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ALGORITHMIC AND ANALYTICAL APPROACHES TO THE SPLIT FEASIBILITY PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. The split feasibility problem and fixed point problem is considered. New algorithm is presented for solving this split problem. Some analytical techniques are demonstrated and strong convergence results are obtained.

1. Introduction

Recently, some split problems have been presented and studied by some authors. See, e.g., [1]-[16]. In which the following split feasibility problem is now well-known: Finding a point x^* such that

$$x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{1.1}$$

where C and Q are two closed convex subsets of two Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. The prototype of this problem was first introduced by Censor and Elfving [3] in the finite dimensional Hilbert spaces. The background of the split feasibility (1.1) is based on the field of intensity-modulated radiation therapy when one attempts to describe physical dose constraints and equivalent uniform dose constraints within a single model. Censor and Elfving [3] used the simultaneous multi-projections algorithm to solve the split feasibility problem (1.1) where $C \in \mathbb{R}^N$ and $Q \in \mathbb{R}^M$. Their algorithms, as well as others, see, e.g., Byrne [2], involve matrix inversion at each iterative step. Calculating inverses of matrices is very time-consuming, particularly if the dimensions are large. Therefore, a new algorithm for solving the split feasibility problem was devised by Byrne [1], called the CQ-algorithm:

$$x_{n+1} = P_C(x_n - \tau A^*(I - P_Q)Ax_n), n \in \mathbb{N},$$

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where $\tau \in \left(0, \frac{2}{L}\right)$ with L being the largest eigenvalue of the matrix A^*A, I is the unit matrix or operator and P_C and P_Q denote the orthogonal projections onto C and Q, respectively. Consequently, CQ algorithm has been extensively by many mathematicians, see, for instance, [5-10]. Especially, in [12], Xu gave a continuation of the study on the CQ algorithm and its convergence. He applied Mann's algorithm to the split feasibility problem (1.1) and proposed an averaged CQ algorithm:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C(x_n - \tau A^*(I - P_Q)Ax_n), n \in \mathbb{N},$$

which was proved to be weakly convergent to a solution of the split feasibility problem (1.1) under suitable choices of iterative parameters. Xu [12] further suggested a single step regularized method:

$$x_{n+1} = P_C((1 - \alpha_n\gamma_n)x_n - \gamma_n A^*(I - P_Q)Ax_n), n \in \mathbb{N}. \tag{1.2}$$

Xu proved that the sequence $\{x_n\}$ generated by (1.2) converges in norm to the minimum-norm solution of the split feasibility problem (1.1) provided the parameters $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\alpha_n \to 0$ and $0 < \gamma_n \le \frac{\alpha_n}{\|A\|^2 + \alpha_n}$;

(ii) $\sum_n \alpha_n\gamma_n = \infty$;

(iii) $\frac{|\gamma_{n+1} - \gamma_n| + \gamma_n|\alpha_{n+1} - \alpha_n|}{(\alpha_{n+1}\gamma_{n+1})^2} \to 0$.

Next, we concern the following problem: Find hierarchically a fixed point of a nonexpansive mapping T with respect to another mapping S, namely

$$\text{Find } \tilde{x} \in Fix(T) \text{ such that } \langle \tilde{x} - S\tilde{x}, \tilde{x} - x \rangle \le 0, \forall x \in Fix(T). \tag{1.3}$$

It is not hard to check that (1.3) is equivalent to the fixed point problem

$$\text{Find } \tilde{x} \in C \text{ such that } \tilde{x} = P_{Fix(T)} \cdot S\tilde{x},$$

where $P_{Fix(T)}$ stands for the metric projection on the closed convex set $Fix(T)$. By using the definition of the normal cone to $Fix(T)$, i.e.,

$$N_{Fix(T)} : x \mapsto \begin{cases} \{u \in H; (\forall y \in Fix(T)) \ \langle y - x, u \rangle \le 0\}, & \text{if } x \in Fix(T); \\ \emptyset, & \text{otherwise.} \end{cases}$$

