TOTAL SCALAR CURVATURE AND HARMONIC CURVATURE

Gabjin Yun, Jeongwook Chang and Seungsu Hwang*

Abstract. On a compact n-dimensional manifold, it has been conjectured that a critical point metric of the total scalar curvature, restricted to the space of metrics with constant scalar curvature of unit volume, will be Einstein. This conjecture was proposed in 1984 by Besse, but has yet to be proved. In this paper, we prove that if the manifold with the critical point metric has harmonic curvature, then it is isometric to a standard sphere.

1. INTRODUCTION

Let M be an n-dimensional compact manifold and $\mathcal{M}_1$ be the set of all smooth Riemannian structures of unit volume on M. The scalar curvature s_g is a non-linear function of the metric g. Its linearization at g in the direction of the symmetric two-tensor h is given by

$$s'_g(h) = -\Delta_g tr h + \delta_g \delta_g h - g(h, r_g).$$

Here, Δ_g is the negative Laplacian of g, r_g is its Ricci tensor, and δ_g is the metric dual of the map on the bundle of symmetric tensors induced by covariant differentiation. The dual linearized operator s'^*_g of s'_g is given by

$$(1) \qquad s'^*_g(f) = D_g df - g\Delta_g f - f r_g,$$

where $D_g df$ is the Hessian of f.

The following formulae are important for the analysis of the map

$$\mathcal{S} : g \mapsto \int_M s_g dv_g$$

when $\mathcal{S}$ is defined over $\mathcal{C}$, a subset of $\mathcal{M}_1$, consisting of constant scalar curvature metrics. The Euler-Lagrange equation of $\mathcal{S}$ restricted to $\mathcal{C}$ may be written as the following critical point equation(CPE, hereafter):

$$(2) \qquad\qquad\qquad z_g = s_g'^*(f),$$

where z_g is the traceless Ricci tensor defined by $z_g = r_g - \frac{s_g}{n} g$ and f is a function on M with a vanishing mean value. It is obvious that a solution g of the CPE is Einstein if f is trivial. Therefore, from now on, we consider only the case when (g, f) is a non-trivial solution of the CPE.

In 1987 Besse proposed the following conjecture in [1]:

Conjecture I. A solution of the critical point equation (2) is Einstein.

There are some partial answers to Conjecture I. Among them, Lafontaine showed in [8] that if a solution metric g of the CPE is locally conformally flat and $\ker s_g'^* \neq 0$, then such a metric is Einstein. Later Bessières, Lafontaine, and Rozoy showed that if a solution metric g of the 3-dimensional CPE is locally conformally flat, Conjecture I is true without the need for a condition on $\ker s_g'^*$ ([2]). Recently we proved the following result:

Theorem 1.1. [3]. *Let (g, f) be a non-trivial solution of the CPE on an n-dimensional compact Riemannian manifold M. If the Ricci tensor of g is parallel, then (M, g) is isometric to a standard sphere.*

The geometric structure of an Einstein solution is known to be simple; Obata showed that such a solution is isometric to a standard n-sphere([10]). For more details, we refer the reader to [1] and [7].

In this paper, we answer Conjecture I for harmonic curvature, which is a generalization of conformal flatness or parallel Ricci tensor conditions. We say that (M, g) has harmonic curvature if the divergence of the Riemann curvature vanishes, i.e., $\delta R = 0$. A locally conformally flat metric with constant scalar curvature has harmonic curvature (see Section 3). It is also well known that every manifold with a parallel Ricci tensor has harmonic curvature. However, there are examples of compact and noncompact Riemannian manifolds with $\delta R = 0$ and $\nabla r_g \neq 0$ (see [4] and Theorem 5.2 in [6]). By virtue of Theorem 1.1, it is natural to ask whether a solution metric of the CPE which has harmonic curvature is Einstein. In the following we show that the answer to this question is affirmative.

Theorem 1.2. *Let* (g, f) *be a non-trivial solution of the CPE on an* n-*dimensional compact Riemannian manifold* M. *If* (M, g) *has harmonic curvature, then* M *is isometric to a standard sphere.*

This result is a good progress to solve Conjecture I. The remaining part is to show that the solution metric in Conjecture I is harmonic.

We remark that Derdzinski has given a local characterization of the analytic metrics with harmonic curvature, assuming that the Ricci tensor is not parallel and has less than three distinct eigenvalues at any point; such a compact analytic manifold is covered isometrically by certain warped products ([4]). In view of Remark 3.5, the eigenvalues of the Ricci tensor has less than three distinct eigenvalues if (M, g) has harmonic curvature and (g, f) is a non-trivial solution of the CPE. However, since a CPE solution cannot be realized on a warped product space ([3]), we may conclude that the Ricci tensor should be parallel in this case.

As an immediate consequence of Theorem 1.2, we have the following generalization of Lafontaine's result([8]), which does not need a condition on $\ker s_g'^*$.

Corollary 1.3. *Let* (g, f) *be a non-trivial solution of the CPE on an* n-*dimensional compact Riemannian manifold* M. *If* g *is locally conformally flat, then* (M, g) *is isometric to a standard sphere.*

While Lafontaine's proof relies on the classification of locally conformally flat metrics with $\ker s_g'^* \neq 0$, our work directly shows that g is Einstein by Theorem 4.1 if g is locally conformally flat (see Remark 3.5).

This paper is organized as follows. In Section 2, we analyze the critical point equations. In Section 3, we formulate the critical point equations under harmonic curvature conditions. In particular, we decompose the traceless Ricci tensor z_g in Lemma 3.4 into tangential component of the level sets of f and their orthogonal complements. Using this decomposition, we prove Theorem 4.1, a special case of Theorem 1.2, which has the condition that one component $\mathcal{W}_N$ of the decomposition of z_g vanishes. Finally, we will complete the proof of Theorem 1.2 in Section 5 by showing that $\mathcal{W}_N$ is identically zero on M.

2. Preliminaries

Let (M, g, f) be a non-trivial solution of the CPE. Taking the trace of (2) gives

$$(3) \qquad \Delta_g f = -\frac{s_g}{n-1} f.$$

Thus, by (1), (2) and (3), the CPE may be written as

$$(4) \qquad (1 + f) z_g = D_g df + \frac{s_g f}{n(n-1)} g.$$

Here the scalar curvature s_g is constant from our stated assumptions.

Proposition 2.1. *Let (g, f) be a non-trivial solution of (4), and $B = \{x \in M^n \,|\, f(x) = -1\}$. Then B has measure zero.*

Proof. Let B' be the set of critical points of f in B. Then $B \setminus B'$ is a union of hypersurfaces. For $p \in B'$, from (4) we have

$$Ddf_p(\xi, \xi) = \frac{s_g}{n(n-1)} g_p(\xi, \xi) > 0$$

for any nonzero tangent vector ξ in the tangent space $T_p M$ at p. Thus p is a non-degenerate critical point of f. Such non-degenerate critical points are isolated, and thus the set B' should be finite. Therefore $B = B' \cup (B \setminus B')$ has measure zero. ∎

Let $\mathrm{Crit}(f) = \{x \in M \,|\, df(x) = 0\}$. For this set, we observe the following fact.

