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HOMOCLINIC SOLUTIONS FOR A CLASS OF NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH TIME-VARYING DELAYS

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Abstract. In this paper, by using Mawhin's continuation theorem of coincidence degree theory, we obtain some sufficient conditions for the existence of homoclinic solutions for a class of nonlinear second-order differential equations with time-varying delays. Moreover, we give an example to illustrate the feasibility of obtained results. Our results are completely new.

1. INTRODUCTION

In the past few years, there has been increasing interest in studying the existence of solutions, such as periodic solutions, almost periodic solutions or anti-periodic solutions, of differential equations. Since homoclinic orbits play an important role in nonlinear dynamical systems, problems of existence of homoclinic solutions are of utmost importance in the study of differential equations. Recently, the existence and multiplicity of homoclinic solutions has become one of the most important problems in the research of differential systems. And there have been many results on the existence of homoclinic solutions for first order or second order differential equations (see [1-10] and references cited therein).

Most existing results on the existence of homoclinic solutions for differential equations are obtained by using critical point theory (or variation method). For example, in [11], authors studied the existence of homoclinic orbits for the following second order Hamiltonian system

$$q''(t) + V_q(t, q) = f(t).$$

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In [12], by using the Mountain Pass theorem, authors discussed the existence of homoclinic solutions for the following second-order Hamiltonian system

$$x''(t) - L(t)x(t) + \nabla w(t, x(t)) = 0.$$

And in [13], by means of variation method, author presented an existence result of homoclinic solutions to the following nonlinear second-order differential equation

$$x'' + 2f(t)x' + \beta(t)x + g(t, x) = 0, t \in R.$$

Recently, in [14], by using Mawhin's continuation theorem, authors obtained some sufficient conditions ensuring the existence of homoclinic solutions for the following differential equation

$$u''(t) + g(u'(t)) + h(x(t)) = f(t).$$

This equation is important in the applied sciences such as nonlinear vibration of masses, see [15-17] and the references therein.

However, it is well known that more realistic models should include some of the past states of these systems, that is, ideally, a real system should be modeled by differential equations with time delays. Therefore, the research on delay differential equations has much significance. To the best of our knowledge, few papers have been published on the existence of homoclinic solutions for differential equations with delays.

Motivated by above mentioned, in this paper, applying the coincidence degree theory, we study the existence of homoclinic solutions for the following nonlinear differential equation with time-varying delays

$$\begin{aligned} &x''(t) + a(t)g(x'(t)) \\ &+\sum_{i=1}^{n} b_i(t)h_i\big(x(t-\tau_1(t)), x(t-\tau_2(t)), \dots, x(t-\tau_n(t))\big) = f(t), \end{aligned} \tag{1.1}$$

where $t \in R$, $g, f \in C(R, R)$, $h_i \in C(R^n, R)$, $\tau_i \in C(R, (0, +\infty))$, $a, b_i \in C(R, [0, +\infty))$ with $a^- \le a \le a^+$, $b_i^- \le b_i \le b_i^+$, a^-, a^+, b_i^-, b_i^+ are all positive constants and $a(t), b_i(t)$ are all $2T$-periodic, $T > 0$ ia a given constant, $i = 1, 2, \dots, n$. In order to investigate the homoclinic solutions of (1.1), firstly, we study the existence of $2kT$-periodic solutions of the following equation for each $k \in N$:

$$\begin{aligned} &x''(t) + a(t)g(x'(t)) \\ &+\sum_{i=1}^{n} b_i(t)h_i\big(x(t-\tau_1(t)), x(t-\tau_2(t)), \dots, x(t-\tau_n(t))\big) = f_k(t), \end{aligned} \tag{1.2}$$

where $f_k : R \to R$ is a $2kT$-periodic function, $k \in N$ and

$$f_k(t) = \begin{cases} f(t), \quad t \in [-kT, kT - \varepsilon), \\ f(kT - \varepsilon) + \frac{f(-kT) - f(kT-\varepsilon)}{\varepsilon}(t - kT + \varepsilon), \quad t \in [kT - \varepsilon, kT], \end{cases}$$

$\varepsilon \in (-T, T)$ is a constant independent of $k \in N$. Then a homoclinic solution of (1.1) is obtained as a limit point of the set $\{x_k(t)\}$, where $\{x_k(t)\}$ is a $2kT$-periodic solution of (1.2) for each $k \in N$.

