0, 1\}$. Thus R is commutative if and only if $J(R)$ is commutative. Then, by applying the argument for the case of $|J(R)| = 8$ in the proof of [13, Theorem 2.3], we have that R is isomorphic to R_2 (when $char(R) = 2$) or R_3 (when $char(R) = 4$) above. ■

Question. What are the shapes of non-Abelian right ideal-McCoy rings R such that $|R| = 16$ and $|J(R)| = 4$?

Every ring R_i above is actually strongly left and right McCoy, and thus we get the following with the help of Theorem 2.7. A strongly McCoy ring means a strongly left and right McCoy ring.

Corollary 2.8. *R is a minimal noncommutative Abelian right McCoy ring if and only if R is a minimal noncommutative Abelian right ideal-McCoy ring if and only if R is a minimal noncommutative Abelian strongly right McCoy ring if and only if R is a minimal noncommutative Abelian strongly McCoy ring.*

Considering Corollary 2.8, one may conjecture that the ideal-McCoy property is left-right symmetric for the case of finite rings. However the answer is negative by Example 1.2, letting K be a finite field.

Finally, we deal with some kinds of ring extensions over right ideal-McCoy rings. Camillo and Nielsen showed, in [2, Lemma 4.1], that a direct product of rings R_i ($i \in I$) is right McCoy if and only if so is every R_i. They also showed, in [2, Proposition 4.3], that If I is an infinite set then the direct sum of rings R_i ($i \in I$) is right McCoy. Also Hong, et al. [9, Proposition 2.6] proved that the class of (strongly) right McCoy rings is closed under direct limits. Π and Σ denote direct product and direct sum, respectively.

Remark 2.9. (1) The class of right ideal-McCoy rings is closed under direct limits.

(2) Let $R = \prod_{i \in I} R_i$ be the direct product of rings R_i. Then R is right ideal-McCoy if and only if R_i is right ideal-McCoy for every $i \in I$.

(3) Let $R = \prod_{i \in I} R_i$ be the direct product of rings R_i. Then R is strongly right McCoy if and only if R_i is strongy right McCoy for every $i \in I$.

(4) Let $R = \sum_{i \in I} R_i$ be a direct sum of rings R_i. Then R (possibly without identity) is right ideal-McCoy if and only if R_i is right ideal-McCoy for every $i \in I$.

(5) Let $R = \sum_{i \in I} R_i$ be a direct sum of rings R_i. Then R (possibly without identity) is strongly right McCoy if and only if R_i is strongly right McCoy for every $i \in I$.

Lei, et al. [17, Theorem 1] proved that a ring R is right McCoy if and only if so is $R[x]$. Also Hong, et al. [9, Proposition 2.4] proved that if $R[x]$ is strongly right McCoy then so is R. By a similar method as in the proof of [9, Proposition 2.4], we can get that if $R[x]$ is right ideal-McCoy then so is R.

However we do not know whether $R[x]$ is strongly right ideal-McCoy if R is a strongly right McCoy ring.

Question. If R is a right ideal McCoy ring then is $R[x]$ right ideal-McCoy?

A ring R is called *right* (resp. *left*) *Ore* if given $a, b \in R$ with b (resp. a) regular there exist $a_1, b_1 \in R$ with b_1 (resp. a_1) regular such that $ab_1 = ba_1$ (resp. $a_1 b = b_1 a$). Note that R is a right (resp. left) Ore ring if and only if the classical right (resp. left) quotient ring of R exists. There exist many reduced rings which are neither right nor left Ore as can be seen by the free algebra in two indeterminates over a field (this ring is a domain but cannot have its classical right (left) quotient ring).

Hong, et al. [9, Theorem 2.1] proved that letting R be a right Ore ring with the classical right quotient ring Q then R is strongly right McCoy if and only if so is Q, and R is right McCoy if and only if so is Q.