Proposition 2.2. *The measure of $\mathrm{Crit}(f)$ is zero.*

Proof. We first claim that $\mathrm{Crit}(f) \cap B$ is finite. If $p \in \mathrm{Crit}(f) \cap B$, we observe that as in the proof of Proposition 2.1 that p is an isolated point, and thus $\mathrm{Crit}(f) \cap B$ is finite since B is compact. To prove that there is no open set in $\mathrm{Crit}(f) \cap (M \setminus B)$, it suffices to prove that g and f are analytic in $M \setminus B$. Since the coefficients, in local harmonic coordinates, of the coupled elliptic system (3) and (4) are real-analytic, it follows from Theorem 6.6.1 in [9] that g and $h = 1 + f$ are in fact real-analytic where $f \neq -1$. ∎

3. Harmonic Curvature

In this section we study harmonic curvature and its relation to the CPE. For an n-dimensional Riemannian manifold (M, g), the second Bianchi identity yields the well-known divergence formulae $\delta r_g = -\frac{1}{2} ds_g$ and

$$(5) \qquad \delta R = -d^D r_g,$$

where d^D is the first-order differential operator from $C^\infty(S^2 M)$ into $C^\infty(\Lambda^2 T^* M \otimes T^* M)$ defined by

$$d^D \omega(x, y, z) = D_x \omega(y, z) - D_y \omega(x, z)$$

for a two form ω. Consequently

$$(6) \qquad \delta \mathcal{W} = -\frac{n-3}{n-2} d^D \left(r_g - \frac{s_g}{2(n-1)} g \right),$$

where $\mathcal{W}$ is its Weyl conformal curvature tensor.

We say that (M, g) has harmonic curvature if the divergence of the Riemann curvature vanishes, i.e., $\delta R = 0$. When $n = 3$, (M, g) is locally conformally flat and has

constant scalar curvature. When $n \geq 4$, (M, g) has harmonic Weyl tensor ($\delta \mathcal{W} = 0$) and constant scalar curvature. Moreover, it is equivalent to $d^D r_g = 0$, in other words, r_g is a Codazzi tensor. Harmonic curvature is a generalization of locally conformally flatness of a metric. For $n \geq 4$, a metric is locally conformally flat if and only if $\mathcal{W} = 0$. Thus, by (6) and the definition of harmonic curvature, a locally conformally flat metric with constant scalar curvature has harmonic curvature for $n \geq 3$.

The product of a 1-form β and a symmetric two form η can be defined by $\beta \wedge \eta(x, y, z) = \beta(x)\eta(y, z) - \beta(y)\eta(x, z)$. Then we have the following equation.

Lemma 3.1. *Let (M, g) have harmonic curvature and (g, f) be a non-trivial solution of the CPE. Then*

(7)
$$(n - 2)\tilde{i}_{\nabla f} \mathcal{W} = (n - 1)df \wedge z + i_{\nabla f} z \wedge g.$$

Here, we define $\tilde{i}$ by $\tilde{i}_\xi \omega(X, Y, Z) = \omega(X, Y, Z, \xi)$ for a 4-tensor ω, and i by $i_\xi z(X) = z(\xi, X)$.

Proof. From (4) we have

$$(1 + f)d^D r_g(X, Y, Z) = \langle R(\nabla f, Z)Y, X \rangle + \left(\frac{s_g}{n - 1} df \wedge g - df \wedge r_g \right)(X, Y, Z).$$

Now from

$$\mathcal{W}(X, Y, Z, W) = R(X, Y, Z, W) - \frac{1}{n - 2}(g(X, Z)r(Y, W) + g(Y, W)r(X, Z)$$
$$- g(Y, Z)r(X, W) - g(X, W)r(Y, Z))$$
$$+ \frac{s_g}{(n - 1)(n - 2)}(g(X, Z)g(Y, W) - g(Y, Z)g(X, W)),$$

$$\tilde{i}_{\nabla f} R = \tilde{i}_{\nabla f} \mathcal{W} - \frac{1}{n - 2} i_{\nabla f} r_g \wedge g + \frac{s_g}{(n - 1)(n - 2)} df \wedge g - \frac{1}{n - 2} df \wedge r_g.$$

We can obtain

$$(1 + f)d^D r_g = \tilde{i}_{\nabla f} \mathcal{W} - \frac{1}{n - 2} i_{\nabla f} r_g \wedge g + \frac{s_g}{(n - 2)} df \wedge g - \frac{n - 1}{n - 2} df \wedge r_g$$
$$= \tilde{i}_{\nabla f} \mathcal{W} - \frac{n - 1}{n - 2} df \wedge z - \frac{1}{n - 2} i_{\nabla f} z \wedge g.$$

Then equation (7) follows from the harmonicity of the metric g. ∎

Throughout the rest of this paper, *we assume that (M, g) has harmonic curvature and (g, f) is a non-trivial solution of the CPE*. As immediate consequences of (7), we have the following two results.

Lemma 3.2. *For each regular value c of f and a tangent vector X to $f^{-1}(c)$,*

$$z(X, \nabla f) = 0 \tag{8}$$

on $f^{-1}(c)$.

Proof. Note that X is orthogonal to ∇f. Applying the triple $(X, \nabla f, \nabla f)$ into (7) gives

$$0 = (n-2)\mathcal{W}(X, \nabla f, \nabla f, \nabla f) = (2-n)z(X, \nabla f)|df|^2,$$

since $df(X) = 0$. This implies that (8) is true. ∎

Lemma 3.3. *On M we have*

$$|df|^2 i_{\nabla f} z = z(\nabla f, \nabla f)\, df. \tag{9}$$

Proof. From $d^D r_g = 0$ and (7), applying the triple $(\nabla f, Y, \nabla f)$ into (7) with an arbitrary vector Y gives

$$\begin{aligned}
0 &= (n-2)\mathcal{W}(\nabla f, Y, \nabla f, \nabla f) \\
&= (n-1)(|df|^2 z(Y, \nabla f) - df(Y)z(\nabla f, \nabla f)) \\
&\quad + z(\nabla f, \nabla f)df(Y) - z(\nabla f, Y)|df|^2.
\end{aligned}$$

Therefore we obtain

$$|df|^2 z(Y, \nabla f) = df(Y)z(\nabla f, \nabla f).$$

Thus equation (9) holds. ∎

On $M \setminus \mathrm{Crit}(f)$, we can define $N := df/|df|$ and $\alpha = z(N, N)$. Then we can rewrite the equation (9) as

$$i_{\nabla f} z = \alpha\, df. \tag{10}$$

Taking the divergence of (10) gives

$$\delta(i_{\nabla f} z) = -\langle d\alpha, df \rangle - \alpha \Delta f. \tag{11}$$