We easily prove that (1.3) is equivalent to the variational inclusion

$$0 \in (I - S)\tilde{x} + N_{Fix(T)}\tilde{x}.$$

In order to solve (1.3), Moudafi et al. [17]-[18] suggested a well-known viscosity algorithm and obtained convergence result. Further, Marino and Xu [19] suggested the following general iterative algorithm to minimize a quadratic function $\frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle$ over the set of fixed points of nonexpansive mapping T, where B is a strongly positive linear bounded operator and b is a given point:

$$x_{n+1} = \alpha_n \sigma f(x_n) + (I - \alpha_n B)Tx_n, \forall n \in \mathbb{N}. \tag{1.4}$$

Subsequently, algorithm (1.4) and its variant have been extensively studied. Please consult [23]-[30].

Motivated and inspired by the works in this direction, in this paper we will devote to study the split feasibility problem and fixed point problem. In section 2 we recall some basic concepts and cite some useful lemmas. In section 3, we first introduce our problem and construct our iterative algorithm for the studied problem. In section 4, we give convergence analysis of the suggested algorithm.

2. Basic Concepts and Useful Lemmas

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H.

Definition 2.1. A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|,$$

for all $x, y \in C$.

We will use $Fix(T)$ to denote the set of fixed points of T, that is, $Fix(T) = \{x \in C : x = Tx\}$.

Definition 2.2. A mapping $f : C \to C$ is called *contractive* if

$$\|f(x) - f(y)\| \le \rho\|x - y\|,$$

for all $x, y \in C$ and for some constant $\rho \in (0, 1)$. In this case, we call f is a ρ-contraction.

Definition 2.3. A linear bounded operator $B : H \to H$ is called *strongly positive* if there exists a constant $\gamma > 0$ such that

$$\langle Bx, x\rangle \ge \gamma\|x\|^2,$$

for all $x, y \in H$.

Definition 2.4. We call $P_C : H \to C$ the metric projection if for each $x \in H$

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}.$$

It is well known that the metric projection $P_C : H \to C$ is characterized by:

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0$$

for all $x \in H$, $y \in C$. From this, we can deduce that P_C is firmly-nonexpansive, that is,

$$\|P_C(x) - P_C(y)\|^2 \leq \langle x - y, P_C(x) - P_C(y)\rangle \tag{2.5}$$

for all $x, y \in H$. Hence P_C is also nonexpansive.

It is well-known that in a real Hilbert space H, the following two equalities hold:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \tag{2.6}$$

for all $x, y \in H$ and $t \in [0, 1]$, and

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \tag{2.7}$$

for all $x, y \in H$. It follows that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \tag{2.8}$$

for all $x, y \in H$.

Lemma 2.5. ([20]). *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$$

for all $n \geq 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.6. ([21]). *Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I - S)x_n \to y$ strongly, then $(I - S)x^* = y$.*

Lemma 2.7. ([22]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, n \in \mathbb{N},$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Problems and Constructed Algorithms

In this section, we first introduce our problem and consequently suggest our algorithm for solving this problem. Now we give the assumptions on the underlying spaces, involved operators and additional parameters which will be used in the next section, throughout.

1. Underlying Spaces:

(S1): H_1 and H_2 are two real Hilbert spaces;

(S2): $D \subset H_1$ and $E \subset H_2$ are two nonempty closed convex sets.

(S3): $C \subset D$ and $Q \subset E$ are two nonempty closed convex sets.

2. Involved Operators:

(O1): $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*;

(O2): B is a strongly positive bounded linear operator on H_1 with coefficient $\gamma > 0$;

(O3): $f : D \to D$ is a ρ-contraction;

(O4): $S : Q \to Q$ and $T : C \to C$ are two nonexpansive mappings.