Let $C_{2kT} = \{x \in C(R, R) | x(t + 2kT) = x(t)\}$ with norm $||x||_{C_{2kT}} = \max\limits_{t\in[-kT,kT]} |x(t)|$ and $C^1_{2kT} = \{x \in C^1(R, R) | x(t+2kT) = x(t)\}$ with norm $||x||_{C^1_{2kT}} = \max\{||x||_{C_{2kT}}, ||x'||_{C_{2kT}}\}$. Then both C_{2kT} and C^1_{2kT} are Banach spaces. For $x \in C_{2kT}$, we denote $||x||_p = \big(\int_{-kT}^{kT} |x(t)|^p\big)^{\frac{1}{p}}$, where $p \in (1, +\infty)$. It is obvious that norms $||x||_{C_{2kT}}$ and $||x||_p$ are equivalent. Further, we assume that there is an integer m_0 such that $\{\tau_i(t) : t \in [0, 2T]\} \subset [(m_0 - 1)T, (m_0 + 1)T]$, $i = 1, 2, \ldots, n$, denote $\theta := \max\limits_{1\le i\le n} \max\limits_{t\in[0,2T]} |\tau_i(t) - m_0T|$.

Throughout this paper, we assume the following conditions hold

(H_1) $\sup\limits_{t\in R} |f(t)| < \infty$, $\int_R |f(t)|^2 \mathrm{d}t < \infty$, $\int_{\vartheta\in R^n} |h_i(\vartheta)|^2 \mathrm{d}\vartheta < \infty$, $i = 1, 2, \ldots, n$;

(H_2) there exist positive constants n_i and $L^{(i)}_k$ such that $uh_i(u, u, \ldots, u) \le -n_i|u|^2$ and

$$|h_i(u_1, u_2, \ldots, u_n) - h_i(v_1, v_2, \ldots, v_n)| \le L^{(i)}_1 |u_1 - v_1| + L^{(i)}_2 |u_2 - v_2| + \cdots + L^{(i)}_n |u_n - v_n|, \forall u, u_i, v_i \in R, i, k = 1, 2, \ldots, n;$$

(H_3) $|g(u)| \le m_1|u|$, $ug(u) \ge mu^2$, where m satisfies $a^-m > \sqrt{2}b^+L\theta$, $b^+ = \max\limits_{1\le i\le n} b^+_i$ and $L = \max\limits_{1\le i\le n} \sum\limits_{k=1}^{n} L^{(i)}_k$, m_1 is a positive constant, $\forall u \in R$.

2. Preliminaries

In this section, we state some preliminary results.

Let X, Y be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if dim Ker L=codim Im $L < +\infty$ and Im L is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that Im P=Ker L, Ker Q=Im L=Im $(I - Q)$, it follows that mapping $L|_{\operatorname{Dom} L\cap \operatorname{Ker} P} : (I - P)X \to$Im L is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Since Im Q is isomorphic to Ker L, there exists an isomorphism $J : \operatorname{Im} Q \to$ Ker L.

Lemma 2.1. [18]. *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Y$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume*

(a) *for each $\lambda \in (0, 1), x \in \partial\Omega \cap \operatorname{Dom} L, Lx \ne \lambda Nx$;*

(b) *for each $x \in \partial\Omega \cap$ Ker $L, QNx \ne 0$;*

(c) $\deg(JNQ, \Omega\cap \mathrm{Ker}\ L, 0) \neq 0$.

Then $Lx = Nx$ *has at least one solution in* $\overline{\Omega}\cap \mathrm{Dom}\, L$.

Lemma 2.2. [19]. *Let* $r > 0$ *and* $u \in W^{1,p}(R, R)$. *Then for any* $t \in R$, *the following inequality holds*

$$|u(t)| \leq (2r)^{-\frac{1}{q}}\Big(\int_{t-r}^{t+r}|u(s)|^q \mathrm{d}s\Big)^{\frac{1}{q}} + r(2r)^{-\frac{1}{p}}\Big(\int_{t-r}^{t+r}|u'(s)|^p \mathrm{d}s\Big)^{\frac{1}{p}},$$

where $p, q \in (1, +\infty)$ *are constants.*

Lemma 2.3. [20]. *Let* $0 \leq \alpha \leq 2T$ *be a constant and* s *be* $2T$*-periodic with* $\max\limits_{t\in[0,2T]}|s(t)| \leq \alpha$. *Then for any* $x \in C^1_{2T}$, *we have*

$$\int_0^{2T}|x(t) - x(t - s(t))|^2 \mathrm{d}t \leq 2\alpha^2\int_0^{2T}|x'(t)|^2 \mathrm{d}t.$$

Lemma 2.4. [14]. *Let* (H_1) *hold. Then* $||f_k||_{C_{2kT}}$ *and* $||f_k||_p$ *are constants independent of* $k \in N$, *where* $p > 1$ *is a constant.*

3. Main Results

In this section, we will state and prove the existence of homoclinic solutions for (1.1).