On the other hand, if $\{E_i\}_{i=1,\ldots,n}$ is a local orthonormal basis of vector fields,

$$\begin{aligned}
\delta(r_g(d\varphi, \cdot)) &= -\sum_i (D_{E_i} r_g(d\varphi))(E_i) = -\sum_i E_i(r(d\varphi, E_i)) \\
&= -\langle Dd\varphi, r_g \rangle + \delta r_g(d\varphi) = -\langle Dd\varphi, r_g \rangle - \frac{1}{2}\langle ds_g, d\varphi \rangle
\end{aligned}$$

for any smooth function φ. Thus, since s_g is constant, from (4) we have

$$(12) \quad \delta(i_{\nabla f}z) = \delta(r(df, \cdot)) + \frac{s_g}{n}\Delta f = -\langle Ddf, r\rangle - \frac{s_g{}^2}{n(n-1)}f = -(1+f)|z|^2.$$

Therefore, by (11) and (12) we obtain

$$(13) \qquad\qquad (1+f)|z|^2 = -\frac{s_g}{n-1}\alpha f + \langle d\alpha, df\rangle.$$

On the other hand, by Lemma 3.1 and Lemma 3.3 we have

$$-df \wedge z = \frac{\alpha}{n-1}\, df \wedge g - \frac{n-2}{n-1}\,\tilde{i}_{\nabla f}\mathcal{W}.$$

Thus, applying the triple $(E_i, \nabla f, E_j)$ into the above equation gives the following orthogonal decomposition of z.

Lemma 3.4. *Let $\{E_i\}_{i=1}^n$ be a local orthonormal frame field on $M \setminus \mathrm{Crit}(f)$ with $E_n = N = \nabla f/|\nabla f|$. Then we have*

$$(14) \qquad\qquad z_{ij} = -\frac{\alpha}{n-1}\delta_{ij} - \frac{n-2}{n-1}\mathcal{W}_{ij},$$

for $i, j = 1, ..., n-1$, where $\mathcal{W}_{ij} = \mathcal{W}_N(E_i, E_j) \equiv \mathcal{W}(E_i, N, E_j, N)$. Thus

$$(15) \qquad\qquad |z|^2 = \frac{n}{n-1}\alpha^2 + \left(\frac{n-2}{n-1}\right)^2 |\mathcal{W}_N|^2.$$

Note that equation (15) follows from $\mathrm{tr}\,\mathcal{W}_N = 0$.

Remark 3.5. Using our approach, we can easily recover Lafontaine's work, which says that, if (M, g) is locally conformally flat and (g, f) is a non-trivial solution of the CPE, then (M, g) is Einstein. In fact, if (M, g) is (locally) conformally flat, then by Lemma 3.1 and 3.2, z_g has two eigenvalues with multiplicities 1 and $n - 1$, respectively. Namely, choosing an orthonormal frame $\{E_i\}$ with $E_n = \nabla f/|\nabla f|$ and letting $\alpha = z_g(E_n, E_n)$, we have

$$(16) \qquad\qquad z_g(E_i, E_j) = -\frac{\alpha}{n-1}\delta_{ij} \quad \text{for} \quad 1 \le i, j \le n-1.$$

In this case, g is Einstein by Theorem 4.1.

Note that, if (M, g) has harmonic curvature, we may also recover (16) by (15) with Theorem 5.1. In other words, z_g also has at most two eigenvalues if (M, g) has harmonic curvature.

For a real number c, we denote L_c as a connected component of $f^{-1}(c)$. The following lemma implies that the functions $|z|^2$ and $|\mathcal{W}_N|^2$ are constant on L_c due to (13) and (15) since α, $|df|$, $\langle d\alpha, df \rangle$ are constant on L_c.

Lemma 3.6. *For each regular value c of f, α, $|df|$, and $\langle d\alpha, df \rangle$ are constant on L_c.*

Proof. From (4) and Lemma 3.2, it is easy to see that $D_N N = 0$;

$$D_N N = \sum_{i=1}^{n-1} \langle D_N N, E_i \rangle E_i = \frac{1}{|df|} \sum_{i=1}^{n-1} \langle D_N df, E_i \rangle E_i = \frac{1+f}{|df|} \sum_{i=1}^{n-1} z(N, E_i) E_i = 0.$$

Now, since $d^D r_g = 0$ and the scalar curvature s_g is constant, $d^D z = 0$. Thus for a tangent vector X to L_c, from (8) and $D_N N = 0$ we obtain

$$\begin{aligned}
X(\alpha) = X(z(N,N)) &= D_X z(N,N) + 2z(D_X N, N) \\
&= D_N z(X, N) = N(z(X,N)) - z(D_N X, N) - z(X, D_N N) \\
&= -\langle D_N X, N \rangle \alpha = 0.
\end{aligned}$$

This implies the constancy of α on $f^{-1}(c)$. In particular, $d\alpha = N(\alpha)N$. The second part follows easily; for a tangent vector ξ to the level set of f,

$$\xi|df|^2 = 2\langle D_\xi df, df \rangle = 2(1+f)z(\xi, df) - \frac{2sf}{n(n-1)}\xi(f) = 0$$

by Lemma 3.2. Also, since α is constant on each level sets of f, the third statement follows from

$$\begin{aligned}
0 = NX(\alpha) = N\langle d\alpha, X \rangle &= \langle D_N d\alpha, X \rangle \\
&= \langle D_X d\alpha, N \rangle = X\langle d\alpha, N \rangle,
\end{aligned}$$

where we used Lemma 3.2 in the last equality. ∎

It is clear that α and $d\alpha$ are defined on $M \setminus \mathrm{Crit}(f)$. For the rest of this section, we discuss the extension of α and $d\alpha$ onto all of M. Since we have $|\alpha| \leq |z|$, α can even be defined on the measure zero set $\mathrm{Crit}(f)$. In particular, if $z(x_0) = 0$ for $x_0 \in \mathrm{Crit}(f)$, $\alpha(x_0)$ can be continuously defined to be zero since $\lim_{x \to x_0} |\alpha| \leq \lim_{x \to x_0} |z| = 0$.

Let $G = |df|^2 + \frac{s}{n(n-1)}f^2$. It is easy to see that G is continuous on M. To investigate the properties of α and the structure of level sets of f, we need some technical properties of the function G. To find these, we are going to use a conformal metric $(1+f)^{-2}g$ of g which may be singular somewhere. We remark that if the scalar curvature is identically zero, the equation (4) reduces to a static space-time metric $h\, r_g = D_g dh$ with $h = 1 + f \geq 0$. In studying static metrics, it is often very useful to consider the conformal change of the metric. For example, in dimension 4, the Ricci curvature of the conformal change $h^{-2}g$ of a static metric g becomes non-negative.

Lemma 3.7. *For the function* $G = |df|^2 + \frac{s}{n(n-1)}f^2$ *and the conformal metric* $\tilde{g} = h^{-2}g$ *with* $h = 1 + f$, *we have*

(17) $$\tilde{\Delta}G + \frac{(n-3)}{h}\tilde{g}(dG, df) = 2h^4|z|^2,$$

where $\tilde{\Delta}$ *is the Laplacian of* $\tilde{g}$. *In particular, if* $x_0 \in \mathrm{Crit}(f)$ *and* $z(x_0) \neq 0$, *then* G *has its local minimum at* x_0.