3. Additional Parameters:

(P1): $\delta \in (0, \frac{1}{\|A\|^2})$ and $\sigma > 0$ are two constants;

(P2): $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0, 1)$.

In this paper, we devote to study the following split feasibility problem and fixed point problem:

Problem 3.1. Find $x^* \in C \cap Fix(T)$ such that $Ax^* \in Q \cap Fix(S)$.

Remark 3.2. It is obvious this problem contains the split feasibility problem (1.1) as a special case. In fact, if we can take $T = I$ and $S = I$, then $Fix(T) = C$ and $Fix(S) = Q$.

In order to solve Problem 3.1, we construct the following algorithm:

Algorithm 3.3. Taking $x_0 \in H_1$ arbitrarily, we define a sequence $\{x_n\}$ by the following:

$$\begin{cases} v_n = TP_C(x_n - \delta A^*(I - SP_Q)Ax_n), \\ x_{n+1} = \alpha_n \sigma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n B)v_n, \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$.

4. Convergence Analysis

In this section, we give the convergence analysis of the algorithm (3.1) and obtain our main results. We use Γ to denote the solution set of Problem 3.1, i.e.,

$$\Gamma = \{x^*|x^* \in C \cap Fix(T), Ax^* \in Q \cap Fix(S)\}.$$

Theorem 4.1. *Suppose* $\Gamma \neq \emptyset$. *Assume the following conditions hold:*

$(A1) : \lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

$(A2) : 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

$(A3) : \sigma\rho < \gamma$.

Then the sequence $\{x_n\}$ *generated by algorithm* (3.1) *converges strongly to* $p = Proj_{\Gamma}(\sigma f + I - B)p$ *which solves the following VI:*

$$\langle(\sigma f - B)x, y - x\rangle \leq 0, \forall y \in \Gamma. \tag{4.1}$$

Remark 4.2. It is clear that the solution of (4.1) is unique.

Proof. Set $z_n = P_Q Ax_n$, $y_n = x_n - \delta A^*(I - SP_Q)Ax_n$ and $u_n = P_C(x_n - \delta A^*(I - SP_Q)Ax_n)$ for all $n \in \mathbb{N}$. Then $u_n = P_C y_n$. Let $p = P_{\Gamma}(\sigma f + I - B)p$. Then, we have $p \in C \cap Fix(T)$ and $Ap \in Q \cap Fix(S)$. By these facts and the firmly-nonexpansivity of P_C and P_Q, we have the following conclusions:

$$\text{(i):}\quad \|z_n - Ap\| = \|P_Q Ax_n - Ap\| \leq \|Ax_n - Ap\|, \tag{4.2}$$

$$\text{(ii):}\quad \|u_n - p\| = \|P_C y_n - p\| \leq \|y_n - p\|, \tag{4.3}$$

$$\text{(iii):}\quad \|Sz_n - Ap\|^2 \leq \|z_n - Ap\|^2 \leq \|Ax_n - Ap\|^2 - \|z_n - Ax_n\|^2, \tag{4.4}$$

$$\text{(iv):}\quad \|u_{n+1} - u_n\| = \|P_C y_{n+1} - P_C y_n\| \leq \|y_{n+1} - y_n\|, \tag{4.5}$$

and

$$\text{(v):}\quad \|z_{n+1} - z_n\| = \|P_Q Ax_{n+1} - P_Q x_n\| \leq \|Ax_{n+1} - Ax_n\|. \tag{4.6}$$

From (3.1), we have

$$\begin{aligned} &\|x_{n+1} - p\| \\ &= \|\alpha_n(\sigma f(x_n) - Bp) + \beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_n B)(Tu_n - p)\| \\ &\leq \alpha_n\sigma\|f(x_n) - f(p)\| + \alpha_n\|\sigma f(p) - Bp\| + \beta_n\|x_n - p\| \\ &\quad +(1 - \beta_n - \alpha_n\gamma)\|u_n - p\|. \end{aligned} \tag{4.7}$$