Theorem 3.1. *Let* (H_1), (H_2) *and* (H_3) *hold. Then* (1.2) *has at least one* $2kT$*-periodic solution for each* $k \in N$.

Proof. Set $X = C_{2kT}$ and $Y = C^1_{2kT}$. Let

$$L : \mathrm{Dom}\ L\cap X \to Y, \quad Lx = x'', \ x \in X,$$

where $\mathrm{Dom}\, L = \{x \in C^2(R, R)|x(t + 2kT) = x(t)\}$. Clearly, Ker $L = R$ and Im $L = \big\{y \in Y|\int_{-kT}^{kT} y(s)\mathrm{d}s = 0\big\}$. Thus L is a Fredholm operator with index zero. Set

$$P : X \to \mathrm{Ker}\ L, \quad Px = \frac{1}{2kT}\int_0^{2kT} x(s)\mathrm{d}s, \ x \in X$$

and

$$Q : Y \to Y/\mathrm{Im}\ L, \quad Qy = \frac{1}{2kT}\int_0^{2kT} y(s)\mathrm{d}s, \ y \in Y.$$

We can obtain that the inverse K_P : Im $L \to$ Ker $P\cap$ Dom L of L_P exists and is given by

$$K_P(y) = \int_0^{2kT} G(t, s)y(s)\mathrm{d}s,$$

where

$$G(t,s)=\begin{cases}\dfrac{s(2kT-t)}{2kT}, & 0\le s<t\le 2kT,\\ \dfrac{t(2kT-s)}{2kT}, & 0\le t<s\le 2kT.\end{cases}$$

Define $N : X \to Y$ as follows

$$N(x)=-a(t)g(x'(t))-\sum_{i=1}^{n} b_i(t)h_i\big(x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_n(t))\big)+f_k(t).$$

For any open bounded set $\Omega\subset X$, it is easy to verify that N is L-compact on $\bar{\Omega}$. Now, we are in the position of searching for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation

$$Lx=\lambda Nx,\ \ \lambda\in(0,1),$$

we have

$$\begin{aligned}&x''(t)+\lambda a(t)g(x'(t))\\ &+\lambda\sum_{i=1}^{n} b_i(t)h_i\big(x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_n(t))\big)=\lambda f_k(t).\end{aligned}\tag{3.1}$$

Suppose that $x(t)\in X$ is a solution of system (3.1) for a certain $\lambda\in(0,1)$. Multiplying both sides of (3.1) by $x'(t)$ and integrating on the interval $[-kT,kT]$, we have that

$$\begin{aligned}&\int_{-kT}^{kT} x'(t)f_k(t)\mathrm{d}t\\ =&\int_{-kT}^{kT} a(t)x'(t)g(x'(t))\mathrm{d}t\\ &+\int_{-kT}^{kT} x'(t)\sum_{i=1}^{n} b_i(t)h_i\big(x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_n(t))\big)\mathrm{d}t.\end{aligned}$$

Hence, in view of (H_1) and (H_3), we obtain

$$\begin{aligned}&a^{-}m\int_{-kT}^{kT}(x'(t))^2\mathrm{d}t\\ \le&\Big|\int_{-kT}^{kT} x'(t)\sum_{i=1}^{n} b_i(t)h_i\big(x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_n(t))\big)\mathrm{d}t\Big|\\ &+\Big|\int_{-kT}^{kT} x'(t)f_k(t)\mathrm{d}t\Big|\\ \le&\sum_{i=1}^{n} b_i^{+}\Big(\int_{-kT}^{kT}|x'(t)||h_i\big(x(t),x(t),\ldots,x(t)\big)\\ &-h_i\big(x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_n(t))\big)|\mathrm{d}t\\ &+\int_{-kT}^{kT}|x'(t)||h_i\big(x(t),x(t),\ldots,x(t)\big)|\mathrm{d}t\Big)+\Big|\int_{-kT}^{kT} x'(t)f_k(t)\mathrm{d}t\Big|.\end{aligned}\tag{3.2}$$