Proof.   Making a conformal change (c.f. see [2]) gives

$$\tilde{\Delta}\left(|df|^2 + \frac{s}{n(n-1)}f^2\right) = 2h^4|z|^2 - 2(n-3)h^2 z(df, df),$$

since

$$\tilde{\Delta}|df|^2 = 2h^4|z|^2 - \frac{2s}{n(n-1)}h^2|df|^2 + \frac{2s^2}{n(n-1)^2}f^2h^2$$
$$-2(n-3)h^2 z(df, df) + \frac{2(n-2)}{n(n-1)}sfh|df|^2$$

and

$$\tilde{\Delta}f^2 = -\frac{2s}{n-1}h^2f^2 + 2|df|^2h^2 - 2(n-2)fh|df|^2.$$

Thus the function G satisfies (17).

Note that, for any tangent vector ξ at x_0,

(18) $$\xi(G) = 2\langle D_\xi df, df\rangle + \frac{2sf}{n(n-1)}\langle \xi, df\rangle = 2hz(\xi, df).$$

Therefore $dG = 2h\, i_{\nabla f}z$.

If $x_0 \in \mathrm{Crit}(f)$, we have $\xi(G)(x_0) = 0$ by (18). Also by (17) and the assumption that $z(x_0) \neq 0$, $\tilde{\Delta}G = 2h^4|z|^2 > 0$ at x_0. Since G is constant on each level sets of f, we may conclude that G has its local minimum at x_0. ∎

Remark 3.8. By Lemma 3.7 we can apply the maximum principle to $G = |df|^2 + \frac{s}{n(n-1)}f^2$ on the open set $M^\epsilon = \{x \in M \mid 1 + f(x) > \epsilon\}$ for an arbitrary small positive number ϵ to conclude that G achieves its maximum on $B = \{x \in M \mid f(x) = -1\}$. Similarly, we may conclude that G on the set $M_{-\epsilon} = \{x \in M \mid 1 + f(x) < -\epsilon\}$ also achieves its local maximum on B.

Let $x_1 \in \mathrm{Crit}(f)$. Note that $\mathrm{Crit}(f)$ has measure zero by Proposition 2.2 . As mentioned above, if $z(x_1) = 0$, then $\alpha(x_1)$ can be continuously defined to be zero. Now we assume that $z(x_1) \neq 0$. Let $M^0 = \{x \in M \mid 1 + f(x) > 0\}$.

Lemma 3.9. *Let $x_1 \in \mathrm{Crit}(f)$. If $x_1 \in M^0$ and $z(x_1) \neq 0$, then x_1 has to be a local maximum point of f. Also, if $f(x_1) < -1$ and $z(x_1) \neq 0$, then x_1 has to be a local minimum point of f. If $f(x_1) = -1$, then x_1 has to be a local minimum point of f.*

Proof. First of all, note that, for a connected level set $L_{f(x_1)}$ containing x_1, we have $|df|(y) = 0$ for every $y \in L_{f(x_1)}$ and G has its local minimum on $L_{f(x_1)}$ by Lemma 3.7.

First we prove the case when $x_1 \in M^0$. Suppose x_1 is not a local maximum point of f. Let x_2 be the (global) maximum point of f. Then G has its local minimum at x_2; near x_2 we have

$$dG = 2\, i_{\nabla f} z = 2h\, \alpha\, df,$$

and $h\alpha$ is negative since, for a small connected neighborhood $\Omega^\epsilon = \{x \in M \mid f(x) > f(x_2) - \epsilon\}$ of x_2 with an arbitrarily small $\epsilon > 0$ (see Fig 1 (a) where x_2 replaces x_1), $h = 1 + f > 0$ on Ω^ϵ, and

$$0 < \int_{\Omega^\epsilon} (1+f)|z|^2 = -\int_{\partial\Omega^\epsilon} \alpha|df| = -\alpha \int_{\partial\Omega^\epsilon} |df|,$$

implying that $\alpha < 0$ on $\partial\Omega^\epsilon = L_{f(x_2)-\epsilon}$. Here, we used the fact that $\int_{\Omega^\epsilon} (1+f)|z|^2 \neq 0$; otherwise, since the metric g and thus the traceless Ricci tensor z are analytic on M^0 as seen in the proof of Proposition 2.2, $z \equiv 0$ on Ω^ϵ, which implies that $z \equiv 0$ on the connected component of M^0 containing x_1. This contradicts the fact that $z(x_1) \neq 0$ for $x_1 \in M^0$.

Now consider a geodesic γ from x_1 to x_2 in M^0 (see Fig 1 (b) where x_1 replaces x_0 and x_2 replaces x_1). Since G also has its local minimum at x_2 by the above argument and the fact that G is constant on the each level sets of f, there exists a point on γ at which G has its local maximum. Then, by the maximum principle of G mentioned in Remark 3.8, G has to be constant along γ, and thus by (17) $z = 0$ on the connected subset of $\{x \in M \mid f(x_1) \leq f(x) \leq f(x_2)\}$ containing γ, contradicting our assumption that $z(x_1) \neq 0$. This completes the proof of the first statement.

The proof of the second statement is similar. The remaining case is when $f(x_1) = -1$. In this case, x_1 has to be a local minimum point of f by Proposition 2.1. ∎

By Lemma 3.9, we may conclude that any critical point x_1 of f with $z(x_1) \neq 0$ should be a local maximum if $f(x_1) > -1$, or a local minimum if $f(x_1) \leq -1$, and since α is constant on each level sets of f, $\alpha = z(df, df)/|df|^2$ can be continuously defined up to x_1 if $z(x_1) \neq 0$. If $z(x_1) = 0$ for $x_1 \in \mathrm{Crit}(f)$, $\alpha = 0$ as discussed above. In other words, α can be extended to a C^0-function on on all of M. Also the differentiation of α on $\mathrm{Crit}(f)$ can be considered in the distribution sense.

4. A Special Case

In this section we prove Theorem 1.2 in the case when $\mathcal{W}_N = 0$ on $M \setminus \mathrm{Crit}(f)$. Then by continuity $\mathcal{W}_N \equiv 0$ on M, and thus, by (15) α is smooth on all of M. Due

to the results obtained by Obata ([10]), it is sufficient for the proof of Theorem 1.2 to prove that $z = 0$ identically on M.

Theorem 4.1. *If $\mathcal{W}_N \equiv 0$ on M, then g is Einstein.*

In the following we shall prove that α is constant by showing that α is superharmonic on M (Lemma 4.6). If α is constant, since, by (12), (15), and the fact that $\int_M f = 0$,

$$\frac{n}{n-1}\alpha^2 = \frac{n}{n-1}\int_M (1+f)\,\alpha^2 = \int_M (1+f)|z|^2 = -\int_M \delta(i_{\nabla f}z) = 0,$$

implying that $\alpha \equiv 0$ on M.