Using (2.7), we get

$$\begin{aligned} &\|y_n - p\|^2 \\ &= \|x_n - p + \delta A^*(Sz_n - Ax_n)\|^2 \\ &= \|x_n - p\|^2 + \delta^2\|A^*(Sz_n - Ax_n)\|^2 + 2\delta\langle x_n - p, A^*(Sz_n - Ax_n)\rangle. \end{aligned} \tag{4.8}$$

Since A is a linear operator with its adjoint A^*, we have

$$
\begin{aligned}
&\langle x_n - p, A^*(Sz_n - Ax_n)\rangle \\
&= \langle A(x_n - p), Sz_n - Ax_n\rangle \\
&= \langle Ax_n - Ap + Sz_n - Ax_n - (Sz_n - Ax_n), Sz_n - Ax_n\rangle \\
&= \langle Sz_n - Ap, Sz_n - Ax_n\rangle - \|Sz_n - Ax_n\|^2.
\end{aligned} \tag{4.9}
$$

Again using (2.7), we obtain

$$
\begin{aligned}
&\langle Sz_n - Ap, Sz_n - Ax_n\rangle \\
&= \frac{1}{2}(\|Sz_n - Ap\|^2 + \|Sz_n - Ax_n\|^2 - \|Ax_n - Ap\|^2).
\end{aligned} \tag{4.10}
$$

By (4.4), (4.9) and (4.10), we get

$$
\begin{aligned}
&\langle x_n - p, A^*(Sz_n - Ax_n)\rangle \\
&= \frac{1}{2}(\|Sz_n - Ap\|^2 + \|Sz_n - Ax_n\|^2 - \|Ax_n - Ap\|^2) \\
&\quad -\|Sz_n - Ax_n\|^2 \\
&\le \frac{1}{2}(\|Ax_n - Ap\|^2 - \|z_n - Ax_n\|^2 + \|Sz_n - Ax_n\|^2 \\
&\quad -\|Ax_n - Ap\|^2) - \|Sz_n - Ax_n\|^2 \\
&= -\frac{1}{2}\|z_n - Ax_n\|^2 - \frac{1}{2}\|Sz_n - Ax_n\|^2.
\end{aligned} \tag{4.11}
$$

Substituting (4.11) into (4.8) to deduce

$$
\begin{aligned}
\|y_n - p\|^2 &\le \|x_n - p\|^2 + \delta^2\|A\|^2\|Sz_n - Ax_n\|^2 \\
&\qquad +2\delta(-\frac{1}{2}\|z_n - Ax_n\|^2 - \frac{1}{2}\|Sz_n - Ax_n\|^2) \\
&= \|x_n - p\|^2 + (\delta^2\|A\|^2 - \delta)\|Sz_n - Ax_n\|^2 - \delta\|z_n - Ax_n\|^2 \\
&\le \|x_n - p\|^2.
\end{aligned} \tag{4.12}
$$

It follows that

$$
\|y_n - p\| \le \|x_n - p\|.
$$

Thus, from (4.7), we get

$$
\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n\sigma\rho\|x_n - p\| + \alpha_n\|\sigma f(p) - Bp\| \\
&\qquad +\beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\gamma)\|x_n - p\| \\
&= [1 - (\gamma - \sigma\rho)\alpha_n]\|x_n - p\| + \alpha_n\|\sigma f(p) - Bp\| \\
&\le \max\left\{\|x_n - p\|, \frac{\sigma f(p) - Bp\|}{\gamma - \sigma\rho}\right\}.
\end{aligned}
$$

The boundedness of the sequence $\{x_n\}$ yields.