Using Lemma 2.3, for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
&\int_{-kT}^{kT} |x'(t)||h_i\big(x(t), x(t), \ldots, x(t)\big) \\
&\qquad -h_i\big(x(t-\tau_1(t)), x(t-\tau_2(t)), \ldots, x(t-\tau_n(t))\big)|\mathrm{d}t \\
\leq\ & \int_{-kT}^{kT} |x'(t)|\big(L_1^{(i)}|x(t)-x(t-\tau_1(t))| + L_2^{(i)}|x(t)-x(t-\tau_2(t))| + \cdots \\
&+L_n^{(i)}|x(t)-x(t-\tau_n(t))|\big)\mathrm{d}t \\
\leq\ & L_1^{(i)}\Big(\int_{-kT}^{kT}|x'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_{-kT}^{kT}|x(t)-x(t-\tau_1(t))|^2\mathrm{d}t\Big)^{\frac{1}{2}} + L_2^{(i)}\Big(\int_{-kT}^{kT}|x'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \\
&\times\Big(\int_{-kT}^{kT}|x(t)-x(t-\tau_2(t))|^2\mathrm{d}t\Big)^{\frac{1}{2}} + \cdots + L_n^{(i)}\Big(\int_{-kT}^{kT}|x'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \\
&\times\Big(\int_{-kT}^{kT}|x(t)-x(t-\tau_n(t))|^2\mathrm{d}t\Big)^{\frac{1}{2}} \\
\leq\ & ||x'||_2\sum_{i=1}^{n} L_i\Big(2\theta^2\int_{-kT}^{kT}|x'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \\
=\ & \sqrt{2}\sum_{k=1}^{n} L_k^{(i)}\theta||x'||_2^2.
\end{aligned}$$

Thus, from (3.2), we have

$$a^- m||x'||_2^2 \leq \sqrt{2}b^+L\theta||x'||_2^2 + b^+H||x'||_2 + ||f_k||_2||x'||_2,$$

where $H = \max\limits_{1\leq i\leq n}||h_i||_2$. It follows that

$$||x'||_2 \leq \frac{b^+H + ||f_k||_2}{a^- m - \sqrt{2}b^+L\theta} := \alpha_1.$$

Since $||x'||_2$ and $||x'||_{C_{2kT}}$ are equivalent, there exist a constant $c_1 > 0$ such that $||x'||_{C_{2kT}} \leq c_1||x'||_2 = c_1\alpha_1$.

Multiplying both sides of (3.1) by $x(t)$ and integrating on the interval $[-kT, kT]$, we have that

$$\begin{aligned}
&\lambda\int_{-kT}^{kT} x(t)f_k(t)\mathrm{d}t \\
=\ & \int_{-kT}^{kT} x(t)x''(t)\mathrm{d}t + \lambda\int_{-kT}^{kT} a(t)x(t)g(x'(t))\mathrm{d}t \\
&+\lambda\int_{-kT}^{kT} x(t)\sum_{i=1}^{n} b_i(t)h_i\big(x(t-\tau_1(t)), x(t-\tau_2(t)), \ldots, x(t-\tau_n(t))\big)\mathrm{d}t.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\int_{-kT}^{kT}\big(x'(t)\big)^2\mathrm{d}t\\&=\lambda\int_{-kT}^{kT}a(t)x(t)g(x'(t))\mathrm{d}t-\lambda\int_{-kT}^{kT}x(t)f_k(t)\mathrm{d}t\\&\quad+\lambda\int_{-kT}^{kT}x(t)\sum_{i=1}^{n}b_i(t)h_i\big(x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_n(t))\big)\mathrm{d}t.\end{aligned}$$