When $\mathcal{W}_N \equiv 0$, by (13) and (15) we have

(19)
$$\frac{n}{n-1}(1+f)\,\alpha^2 = -\frac{s}{n-1}\,\alpha f + N(\alpha)|df|,$$

since $N(\alpha)|df| = \langle d\alpha, df \rangle$. By virtue of Lemma 3.6, we denote $\alpha' = N(\alpha)$ and $\alpha'' = NN(\alpha)$.

For the proof of Theorem 4.1, we need the following Lemma 4.2 and Lemma 4.4.

Lemma 4.2. *If $\mathcal{W}_N \equiv 0$ on M, we have the following equalities*

(20)
$$\alpha' = \frac{n}{n-1}\,\alpha\,\delta N,$$

(21)
$$\alpha'' = \frac{n\alpha}{n-1}\left(\alpha + \frac{s}{n}\right) + \frac{n+1}{n-1}\,\alpha'\,\delta N,$$

(22)
$$\Delta\,\alpha = \frac{n\alpha}{n-1}\left(\alpha + \frac{s}{n}\right) + \frac{2}{n-1}\,\alpha'\,\delta N.$$

Proof. From the definition of divergence and $D_N N = 0$,

$$\delta N = -\sum_{i=1}^{n-1}\langle D_{E_i}N, E_i\rangle$$

$$= -\frac{1}{|df|}\left((1+f)\sum_{i=1}^{n-1} z(E_i, E_i) - \frac{sf}{n}\right) = \frac{1}{|df|}\left((1+f)\alpha + \frac{sf}{n}\right).$$

Thus by (19) we obtain

$$\alpha'|df| = \frac{n}{n-1}\,\alpha\left((1+f)\alpha + \frac{s}{n}f\right) = \frac{n}{n-1}\,\alpha\,|df|\,\delta N.$$

Taking the derivative in the direction N of (19) gives

$$(n-1)\alpha'' = n\alpha^2 + s\,\alpha + (n+1)\alpha'\delta N,$$

where we used (20) and the fact that

$$(23) \qquad N(|df|) = \langle D_N df, N \rangle = (1 + f)\alpha - \frac{sf}{n(n-1)}.$$

The last equation for the Laplacian of α follows from the following observation

$$\Delta \alpha = -\delta(d\alpha) = -\delta(\alpha' N) = \alpha'' - \alpha' \delta N. \qquad \blacksquare$$

Remark 4.3. Equation (23) holds on $M \setminus \mathrm{Crit}(f)$ without any condition on $\mathcal{W}_N$.

The following is a special case of Lemma 5.2. However, we include the proof for the sake of the completeness.

Lemma 4.4. *If $\mathcal{W}_N \equiv 0$ on M, $\alpha \le 0$ on M.*

Proof. Let p be a maximum point of α. Then $\alpha'(p) = 0$ and $\Delta \alpha(p) \le 0$. From

$$0 \ge \Delta \alpha(p) = \alpha''(p) = \frac{n}{n-1}\alpha^2(p) + \frac{s}{n-1}\alpha(p),$$

$(\alpha(p))^2 + \frac{s}{n}\alpha(p) \le 0$, which implies that $-\frac{s}{n} \le \alpha(p) \le 0$. Thus we may conclude that α is always non-positive on M. $\blacksquare$

Let $H = \{x \in M \,|\, \Delta \alpha(x) \le 0\,\}$ and $\Omega = \{x \in M \,|\, \alpha(x) + \frac{s}{n} < 0\,\}$. The following lemma gives us a good understanding of the set Ω.

Lemma 4.5. *If $\mathcal{W}_N \equiv 0$ on M, $\overline{\Omega} \ne M$ and $M \setminus \Omega \subsetneq H$. In particular, if $H \subsetneq M$, Ω can be written as the disjoint union of $M \setminus H$ and $\Omega \cap H$.*

Proof. Note that $\overline{\Omega} \subsetneq M$; otherwise, on the set $M^{+1} = \{x \in M \,|\, 1 + f(x) > 1\}$

$$\delta N = \frac{1}{|df|}\left[\alpha + f(\alpha + \frac{s}{n})\right] < 0,$$

and thus we obtain the following contradiction;

$$0 > \int_{M^{+1}} \delta N = -\int_{M^{+1}} \mathrm{div}(N) = -\int_{\partial M^{+1}} \langle N, -N \rangle = \mathrm{vol}(\partial M^{+1}).$$

Thus $M \setminus \Omega$ is a non-empty set. In particular, $M \setminus \Omega$ is a subset of H. For the proof of this fact, we need to show that $\Delta \alpha \le 0$ on $M \setminus \Omega$. This follows from (22) and the facts that $\alpha \le 0$ and $\alpha' \delta N = \frac{n}{n-1}\alpha(\delta N)^2 \le 0$ on M by Lemma 4.4 and (20).

Also note that we have $\alpha = -\frac{s}{n}$ on $\partial \Omega$ and thus

$$\alpha' = \frac{n}{n-1}\alpha \delta N = \frac{n}{n-1}\frac{\alpha^2}{|df|} = \frac{s^2}{n(n-1)}\frac{1}{|df|} > 0$$

on $\partial\Omega$. Therefore, the outward unit normal to $\partial\Omega$ is given by

$$\frac{\nabla\alpha}{|\nabla\alpha|} = \frac{\alpha'N}{|\alpha'N|} = \frac{\alpha'}{|\alpha'|}N = N.$$

The positivity of α' on $\partial\Omega$ also implies that $M \setminus \Omega \neq H$, since $\delta N = -\frac{s}{n}\frac{1}{|df|} < 0$ on $\partial\Omega$ and by (22)

$$\Delta\alpha = \frac{2}{n-1}\,\alpha'\,\delta N < 0. \qquad \blacksquare$$

Lemma 4.6. *If $\mathcal{W}_N \equiv 0$ on M, $H = M$.*

Proof. By Lemma 4.5, $M \setminus H \subset \Omega$. We claim that Ω has measure zero, implying the proof of our lemma.

Suppose that the n-dimensional measure of Ω is positive. First we observe that $\alpha' \neq 0$ in $\Omega \cap H$; if there is a point $x_0 \in \Omega \cap H$ such that $\alpha'(x_0) = 0$, then, since $\alpha < -\frac{s}{n}$ on Ω

$$\Delta\alpha(x_0) = \frac{n}{n-1}\,\alpha(x_0)\left(\alpha(x_0) + \frac{s}{n}\right) > 0,$$

contradicting the fact that $\Delta\alpha \leq 0$ on H. Thus, from the fact that $\alpha' > 0$ on $\partial\Omega$, $\alpha' > 0$ on $\Omega \cap H$. For $x \in M \setminus H = \Omega \setminus (\Omega \cap H)$, $\Delta\alpha(x) > 0$, and, since α is constant on each level sets of f, we have $\alpha' \geq 0$ on the all of Ω; otherwise there exists some point $y \in M \setminus H$ such that α has a local maximum at y, which is impossible by the maximum principle. Also, due to the fact that $\alpha' > 0$ on $\partial\Omega$, $\int_\Omega \alpha' > 0$.