Next, we estimate $\|u_{n+1}-u_n\|$. According to (2.7) and (4.5), we have

$$
\begin{aligned}
\|u_{n+1}-u_n\|^2 \le\ & \|y_{n+1}-y_n\|^2\\
=\ & \|x_{n+1}-x_n+\delta[A^*(SP_Q-I)Ax_{n+1}-A^*(SP_Q-I)Ax_n]\|^2\\
=\ & \|x_{n+1}-x_n\|^2+\delta^2\|A^*[(SP_Q-I)Ax_{n+1}-(SP_Q-I)Ax_n]\|^2\\
& +2\delta\langle x_{n+1}-x_n, A^*[(SP_Q-I)Ax_{n+1}-(SP_Q-I)Ax_n]\rangle\\
\le\ & \|x_{n+1}-x_n\|^2+\delta^2\|A\|^2\|Sz_{n+1}-Sz_n-(Ax_{n+1}-Ax_n)\|^2\\
& +2\delta\langle Ax_{n+1}-Ax_n, Sz_{n+1}-Sz_n-(Ax_{n+1}-Ax_n)\rangle\\
=\ & \|x_{n+1}-x_n\|^2+\delta^2\|A\|^2\|Sz_{n+1}-Sz_n-(Ax_{n+1}-Ax_n)\|^2\\
& +2\delta\langle Sz_{n+1}-Sz_n, Sz_{n+1}-Sz_n-(Ax_{n+1}-Ax_n)\rangle\\
& -2\delta\|Sz_{n+1}-Sz_n-(Ax_{n+1}-Ax_n)\|^2\\
=\ & \|x_{n+1}-x_n\|^2+\delta^2\|A\|^2\|Sz_{n+1}-Sz_n-(Ax_{n+1}-Ax_n)\|^2\\
& +\delta(\|Sz_{n+1}-Sz_n\|^2+\|Sz_{n+1}-Sz_n-(Ax_{n+1}-Ax_n)\|^2\\
& -\|Ax_{n+1}-Ax_n\|^2)-2\delta\|Sz_{n+1}-Sz_n-(Ax_{n+1}-Ax_n)\|^2\\
=\ & \|x_{n+1}-x_n\|^2+(\delta^2\|A\|^2-\delta)\|Sz_{n+1}-Sz_n-(Ax_{n+1}-Ax_n)\|^2\\
& +\delta(\|Sz_{n+1}-Sz_n\|^2-\|Ax_{n+1}-Ax_n\|^2)\\
\le\ & \|x_{n+1}-x_n\|^2+(\delta^2\|A\|^2-\delta)\|Sz_{n+1}-Sz_n-(Ax_{n+1}-Ax_n)\|^2\\
& +\delta(\|z_{n+1}-z_n\|^2-\|Ax_{n+1}-Ax_n\|^2).
\end{aligned} \tag{4.13}
$$

Since $\delta\in(0,\frac{1}{\|A\|^2})$, we derive by virtue of (2.7) and (4.13) that

$$\|u_{n+1}-u_n\|\le\|x_{n+1}-x_n\|. \tag{4.14}$$

From (3.1), we write $x_{n+1}=\beta_n x_n+(1-\beta_n)w_n$ where $w_n=Tu_n+\frac{\alpha_n}{1-\beta_n}(\sigma f(x_n)-BTu_n)$ for all $n\in\mathbb{N}$. Then, we have

$$
\begin{aligned}
\|w_{n+1}-w_n\| =\ & \|Tu_{n+1}-Tu_n+\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\sigma f(x_{n+1})-BTu_{n+1})\\
& -\frac{\alpha_n}{1-\beta_n}(\sigma f(x_n)-BTu_n)\|\\
\le\ & \|Tu_{n+1}-Tu_n\|+\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\sigma f(x_{n+1})-BTu_{n+1}\|\\
& +\frac{\alpha_n}{1-\beta_n}\|\sigma f(x_n)-BTu_n\|\\
\le\ & \|u_{n+1}-u_n\|+\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\sigma f(x_{n+1})-BTu_{n+1}\|\\
& +\frac{\alpha_n}{1-\beta_n}\|\sigma f(x_n)-BTu_n\|
\end{aligned}
$$

$$\leq \|x_{n+1} - x_n\| + \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\sigma f(x_{n+1}) - BTu_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\sigma f(x_n) - BTu_n\|.$$