Define $F_k(t)=\int_0^t f_k(s)\mathrm{d}s$, then $-\int_{-kT}^{kT}x(t)f_k(t)\mathrm{d}t=\int_{-kT}^{kT}x'(t)F_k(t)\mathrm{d}t$. Therefore, we have

$$\begin{aligned}&\int_{-kT}^{kT}\big(x'(t)\big)^2\mathrm{d}t\\&=\lambda\int_{-kT}^{kT}x(t)\sum_{i=1}^{n}b_i(t)h_i\big(x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_n(t))\big)\mathrm{d}t\\&\quad+\lambda\int_{-kT}^{kT}a(t)x(t)g(x'(t))\mathrm{d}t+\lambda\int_{-kT}^{kT}x'(t)F_k(t)\mathrm{d}t\\&=\lambda\int_{-kT}^{kT}x(t)\sum_{i=1}^{n}b_i(t)h_i\big(x(t),x(t),\ldots,x(t)\big)\mathrm{d}t\\&\quad+\lambda\int_{-kT}^{kT}x(t)\sum_{i=1}^{n}b_i(t)\Big[h_i\big(x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_n(t))\big)\\&\quad-h_i\big(x(t),x(t),\ldots,x(t)\big)\Big]\mathrm{d}t+\lambda\int_{-kT}^{kT}a(t)x(t)g(x'(t))\mathrm{d}t\\&\quad+\lambda\int_{-kT}^{kT}x'(t)F_k(t)\mathrm{d}t.\end{aligned}\tag{3.3}$$

In view of (H_2) and Lemma 2.3, for $i=1,2,\ldots,n$, we have

$$\begin{aligned}&\int_{-kT}^{kT}|x(t)|\big|h_i\big(x(t),x(t),\ldots,x(t)\big)\\&\quad-h_i\big(x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_n(t))\big)\big|\mathrm{d}t\\&\le\int_{-kT}^{kT}|x(t)|\big(L_1^{(i)}|x(t)-x(t-\tau_1(t))|+L_2^{(i)}|x(t)-x(t-\tau_2(t))|+\cdots\\&\quad+L_n^{(i)}|x(t)-x(t-\tau_n(t))|\big)\mathrm{d}t\\&\le L_1^{(i)}\Big(\int_{-kT}^{kT}|x(t)|^2\mathrm{d}t\Big)^{\frac12}\Big(\int_{-kT}^{kT}|x(t)-x(t-\tau_1(t))|^2\mathrm{d}t\Big)^{\frac12}\end{aligned}\tag{3.4}$$

$$
\begin{aligned}
&+L_2^{(i)}\Big(\int_{-kT}^{kT}|x(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\\
&\times\Big(\int_{-kT}^{kT}|x(t)-x(t-\tau_2(t))|^2\mathrm{d}t\Big)^{\frac{1}{2}}+\cdots+L_n^{(i)}\Big(\int_{-kT}^{kT}|x(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\\
&\times\Big(\int_{-kT}^{kT}|x(t)-x(t-\tau_n(t))|^2\mathrm{d}t\Big)^{\frac{1}{2}}\\
\le\ & ||x||_2\sum_{k=1}^{n}L_k^{(i)}\Big(2\theta^2\int_{-kT}^{kT}|x'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\\
=\ & \sqrt{2}\theta\sum_{k=1}^{n}L_k^{(i)}||x||_2||x'||_2
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda\textstyle\int_{-kT}^{kT}a(t)x(t)g(x'(t))\mathrm{d}t\ &\le\ a^+\int_{-kT}^{kT}|x(t)||g(x'(t))|\mathrm{d}t\\
&\le\ a^+m_1\int_{-kT}^{kT}|x(t)||x'(t)|\mathrm{d}t\\
&\le\ a^+m_1\Big(\int_{-kT}^{kT}|x(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_{-kT}^{kT}|x'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\\
&=\ a^+m_1||x||_2||x'||_2.
\end{aligned}
\tag{3.5}
$$

Moreover, we also have

$$
\lambda\int_{-kT}^{kT}x'(t)F_k(t)\mathrm{d}t\le||F_k||_2||x'||_2.
\tag{3.6}
$$

Here in view of Lemma 2.4, we have that $||F_k||_2$ is independent of $k\in N$. From (3.3), we have that

$$
\begin{aligned}
&-\lambda\int_{-kT}^{kT}x(t)\sum_{i=1}^{n}b_i(t)h_i\big(x(t),x(t),\ldots,x(t)\big)\mathrm{d}t\\
\le\ & \lambda\int_{-kT}^{kT}x(t)\sum_{i=1}^{n}b_i(t)\Big[h_i\big(x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_n(t))\big)\\
&-h_i\big(x(t),x(t),\ldots,x(t)\big)\Big]\mathrm{d}t+\lambda\int_{-kT}^{kT}a(t)x(t)g(x'(t))\mathrm{d}t\\
&+\lambda\int_{-kT}^{kT}x'(t)F_k(t)\mathrm{d}t.
\end{aligned}
$$