Note that by the proof of Lemma 4.5 the outward unit normal vector to $\partial\Omega$ is N. Now by (20)

$$\int_\Omega \delta(\alpha N) = \int_\Omega -\langle d\alpha, N \rangle + \alpha\,\delta N = -\frac{1}{n}\int_\Omega \alpha'.$$

On the other hand, by the divergence theorem

$$\int_\Omega \delta(\alpha N) = \int_{\partial\Omega} -\alpha.$$

Consequently, by Lemma 4.4

$$0 < \frac{1}{n}\int_\Omega \alpha' = \int_{\partial\Omega} \alpha \leq 0,$$

which is a contradiction. This completes the proof of Lemma 4.6. $\qquad \blacksquare$

5. THE PROOF OF THEOREM 1.2

This section is devoted to the proof of Theorem 1.2. Due to Theorem 4.1, it suffices to prove that $\mathcal{W}_N$ vanishes identically on M. More precisely,

Theorem 5.1. *Let* (g, f) *be a non-trivial solution of the* CPE. *Assume also that* (M, g) *has harmonic curvature. Then* $\mathcal{W}_N = 0$.

We first show that α is nonnegative on the whole space of M in the following lemma, and then prove that $\mathcal{W}_N = 0$ using Lemma 5.3 and Lemma 5.4.

Lemma 5.2. *Let* (g, f) *be a non-trivial solution of the* CPE. *Assume also that* (M, g) *has harmonic curvature. Then* $\alpha \leq 0$.

Proof. Suppose that $\alpha(x_0) = \max_{x \in M} \alpha(x) > 0$.

Claim 1. We have $-1 < f(x_0) < 0$.
Proof. [of Claim 1] At x_0, we have by (13)

$$(24) \qquad (1 + f)|z|^2 = -\frac{sf}{n-1}\alpha.$$

Thus $f(x_0) \neq 1$. Also $f(x_0) \neq 0$, otherwise $|z|^2(x_0) = 0$ implying that $\alpha(x_0) = 0$, a contradiction. Moreover, at x_0 by (24)

$$0 < \frac{s}{n-1}\alpha = -\frac{1+f}{f}|z|^2,$$
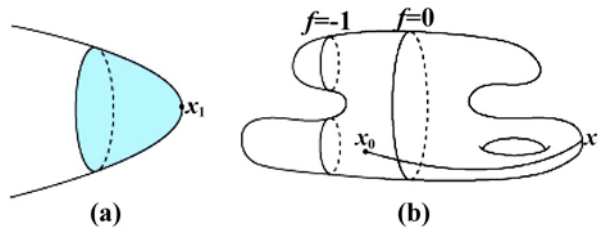
which implies our claim is true. ∎

Fig. 1. Near a critical point.

Now we are ready to derive our contradiction. Choose a maximum point x_1 of f such that a geodesic γ from x_0 to x_1 lies entirely in M^0 (see Fig 1. (b)). It is easy to see that $\alpha < 0$ near every maximum points of f as in the proof of Lemma 3.9. Thus there exist points in γ such that $\alpha = 0$ at those points. Among those points let x_2 be the point at which α changes sign. In other words, $\alpha(x_2) = 0$ and $\alpha > 0$ before the point x_2 and $\alpha < 0$ after x_2 along the geodesic γ from x_0 to x_1. To obtain a contradiction, we need to look into two cases, $|df|(x_2) \neq 0$ and $|df|(x_2) = 0$.

Case I. $|df|(x_2) \neq 0$.

Consider $L_{f(x_2)}$ with $x_2 \in L_{f(x_2)}$. Then, since $|df|(x_2) \neq 0$, there is a connected subset $\Omega_{\epsilon'}$ of M for a sufficiently small $\epsilon' > 0$ such that $f(x_2) - \epsilon' < f(x) < f(x_2)$

for every $x \in \Omega_{\epsilon'}$ with smooth boundaries $\partial \Omega_{\epsilon'} = L_{f(x_2)-\epsilon'} \cup L_{f(x_2)}$. Since

$$0 < \int_{\Omega_{\epsilon'}} (1+f)|z|^2 = \int_{L_{f(x_2)}} \alpha|df| - \int_{L_{f(x_2)-\epsilon'}} \alpha|df|$$

$$= -\int_{L_{f(x_2)-\epsilon'}} \alpha|df| < 0,$$

which is the desired contradiction.

Case II. $|df|(x_2) = 0$.

Note that the critical point x_2 cannot be a local maximum point of f; otherwise on the small connected neighborhood Ω^ϵ of x_2 given by $\Omega^\epsilon = \{x \in M \mid f(x) > f(x_2) - \epsilon\}$ with a sufficiently small ϵ (see Fig. 1 (a) where x_2 replaces x_1), α has to be positive and negative at the same time on $\partial \Omega^\epsilon$, which is impossible since α is constant on $\partial \Omega^\epsilon = L_{f(x_2)-\epsilon}$. Similarly, x_2 cannot be a local minimum point of f. Thus by Lemma 3.9, $z(x_2) = 0$ and f is increasing nearby x_2 along γ from x_0 to x_1.

If one can find a connected subset Ω_ϵ of M such that $\partial \Omega_\epsilon$ is a union of $L_{f(x_2)-\epsilon}$ and $L_{f(x_2)}$ with $x_2 \in L_{f(x_2)}$ as in the proof of Case I, we can obtain the desired contradiction. If that is not possible, we then consider the connected hypersurface components $L^i_{f(x_2)}$ of $f^{-1}(f(x_2))$ containing x_2 with $i = 1, \ldots, k$. Note that $\alpha = 0$ on $L^i_{f(x_2)}$. Then there exists a connected set $\tilde{\Omega}_\epsilon$ with a sufficiently small ϵ such that $f(x_2) - \epsilon < f(x) < f(x_2)$ for $x \in \tilde{\Omega}_\epsilon$ and either $\partial \tilde{\Omega}_\epsilon = L^j_{f(x_2)-\epsilon} \cup L^j_{f(x_2)}$ for some j (see Fig. 2 (a)), or $\partial \tilde{\Omega}_\epsilon = L_{f(x_2)-\epsilon} \cup (\cup_{i=1}^k L^i_{f(x_2)})$ (see Fig. 2 (b)). Then

$$0 < \int_{\tilde{\Omega}_\epsilon} (1+f)|z|^2 = -\int_{L^j_{f(x_2)-\epsilon}} \alpha|df| \quad \text{or} \quad -\int_{L_{f(x_2)-\epsilon}} \alpha|df|,$$

implying that $\alpha < 0$ on $L^j_{f(x_2)-\epsilon}$ or $L_{f(x_2)-\epsilon}$, which are both impossible by the definition of the point x_2.
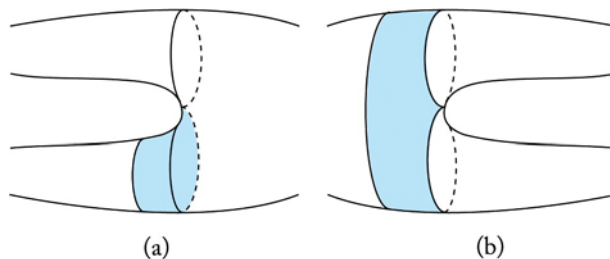