Noting the conditions (A2) and the boundedness of the sequences $\{u_{n+1}\}$, $\{y_{n+1}\}$, $\{z_{n+1}\}$, $\{Ax_n\}$, $\{f(x_n)\}$ and $\{BTu_n\}$, we have

$$\limsup_{n\to\infty}(\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By virtue of Lemma 2.5, we get

$$\lim_{n\to\infty} \|x_n - w_n\| = 0.$$

Hence,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|x_n - w_n\| = 0. \tag{4.15}$$

Since $x_{n+1} - x_n = \alpha_n(\sigma f(x_n) - BTu_n) + (1-\beta_n)(Tu_n - x_n)$, we obtain

$$\|Tu_n - x_n\| \leq \frac{1}{\beta_n}\left\{\alpha_n\|\sigma f(x_n) - BTu_n\| + \|x_{n+1} - x_n\|\right\}.$$

Thus,

$$\lim_{n\to\infty} \|x_n - Tu_n\| = 0. \tag{4.16}$$

Using the firmly-nonexpansivenessity of P_C, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|P_C y_n - p\|^2 \\ &\leq \|y_n - p\|^2 - \|P_C y_n - y_n\|^2 \\ &= \|y_n - p\|^2 - \|u_n - y_n\|^2. \end{aligned} \tag{4.17}$$

Applying (2.8) to (4.7) to deduce

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(\sigma f(x_n) - Bp) + \beta_n(x_n - Tu_n) + (I - \alpha_n B)(Tu_n - p)\|^2 \\ &\leq \|(I - \alpha_n B)(Tu_n - p) + \beta_n(x_n - Tu_n)\|^2 \\ &\quad + 2\alpha_n\langle \sigma f(x_n) - Bp, x_{n+1} - p\rangle \\ &\leq [\|I - \alpha_n B\|\|Tu_n - p\| + \beta_n\|x_n - Tu_n\|]^2 \\ &\quad + 2\alpha_n\|\sigma f(x_n) - Bp\|\|x_{n+1} - p\| \\ &\leq [(1-\alpha_n\gamma)\|u_n - p\| + \beta_n\|x_n - Tu_n\|]^2 \\ &\quad + 2\alpha_n\|\sigma f(x_n) - Bp\|\|x_{n+1} - p\| \\ &= (1-\alpha_n\gamma)^2\|u_n - p\|^2 + \beta_n^2\|x_n - Tu_n\|^2 \\ &\quad + 2(1-\alpha_n\gamma)\beta_n\|u_n - p\|\|x_n - Tu_n\| \\ &\quad + 2\alpha_n\|\sigma f(x_n) - Bp\|\|x_{n+1} - p\| \\ &\leq \|x_n - p\|^2 + \beta_n^2\|x_n - Tu_n\|^2 + 2(1-\alpha_n\gamma)\beta_n\|u_n - p\|\|x_n - Tu_n\| \\ &\quad + 2\alpha_n\|\sigma f(x_n) - Bp\|\|x_{n+1} - p\|. \end{aligned} \tag{4.18}$$

This together with (4.17) imply that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le \|y_n-p\|^2-\|u_n-y_n\|^2+\beta_n^2\|x_n-Tu_n\|^2\\
&\quad+2(1-\alpha_n\gamma)\beta_n\|u_n-p\|\|x_n-Tu_n\|\\
&\quad+2\alpha_n\|\sigma f(x_n)-Bp\|\|x_{n+1}-p\|\\
&\le \|x_n-p\|^2-\|u_n-y_n\|^2+\beta_n^2\|x_n-Tu_n\|^2\\
&\quad+2(1-\alpha_n\gamma)\beta_n\|u_n-p\|\|x_n-Tu_n\|\\
&\quad+2\alpha_n\|\sigma f(x_n)-Bp\|\|x_{n+1}-p\|.
\end{aligned}
\tag{4.19}
$$