Combining with (H_3), (3.4), (3.5) and (3.6), we have that

$$
\begin{aligned}
b^-l\int_{-kT}^{kT}|x|^2\mathrm{d}t\ &\le\ \sqrt{2}b^+L\theta||x||_2||x'||_2+a^+m_1||x||_2||x'||_2+||F_k||_2||x'||_2\\
&\le\ \sqrt{2}b^+L\theta||x||_2\alpha_1+a^+m_1||x||_2\alpha_1+||F_k||_2\alpha_1,
\end{aligned}
$$

where $b^- = \min\limits_{1\le i\le n} b_i^-$ and $l = \min\limits_{1\le i\le n} n_i$. It follows that

$$b^- l||x||_2^2 - (\sqrt{2}b^+ L\theta + a^+ m_1)\alpha_1 ||x||_2 - ||F_k||_2 \alpha_1 \le 0,$$

that is,

$$||x||_2 \le \frac{\sqrt{2}b^+ L\theta + a^+ m_1 + \sqrt{(\sqrt{2}b^+ L\theta + a^+ m_1)^2 \alpha_1^2 + 4b^- l||F_k||_2 \alpha_1}}{2b^- l} := \alpha_2.$$

Since $||x||_2$ and $||x||_{C_{2kT}}$ are equivalent, there exist a constant $c_2 > 0$ such that $||x||_{C_{2kT}} \le c_2 ||x||_2 = c_2\alpha_2$. From what has been discussed above, we finally derive that $||x||_{C^1_{2kT}} \le M_1$, where

$$M_1 = \max\{c_1\alpha_1, c_2\alpha_2\}.$$

Clearly, M_1 is independent of λ and k. Denote $M = M_1 + M_0$, here M_0 is taken sufficiently large such that x^* satisfies $\|x^*\| < M$, where x^* is the solution of the following system

$$\bar{a}g(0) + \sum_{i=1}^{n} \bar{b}_i h_i(x, x, \ldots, x) = \bar{f}_k,$$

where

$$\bar{a} = \frac{1}{2kT}\int_{-kT}^{kT} a(t)\mathrm{d}t,\ \bar{b}_i = \frac{1}{2kT}\int_{-kT}^{kT} b_i(t)\mathrm{d}t,\ \bar{f}_k = \frac{1}{2kT}\int_{-kT}^{kT} f_k(t)\mathrm{d}t,\ i = 1, 2, \ldots, n.$$

Now, we take $\Omega = \{x \in C^1_{2kT} : \|x\|_{C^1_{2kT}} < M\}$. Thus, the condition (a) of Lemma 2.1 is satisfied. When $x \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R$, x is a constant function in R with $|x| = M$. Then we can derive

$$QNx = -\frac{1}{2kT}\int_{-kT}^{kT} \big(a(t)g(0) + \sum_{i=1}^{n} b_i(t)h_i(x, x, \ldots, x) - f_k(t)\big)\mathrm{d}t \ne 0,$$

which implies that the condition (b) of Lemma 2.1 is satisfied. Furthermore, take $J : \operatorname{Im} Q \to \ker L$ such that $J(x) = x$ for $x \in X$. Let $H(x, \mu) = \mu x + (1 - \mu)JQNx$, $\forall x \in \Omega \cap \operatorname{Ker} L$. We have that

$$\deg\{JQN, \Omega \cap \ker L, 0\} = \deg\{I, \Omega \cap \ker L, 0\} = 1 \ne 0,$$

where I is the identity operator. Therefore, the condition (c) of Lemma 2.1 is satisfied. Hence, $Lx = Nx$ has at least one solution in $\operatorname{Dom} L \cap \bar{\Omega}$. Therefore, (1.2) has at least one $2kT$-periodic solution $x_k \in \bar{\Omega}$. This completes the proof. ■

To prove the existence of homoclinic solutions of (1.1), we introduce the following lemma

Lemma 3.1. [19]. *Let* $x_k \in C^1_{2kT}$ *with* $||x_k||_{C_{2kT}} \le M$, $||x'_k||_{C_{2kT}} \le M$, $||x_k||_2 \le \alpha_2$ *and* $||x'_k||_{l+1} \le \alpha_1$ *for all* $k \in N$. *Then there exists a function* $x_0 \in C^1(R, R)$ *such that for each* $[a, b] \subset R$, *there is a subsequence* $\{x_{k_j}\}$ *of* $\{x_k\}_{k\in N}$ *with* $x'_{k_j}(t) \to x'_0(t)$ *uniformly on* $[a, b]$.