Fig. 2. Near a critical point

The contradictions derived in cases $|df|(x_2) \neq 0$ or $|df|(x_2) = 0$ complete the proof of our Lemma. ∎

Note that $\mathcal{W}_N$ is also continuously well defined on all of M as α; if $z(x_0) = 0$ for $x_0 \in \text{Crit}(f)$, $\mathcal{W}_N$ can be defined as zero since $|\mathcal{W}_N| \leq \frac{n-1}{n-2}|z|$, and if $z(x_1) \neq 0$, x_1 is not a critical point of f unless it is a local maximum or minimum point. The differentiation of $\mathcal{W}_N$ on $\text{Crit}(f)$ can also be considered in the distribution sense. We can compute the divergence of $\mathcal{W}_N$ as follows.

Lemma 5.3. *Let $\{E_i\}_{1 \leq i \leq n}$ be an orthonormal frame field with $E_n = N$. Then we have*

$$\delta\mathcal{W}_N(E_i) = 0 \quad and \quad \delta\mathcal{W}_N(N) = -\frac{n-2}{n-1}\frac{1+f}{|df|}|\mathcal{W}_N|^2.$$

Thus, on $M \setminus \text{Crit}(f)$

(25) $$\delta\mathcal{W}_N = \delta\mathcal{W}_N(N)N = \frac{\delta\mathcal{W}_N(N)}{|df|}df.$$

Proof. Since $\delta\mathcal{W} = 0$, for $X = E_j$

$$0 = \delta\mathcal{W}(N, X, N) = -\sum_i D_{E_i}\mathcal{W}(E_i, N, X, N)$$

$$= -\sum_i E_i(\mathcal{W}(E_i, N, X, N)) + \sum_i [\mathcal{W}(D_{E_i}E_i, N, X, N)$$

$$+ \mathcal{W}(E_i, D_{E_i}N, X, N) + \mathcal{W}(E_i, N, D_{E_i}X, N) + \mathcal{W}(E_i, N, X, D_{E_i}N)].$$

Thus

$$\delta\mathcal{W}_N(X) = -\sum_i D_{E_i}\mathcal{W}_N(E_i, X)$$

$$= -\sum_i E_i(\mathcal{W}_N(E_i, X)) + \sum_i (\mathcal{W}_N(D_{E_i}E_i, X) + \mathcal{W}_N(E_i, D_{E_i}X))$$

$$= -\sum_i (\mathcal{W}(E_i, D_{E_i}N, X, N) + \mathcal{W}(E_i, N, X, D_{E_i}N)).$$

Thus, by (7)

$$(n-2)\mathcal{W}(E_i, D_{E_i}N, X, \nabla f) = (n-1)df(E_i)z(D_{E_i}N, X) - df(D_{E_i}N)z(E_i, X)$$

$$+ z(\nabla f, E_i)g(D_{E_i}N, X) - z(\nabla f, D_{E_i}N)g(E_i, X).$$

Note that $df(E_i) = 0$,

$$df(D_{E_i}N) = \langle D_{E_i}N, \nabla f\rangle = \frac{1}{|df|}(1+f)z(E_i, \nabla f) = 0,$$

and $z(\nabla f, E_i) = 0$ by Lemma 3.2, and finally $z(\nabla f, D_{E_i}N) = 0$, from the facts that $|df|$ is constant on each level sets of f, and

$$(26) \qquad D_{E_i}N = \sum_{j=1}^{n-1}\langle D_{E_i}N, E_j\rangle E_j = \frac{1}{|df|}\sum_{j=1}^{n-1}\langle D_{E_i}df, E_j\rangle E_j.$$

Therefore $\mathcal{W}(E_i, D_{E_i}N, X, N) = \frac{1}{|df|}\mathcal{W}(E_i, D_{E_i}N, X, \nabla f) = 0$. Similarly, we have $\mathcal{W}(X, D_{E_i}N, E_i, N) = 0$. Hence we may conclude that $\delta\mathcal{W}_N(X) = 0$.

It still remains to show the second identity is correct. Since $z(N, E_i) = 0$ and $\mathcal{W}_N(N, \cdot) = 0$, we may assume that $E_1, \ldots, E_{n-1}$ diagonalize $\mathcal{W}_N$ at a point p. Then, at p

$$\delta\mathcal{W}_N(N) = -\sum_{i=1}^{n} D_{E_i}\mathcal{W}_N(E_i, N) = \sum_{i=1}^{n-1}\mathcal{W}_N(E_i, D_{E_i}N)$$

$$= \sum_{i=1}^{n-1}\langle D_{E_i}N, E_i\rangle\mathcal{W}_N(E_i, E_i).$$

Here we used the fact that $\mathcal{W}_N(D_{E_i}E_i, N) = 0$. So, from the CPE,

$$\langle D_{E_i}N, E_i\rangle = \frac{1}{|df|}\langle D_{E_i}df, E_i\rangle$$

$$= \frac{1}{|df|}\left((1+f)z(E_i, E_i) - \frac{sf}{n(n-1)}\right)$$

$$= -\frac{1}{(n-1)|df|}\left(\frac{sf}{n} + (1+f)\alpha + (n-2)(1+f)\mathcal{W}_N(E_i, E_i)\right).$$

Hence

$$\delta\mathcal{W}_N(N) = \sum_{i=1}^{n-1}\langle D_{E_i}N, E_i\rangle\mathcal{W}_N(E_i, E_i)$$

$$= -\frac{1}{(n-1)|df|}\left(\frac{sf}{n} + (1+f)\alpha\right)\sum_{i=1}^{n-1}\mathcal{W}_N(E_i, E_i)$$

$$-\frac{n-2}{n-1}\frac{1+f}{|df|}\sum_{i=1}^{n-1}\mathcal{W}_N(E_i, E_i)^2$$

$$= -\frac{n-2}{n-1}\frac{1+f}{|df|}|\mathcal{W}_N|^2,$$

where we used the fact that $\operatorname{tr}\mathcal{W}_N = 0$ in the last equation. ∎

Lemma 5.4. *The differential form $\delta\mathcal{W}_N$ is a closed 1-form on $M \setminus \mathrm{Crit}(f)$.*

Proof. Note that from (26)

$$D_{E_i} N = -\frac{1}{(n-1)|df|} \left(\frac{sf}{n} + (1+f)\alpha + (n-2)(1+f)\mathcal{W}_N(E_i, E_i) \right) E_i$$

for $i = 1, \ldots, n-1$. Since $\mathcal{W}_N(E_i, E_i) = -\frac{n-1}{n-2} \left(\frac{\alpha}{n-1} + z(E_i, E_i) \right)$ by (14), we obtain

$$(27) \qquad D_{E_i} N = \frac{1}{|df|} \left((1+f)z(E_i, E_i) - \frac{sf}{n(n-1)} \right) E_i.$$