Proposition 2.10. *Let R be a right Ore ring with the classical right quotient ring Q. If R is right ideal-McCoy then so is Q.*

Proof. The set of all regular elements in R is denoted by $C(R)$, and [19, Proposition 2.1.16] is referred to freely. Let $F(x)G(x) = 0$ for $F(x), 0 \neq G(x) \in Q[x]$. We can write $F(x) = a_0 u^{-1} + a_1 u^{-1} x + \cdots + a_m u^{-1} x^m$ and $G(x) = b_0 v^{-1} + b_1 v^{-1} x + \cdots + b_n v^{-1} x^n$ for $a_i, b_j \in R$ and $u, v \in C(R)$, where $i = 0, \ldots, m$ and $j = 0, \ldots, n$. Since R is right Ore, there exists $u_1 \in C(R)$ for all j's such that $u^{-1} b_j = b'_j u_1^{-1}$ for some $b'_j \in R$. Next set $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$, $g_1(x) = \sum_{j=0}^{n} b'_j x^j$, and $u_2 = v u_1$. Then $F(x) = f(x) u^{-1}, G(x) = g(x) v^{-1}$, $u^{-1} g(x) = g_1(x) u_1^{-1}$, and

$$F(x)G(x) = f(x) g_1(x) u_1^{-1} v^{-1} = f(x) g_1(x) u_2^{-1},$$

noting that $g(x) \neq 0$ and $g_1(x) \neq 0$. Let B (resp. B') be the ideal of Q (resp. R) generated by the coefficients of $G(x)$ (resp. $g_1(x)$). Since $u^{-1}b_j u_1 = b'_j$ and $b_j = b_j v^{-1} v$, we have $B' \subseteq B$ and $g(x), g_1(x) \in B[x]$. Now from $F(x)G(x) = 0$, we also get $f(x)g_1(x) = 0$. Since R is right ideal-McCoy and $g_1(x) \neq 0$, there exists $0 \neq r \in B'$ such that $f(x)r = 0$. Further, we have

$$0 = f(x)r = f(x)u^{-1}ur = F(x)ur$$

for $0 \neq ur \in B$. Thus Q is right ideal-McCoy. ∎

In the preceding situation, we do not know of any example of a right ideal McCoy ring Q such that R is not right ideal McCoy.

Question. Let R and Q as before. If Q is right ideal-McCoy then is R right ideal-McCoy?

The following can be compared with Proposition 1.1.

Proposition 2.11. *Let R be an algebra with identity over a commutative ring S.*

(1) *R is right ideal-McCoy if and only if so is the Dorroh extension of R by S.*

(2) *R is strongly right McCoy if and only if so is the Dorroh extension of R by S.*

(3) *R is right McCoy if and only if so is the Dorroh extension of R by S.*

(4) *The left versions of (1), (2), and (3) also hold.*

Proof. (1) Let D be the Dorroh extension of R by S, and suppose that $f(x) = \sum_{i=0}^{m}(a_i, b_i)x^i = (f_1(x), f_2(x))$ and $g(x) = \sum_{j=0}^{n}(c_j, d_j)x^j = (g_1(x), g_2(x))$ in $D[x]$ such that $f(x)g(x) = 0$, where $f_1(x) = \sum_{i=0}^{m} a_i x^i$, $f_2(x) = \sum_{i=0}^{m} b_i x^i$, $g_1(x) = \sum_{j=0}^{n} c_j x^j$ and $g_2(x) = \sum_{j=0}^{n} d_j x^j$. Then $(f_1(x)g_1(x) + f_1(x)g_2(x) + f_2(x)g_1(x), f_2(x)g_2(x)) = 0$ and so $f_1(x)g_1(x) + f_1(x)g_2(x) + f_2(x)g_1(x) = 0$ and $f_2(x)g_2(x) = 0$. Note that $s \in S$ is identified with $s1 \in R$, and so S is considered as a subring of R. We refer to [9, Theorem 1.6] freely.

Case 1. ($f_2(x) \neq 0$ and $g_2(x) \neq 0$)

Since $f_2(x)g_2(x) = 0$ and S is commutative, there exists nonzero $\alpha \in \sum_{j=0}^{n} S d_j S$, say $\alpha = \sum_s u_s d_s v_s$, such that $f_2(x)\alpha = 0$. Then $(f_1(x), f_2(x))(-\alpha, \alpha) = 0$ with

$$(-\alpha, \alpha) = (-\sum_s u_s d_s v_s, \sum_s u_s d_s v_s) = \sum_s (-u_s, u_s)(c_s, d_s)(-v_s, v_s) \in \sum_{j=0}^{n} D(c_j, d_j)D.$$

Case 2. ($f_2(x) \neq 0$ and $g_2(x) = 0$)

Since $(f_1(x), f_2(x))(g_1(x), 0) = (f_1(x)g_1(x) + f_2(x)g_1(x), 0) = 0$, $(f_1(x) + f_2(x))g_1(x) = 0$. If $f_1(x) + f_2(x) = 0$, then $(f_1(x), f_2(x))(\beta, 0) = 0$ for any $0 \neq (\beta, 0) \in \sum_{j=0}^{n} D(c_j, 0)D$. If $f_1(x) + f_2(x) \neq 0$, then there exists nonzero