Theorem 3.2. *Let* (H_1), (H_2) *and* (H_3) *hold. Then* (1.1) *has at least one homoclinic solution.*

Proof. By Theorem 3.1, (1.2) has at least one $2kT$-periodic solution $x_k \in \bar{\Omega}$, that is, for each $k \in N$, $||x_k||_{C_{2kT}} \le M$, $||x'_k||_{C_{2kT}} \le M$, $||x_k||_2 \le \alpha_2$ and $||x'_k||_{l+1} \le \alpha_1$. It follows from Lemma 3.1 that there exists a function $x_0 \in C^1(R, R)$ such that for each $[a, b] \subset R$, there is a subsequence $\{x_{k_j}\}$ of $\{x_k\}_{k\in N}$ with $x'_{k_j}(t) \to x'_0(t)$ uniformly on $[a, b]$. In the following, we will show that $x_0(t)$ is just the unique homoclinic solution of (1.1). Firstly, we show that $x_0(t)$ is a solution of (1.1). Since $x_{k_j}(t)$ is a $2k_jT$-periodic solution of (1.2), we have

$$\begin{aligned} &x''_{k_j}(t) + a(t)g(x'_{k_j}(t)) \\ &+\sum_{i=1}^n b_i(t)h_i\big(x_{k_j}(t-\tau_1(t)), x_{k_j}(t-\tau_2(t)), \dots, x_{k_j}(t-\tau_n(t))\big) = f_{k_j}(t), \end{aligned} \tag{3.7}$$

where $t \in [-k_jT, k_jT]$, $j \in N$. Hence, there exists $j_0 \in N$ such that for $j > j_0$ and $t \in [a, b]$, we have

$$\begin{aligned} &x''_{k_j}(t) + a(t)g(x'_{k_j}(t)) \\ &+\sum_{i=1}^n b_i(t)h_i\big(x_{k_j}(t-\tau_1(t)), x_{k_j}(t-\tau_2(t)), \dots, x_{k_j}(t-\tau_n(t))\big) = f(t). \end{aligned} \tag{3.8}$$

Integrating (3.8) from a to $t \in [a, b]$, we obtain

$$\begin{aligned} x'_{k_j}(t) - x'_{k_j}(a) = \int_a^t \Big(&- a(s)g(x'_{k_j}(s)) - \sum_{i=1}^n b_i(s)h_i\big(x_{k_j}(s-\tau_1(s)), \\ &x_{k_j}(s-\tau_2(s)), \dots, x_{k_j}(s-\tau_n(s))\big) + f(s)\Big)\mathrm{d}s. \end{aligned} \tag{3.9}$$

Since $x_{k_j}(t) \to x_0(t)$ and $x'_{k_j}(t) \to x'_0(t)$ uniformly on $[a, b]$ as $j \to \infty$. Let $j \to \infty$ in (3.9), for $t \in [a, b]$, we obtain that

$$\begin{aligned} &x'_0(t) - x'_0(a) \\ = &\int_a^t \Big(- a(s)g(x'_0(s)) \\ &-\sum_{i=1}^n b_i(s)h_i\big(x_0(s-\tau_1(s)), x_0(s-\tau_2(s)), \dots, x_0(s-\tau_n(s))\big) + f(s)\Big)\mathrm{d}s. \end{aligned}$$

In view of $[a, b]$ is arbitrary, we have that $x_0(t)$ is a solution of (1.1).

Next, we will show that $x_0(t) \to 0$ as $t \to \pm\infty$. It is obvious that

$$\begin{aligned}\int_{-\infty}^{+\infty}\Big(|x_0(t)|^2+|x_0'(t)|^2\Big)\mathrm{d}t &= \lim_{k\to+\infty}\int_{-kT}^{kT}\Big(|x_0(t)|^2+|x_0'(t)|^2\Big)\mathrm{d}t \\ &= \lim_{k\to+\infty}\lim_{j\to+\infty}\int_{-k_jT}^{k_jT}\Big(|x_0(t)|^2+|x_0'(t)|^2\Big)\mathrm{d}t \\ &\leq \alpha_2^2+\alpha_1^2.\end{aligned} \tag{3.10}$$