In particular, if $i \neq j$,

$$\langle D_{E_i} E_j, N \rangle = -\langle E_j, D_{E_i} N \rangle = 0$$

and thus

$$\langle [E_i, E_j], N \rangle = \langle D_{E_i} E_j - D_{E_j} E_i, N \rangle = 0.$$

Hence for a regular value c of f, the level set $f^{-1}(c)$ is an integrable hypersurface in M. Since $\delta\mathcal{W}_N(E_i) = 0$ for $1 \leq i \leq n-1$ by Lemma 5.3, it follows from the Frobenius theorem that

$$d\delta\mathcal{W}_N(E_i, E_j) = 0 \quad \text{for} \quad 1 \leq i, j \leq n-1.$$

Next, by Lemma 5.3 and the fact that $f, |df|$ and $|\mathcal{W}_N|^2$ are all constant on each level sets of f, we obtain

$$\begin{aligned}
d\delta\mathcal{W}_N(N, E_i) &= N(\delta\mathcal{W}_N(E_i)) - E_i(\delta\mathcal{W}_N(N)) - \delta\mathcal{W}_N([N, E_i]) \\
&= -\delta\mathcal{W}_N([N, E_i]) = \delta\mathcal{W}_N(D_{E_i}N - D_N E_i).
\end{aligned}$$

By equation (27) and Lemma 5.3, $\delta\mathcal{W}_N(D_{E_i}N) = 0$ and

$$\begin{aligned}
\delta\mathcal{W}_N(D_N E_i) &= \langle D_N E_i, N \rangle \delta\mathcal{W}_N(N) \\
&= -\langle E_i, D_N N \rangle \delta\mathcal{W}_N(N) = 0
\end{aligned}$$

since $D_N N = 0$. Thus, we have $d\delta\mathcal{W}_N(N, E_i) = 0$ and consequently $\delta\mathcal{W}_N$ is a closed 1-form on M. This proves our lemma. ∎

By Lemma 5.4 and Proposition 2.2, therefore, $d\delta\mathcal{W}_N = 0$ almost everywhere. Now we are ready to prove Theorem 5.1.

Considering $\delta \mathcal{W}_N$ as a vector, it follows from Lemma 5.3 that

$$
\begin{aligned}
\langle \delta Ddf, \delta \mathcal{W}_N \rangle &= \langle -d\Delta f - r(df, \cdot), \delta \mathcal{W}_N \rangle \\
&= \frac{s}{n-1} \langle df, \delta \mathcal{W}_N \rangle - r(df, \delta \mathcal{W}_N) \\
&= \frac{s}{n-1} \delta \mathcal{W}_N(N)|df| - |df| \delta \mathcal{W}_N(N) r(N, N) \\
&= \frac{s}{n-1} \delta \mathcal{W}_N(N)|df| - |df| \delta \mathcal{W}_N(N) \left(\alpha + \frac{s}{n} \right) \\
&= a \left(\alpha - \frac{s}{n(n-1)} \right) (1+f)|\mathcal{W}_N|^2,
\end{aligned}
$$

where $a = \frac{n-2}{n-1}$. Integrating this over M^0, we have

$$
\int_{M^0} \langle \delta Ddf, \delta \mathcal{W}_N \rangle = a \int_{M^0} \left(\alpha - \frac{s}{n(n-1)} \right) (1+f)|\mathcal{W}_N|^2.
$$

On the other hand, by Lemma 5.3 and Lemma 5.4 and the divergence theorem,

$$
\begin{aligned}
\int_{M^0} \langle \delta Ddf, \delta \mathcal{W}_N \rangle &= \int_{M^0} \langle Ddf, d\delta \mathcal{W}_N \rangle - \int_{\partial M^0} Ddf(\delta \mathcal{W}_N, N) \\
&= -\int_{\partial M^0} \delta \mathcal{W}_N(N) Ddf(N, N) \\
&= a \int_{\partial M^0} \frac{1+f}{|df|} |\mathcal{W}_N|^2 Ddf(N, N) = 0,
\end{aligned}
$$

where the second equation follows from the fact that $d\delta \mathcal{W}_N = 0$ almost everywhere, and the last equation follows from the definition of $\partial M^0 \subset f^{-1}(-1)$. Thus

$$
\int_{M^0} \left(\alpha - \frac{s}{n(n-1)} \right) (1+f)|\mathcal{W}_N|^2 = 0.
$$

Since $\alpha - \frac{s}{n(n-1)} < 0$ by Lemma 5.2, we may conclude that $\mathcal{W}_N = 0$ on the set $f > -1$. Integrating the same integrand over $f < -1$, we have $\mathcal{W}_N = 0$ on the whole space M. This completes the proof of Theorem 5.1.

<div align="center">REFERENCES</div>

1. A. L. Besse, *Einstein Manifolds*, New York, Springer-Verlag, 1987.

2. L. Bessiéres, J. Lafontiane and L. Rozoy, *Scalar curvature and black holes*, preprint.

3. J. Chang, G. Yun and S. Hwang, Critical point metrics of Total scalar curvature, *Bull. Korean Math. Soc.*, **49(3)** (2012), 655-667.

4. A. Derdzinski, On compact Riemannian manifolds with Harmonic curvature, *Math. Ann.*, **259** (1982), 145-152.

5. A. E. Fischer and J. E. Marsden, Manifolds of Riemannian metrics with prescribed scalar curvature, *Bull. Am. Math. Soc.*, **80** (1974), 479-484.

6. A. Gray, Einstein-like manifolds which are not Einstein, *Geometriae Dedicata*, **7** (1978), 259-280.

7. S. Hwang, Critical points of the scalar curvature functionals on the space of metrics of constant scalar curvature, *Manuscripta Math.*, **103** (2000), 135-142.

8. J. Lafontaine, Sur la géométrie d'une généralisation de l'équation différentielle d'Obata, *J. Math. Pures Appliquées*, **62** (1983), 63-72.

9. C. B. Morrey, *Multiple Integrals in the Calculus of Variations*, Séminaire de Théorie Spectrale et Géométrie, Springer-Verlag Berlin, 1966.

10. M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, *J. Math. Soc. Japan*, **14(3)** (1962), 333-340.

Gabjin Yun
Department of Mathematics
Myong Ji University
San 38-2 Namdong
Yongin, Gyeonggi 449-728
Korea
E-mail: gabjin@mju.ac.kr

Jeongwook Chang
Department of Mathematics Education
Dankook University
126 Jukjeon-dong
Suji-gu Yongin
Gyeonggi 448-701
Korea
E-mail: jchang@dankook.ac.kr

Seungsu Hwang
Department of Mathematics
Chung-Ang University
84 HeukSeok-ro DongJak-gu, Seoul
Korea
E-mail: seungsu@cau.ac.kr