$\beta \in \sum_{j=0}^{n} Rc_jR$, say $\beta = \sum_t u_tc_tv_t$, such that $(f_1(x) + f_2(x))\beta = 0$. Thus $(f_1(x), f_2(x))(\beta, 0) = 0$ with

$$(\beta, 0) = (\sum_t u_tc_tv_t, 0) = \sum_t (u_t, 0)(c_t, 0)(v_t, 0) \in \sum_{j=0}^{n} D(c_j, 0)D.$$

Case 3. ($f_2(x) = 0$ and $g_2(x) \neq 0$)

Since $(f_1(x), 0)(g_1(x), g_2(x)) = (f_1(x)g_1(x) + f_1(x)g_2(x), 0) = 0$, $f_1(x)(g_1(x) + g_2(x)) = 0$. If $g_1(x) + g_2(x) = 0$ (i.e., $d_j = -c_j$ for all j), then $(f_1(x), 0)(\gamma, -\gamma) = 0$ for any $0 \neq (\gamma, -\gamma) \in \sum_{j=0}^{n} D(c_j, -c_j)D$. If $g_1(x) + g_2(x) \neq 0$, then there exists nonzero $\gamma \in \sum_{j=0}^{n} R(c_j + d_j)R$, say $\gamma = \sum_w u_w(c_w + d_w)v_w$, such that $f_1(x)\gamma = 0$. Thus $(f_1(x), 0)(\gamma, 0) = 0$ with

$$(\gamma, 0) = (\sum_w u_w(c_w + d_w)v_w, 0) = \sum_w (u_w, 0)(c_w, d_w)(v_w, 0) \in \sum_{j=0}^{n} D(c_j, d_j)D.$$

Case 4. ($f_2(x) = 0$ and $g_2(x) = 0$)

Since $(f_1(x), 0)(g_1(x), 0) = (f_1(x)g_1(x), 0) = 0$, $f_1(x)g_1(x) = 0$. Since R is right ideal-McCoy, there exists nonzero $\delta \in \sum_{j=0}^{n} Rc_jR$, say $\delta = \sum_l u_lc_lv_l$, such that $f_1(x)\delta = 0$. Thus $(f_1(x), 0)(\delta, 0) = 0$ with

$$(\delta, 0) = (\sum_l u_lc_lv_l, 0) = \sum_l (u_l, 0)(c_l, 0)(v_l, 0) \in \sum_{j=0}^{n} D(c_j, 0)D.$$

By Cases 1, 2, 3, and 4, D is right ideal-McCoy.

Conversely let D be right ideal-McCoy, and suppose that $a(x) = \sum_{i=0}^{m} a_ix^i$ and $b(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$ such that $a(x)b(x) = 0$. Then we also have $f(x)g(x) = 0$ in $D[x]$, letting $f(x) = \sum_{i=0}^{m}(a_i, 0)x^i$ and $g(x) = \sum_{j=0}^{n}(b_j, 0)x^j$. Since D is right ideal-McCoy, there exists nonzero $c \in \sum_{j=0}^{n} D(b_j, 0)D$, say $c = \sum_k (u_k, s_k)(b_k, 0)(v_k, t_k)$ with $(u_k, s_k), (v_k, t_k) \in D$, such that $f(x)c = 0$. Note that every $s \in S$ is identified with $s1 \in R$ and so $R = \{r + s \mid (r, s) \in D\}$. Hence

$$\begin{aligned} c &= \sum_k (u_k, s_k)(b_k, 0)(v_k, t_k) \\ &= \sum_k ((u_k + s_k)b_k, 0)(v_k, t_k) = \sum_k ((u_k + s_k)b_k(v_k + t_k), 0). \end{aligned}$$

Setting $d_k = (u_k + s_k)b_k(v_k + t_k)$ gives

$$\begin{aligned} 0 &= f(x)c = (\sum_{i=0}^{m}(a_i, 0)x^i)(\sum_k (d_k, 0)) \\ &= (\sum_{i=0}^{m}(a_i, 0)x^i)(\sum_k d_k, 0) = (a(x)(\sum_k d_k), 0). \end{aligned}$$

But $0 \neq \sum_k d_k \in \sum_{j=0}^n Rb_jR$, and this shows that R is right ideal-McCoy.

The proofs of (2), (3), and (4) are quite similar to one of (1). ∎

The preceding proposition need not hold for the case that the ring R does not have the identity, as we see in Proposition 1.1.