Hence, we have

$$\int_{|t|\geq\delta}\Big(|x_0(t)|^2+|x_0'(t)|^2\Big)\mathrm{d}t \to 0,\ \delta\to+\infty.$$

It follows that

$$\int_{|t|\geq\delta}|x_0(t)|^2\mathrm{d}t \to 0,\quad \int_{|t|\geq\delta}|x_0'(t)|^2\mathrm{d}t \to 0,\ \delta\to+\infty.$$

By Lemma 2.2, as $t \to \pm\infty$, we have that

$$|x_0(t)| \leq (2r)^{-\frac{1}{2}}\Big(\int_{t-r}^{t+r}|x_0(s)|^2\mathrm{d}t\Big)^{\frac{1}{2}} + r(2r)^{-\frac{1}{2}}\Big(\int_{t-r}^{t+r}|x_0'(s)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \to 0,$$

that is, $x_0(t) \to 0$ as $t \to \pm\infty$.

Finally, we will show that $x_0'(t) \to 0$ as $t \to \pm\infty$. By Theorem 3.1, we have

$$|x_0(t)| \leq M,\ |x_0'(t)| \leq M,\ \forall t \in R.$$

It follows from (1.1) that

$$||x_0''||_{C_{2kT}} \leq a^+g_M + b^+h_M + \sup_{t\in R}|f(t)|,$$

where $g_M = \max_{|x|\leq M}|g(x)|$ and $h_M = \max_{1\leq i\leq n}\max_{|x|\leq M}|h_i(x, x, \ldots, x)|$. By way of contradiction, assume that $x_0'(t) \nrightarrow 0$ as $t \to \pm\infty$. Then there exist a $0 < \varepsilon_0 < \frac{1}{2}$ and a sequence $\{t_k\}$ such that

$$|t_1| < |t_2| < \cdots < |t_k| < |t_{k+1} < \cdots,\ k \in N$$

and

$$|x_0'(t_k)| \geq 2\varepsilon_0,\ k \in N.$$

Then, for $t \in [t_k, t_k + \frac{\varepsilon_0}{1+M}]$, we have

$$|x_0'(t)| = \big|x_0'(t_k) + \int_{t_k}^{t_0} x_0''(s)\mathrm{d}s\big| \geq |x_0'(t_k)| - \int_{t_k}^{t_0}|x_0''(s)|\mathrm{d}s \geq \varepsilon_0.$$

Therefore, we have

$$\int_{-\infty}^{+\infty} |x_0'(t)|^2 \mathrm{d}t \geq \sum_{k=1}^{\infty} \int_{t_k}^{t_k+\frac{\varepsilon_0}{1+M}} |x_0'(t)|^2 \mathrm{d}t = \infty,$$

which contradicts to (3.10). Hence $x_0'(t) \to 0$ as $t \to \pm\infty$. This completes the proof. ■

4. An Example

Example 4.1. Consider the following differential equation

$$\begin{aligned} &x''(t) - 19(2-\cos t)x'(t)(2+\sin x'(t)) \\ &-\frac{3}{2}(1-\frac{1}{2}\sin t)\Big[x\big(t-\frac{3}{4}\sin 2t\big) + x\big(t-\cos 2t\big)\Big] \\ &-\frac{5}{2}(1-\frac{2}{3}\sin t)\Big[x\big(t-\frac{3}{4}\sin 2t\big) + x\big(t-\cos 2t\big)\Big] = \frac{e^{\frac{t}{3}}}{e^{-t}+e^{t}},\ t\in R. \end{aligned} \tag{4.1}$$

It is obvious that $a(t) = 2 - \cos t$, $g(u) = 19u(2+\sin u)$, $b_1(t) = 1 - \frac{1}{2}\sin t$, $b_2(t) = 1 - \frac{2}{3}\sin t$, $h_1(u,v) = -\frac{3}{2}(u+v)$, $h_2(u,v) = -\frac{5}{2}(u+v)$, $\tau_1(t) = \frac{3}{4}\sin 2t$, $\tau_2(t) = \cos 2t$ and $f(t) = \frac{e^{\frac{t}{3}}}{e^{-t}+e^{t}}$. We have that $L = 3$ and we can take $m = 17$. Then it is easy to verify that all conditions in Theorem 3.2 are satisfied. Therefore, (4.1) has at least one homoclinic solution.

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