ACKNOWLEDGMENTS

The authors thank the referee for very careful reading of the manuscript and valuable suggestions. The second named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2011-0009313). The fourth named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2013R1A1A4A01008108). The fifth named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (201306530001).

REFERENCES

1. D. D. Anderson and V. Camillo, Semigroups and rings whose zero products commute, *Comm. Algebra*, **27** (1999), 2847-2852.

2. V. Camillo and P. P. Nielsen, McCoy rings and zero-divisors, *J. Pure Appl. Algebra*, **212** (2008), 599-615.

3. P. M. Cohn, Reversible rings, *Bull. London Math. Soc.*, **31** (1999), 641-648.

4. J. L. Dorroh, Concerning adjunctins to algebras, *Bull. Amer. Math. Soc.*, **38** (1932), 85-88.

5. K. E. Eldridge, Orders for finite noncommutative rings with unity, *Amer. Math. Monthly*, **73** (1966), 512-514.

6. E. H. Feller, Properties of primary noncommutative rings, *Trans. Amer. Math. Soc.*, **89** (1958), 79-91.

7. K. R. Goodearl, *Von Neumann Regular Rings*, Pitman, London-San Francisco-Melbourne, 1979.

8. K. R. Goodearl and R. B. Warfield, Jr. *An Introduction to Noncommutative Noetherian Rings*, Cambridge University Press, 1989.

9. C. Y. Hong, Y. C. Jeon, N. K. Kim and Y. Lee, The McCoy condition on non-commutative rings, *Comm. Algebra*, **39** (2011), 1809-1825.

10. C. Huh, H. K. Kim, N. K. Kim and Y. Lee, Basic examples and extensions of symmetric rings, *J. Pure Appl. Algebra*, **202** (2005), 154-167.

11. Y. C. Jeon, H. K. Kim, N. K. Kim, T. K. Kwak, Y. Lee and D. E. Yeo, On a generalization of the McCoy condition, *J. Korean Math. Soc.*, **47** (2010), 1269-1282.

12. B. O. Kim, T. K. Kwak and Y. Lee, *On Constant Zero-divisors of Linear Polynomials*, submitted.

13. N. K. Kim and Y. Lee, *On Abelian Rings and Generalizations*, submitted.

14. N. K. Kim and Y. Lee, Armendariz rings and reduced rings, *J. Algebra*, **223** (2000), 477-488.

15. R. Kruse and D. Price, *Nilpotent Rings*, Gordon and Breach, NewYork, 1969.

16. J. Lambek, *Lectures on Rings and Modules*, Blaisdell Publishing Company, Waltham, 1966.

17. Z. Lei, J. Chen and Z. Ying, A question on McCoy rings, *Bull. Austral. Math. Soc.*, **76** (2007), 137-141.

18. J. Lambek, On the representation of modules by sheaves of factor modules, *Canad. Math. Bull.*, **14** (1971), 359-368.

19. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, John Wiley & Sons Ltd., 1987.

20. N. H. McCoy, Remarks on divisors of zero, *Amer. Math. Monthly*, **49** (1942), 286-295.

21. P. P. Nielsen, Semi-commutativity and the McCoy condition, *J. Algebra*, **298** (2006), 134-141.

22. V. S. Ramamurthi, Weakly regular rings, *Canad. Math. Bull.*, **16** (1973), 317-321.

23. W. Xue, On strongly right bounded finite rings, *Bull. Austral. Math. Soc.*, **44** (1991), 353-355.

24. W. Xue, Structure of minimal noncommutative duo rings and minimal strongly bounded non-duo rings, *Comm. Algebra*, **20** (1992), 2777-2788.

Yasuyuki Hirano
Department of Mathematics
Naruto University of Education
Takashima, Naruto 772-8502
Japan
E-mail: yahirano@naruto-u.ac.jp

Chan Yong Hong
Department of Mathematics and RIBS
Kyung Hee University
Seoul 131-701
Korea
E-mail: hcy@khu.ac.kr

Hong Kee Kim
Department of Mathematics and RINS
Gyeongsang National University
Jinju 660-701
Korea
E-mail: hkkim@gsnu.ac.kr

Nam Kyun Kim
College of Liberal Arts
Hanbat National University
Daejeon 305-719
Korea
E-mail: nkkim@hanbat.ac.kr

Yang Lee
Department of Mathematics Education
Pusan National University
Pusan 609-735
Korea
E-mail: ylee@pusan.ac.kr