It follows that

$$
\begin{aligned}
&\|u_n-y_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n^2\|x_n-Tu_n\|^2\\
&\quad+2(1-\alpha_n\gamma)\beta_n\|u_n-p\|\|x_n-Tu_n\|+2\alpha_n\|\sigma f(x_n)-Bp\|\|x_{n+1}-p\|\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|+\beta_n^2\|x_n-Tu_n\|^2\\
&\quad+2(1-\alpha_n\gamma)\beta_n\|u_n-p\|\|x_n-Tu_n\|+2\alpha_n\|\sigma f(x_n)-Bp\|\|x_{n+1}-p\|.
\end{aligned}
$$

This together with (4.15), (4.16) and (A1) imply that

$$
\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{4.20}
$$

Returning to (4.18) and using (4.12), we have

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le (1-\alpha_n\gamma)^2\|u_n-p\|^2+\beta_n^2\|x_n-Tu_n\|^2\\
&\quad+2(1-\alpha_n\gamma)\beta_n\|u_n-p\|\|x_n-Tu_n\|+2\alpha_n\|\sigma f(x_n)-Bp\|\|x_{n+1}-p\|\\
&\le \|y_n-p\|^2+\beta_n^2\|x_n-Tu_n\|^2+2(1-\alpha_n\gamma)\beta_n\|u_n-p\|\|x_n-Tu_n\|\\
&\quad+2\alpha_n\|\sigma f(x_n)-Bp\|\|x_{n+1}-p\|\\
&\le \|x_n-p\|^2+(\delta^2\|A\|^2-\delta)\|Sz_n-Ax_n\|^2-\delta\|z_n-Ax_n\|^2\\
&\quad+\beta_n^2\|x_n-Tu_n\|^2+2(1-\alpha_n\gamma)\beta_n\|u_n-p\|\|x_n-Tu_n\|\\
&\quad+2\alpha_n\|\sigma f(x_n)-Bp\|\|x_{n+1}-p\|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(\delta-\delta^2\|A\|^2)\|Sz_n-Ax_n\|^2+\delta\|z_n-Ax_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n^2\|x_n-Tu_n\|^2\\
&\quad+2(1-\alpha_n\gamma)\beta_n\|u_n-p\|\|x_n-Tu_n\|+2\alpha_n\|\sigma f(x_n)-Bp\|\|x_{n+1}-p\|\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|+\beta_n^2\|x_n-Tu_n\|^2\\
&\quad+2(1-\alpha_n\gamma)\beta_n\|u_n-p\|\|x_n-Tu_n\|+2\alpha_n\|\sigma f(x_n)-Bp\|\|x_{n+1}-p\|,
\end{aligned}
$$

which implies that

$$\lim_{n\to\infty} \|Sz_n - Ax_n\| = \lim_{n\to\infty} \|z_n - Ax_n\| = 0. \tag{4.21}$$

So,

$$\lim_{n\to\infty} \|Sz_n - z_n\| = 0. \tag{4.22}$$

Note that

$$\begin{aligned} \|y_n - x_n\| &= \|\delta A^*(SP_Q - I)Ax_n\| \\ &\leq \delta\|A\|\|Sz_n - Ax_n\|. \end{aligned}$$

It follows from (4.21) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{4.23}$$

From (4.16), (4.20) and (4.23), we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{4.24}$$

Now, we show that

$$\limsup_{n\to\infty}\langle(\sigma f - B)p, x_n - p\rangle \leq 0.$$

Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\sigma f - B)p, x_n - p\rangle = \lim_{i\to\infty}\langle(\sigma f - B)p, x_{n_i} - p\rangle \tag{4.25}$$

Since the sequence $\{x_{n_i}\}$ is bounded, we can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup z$. For the sake of convenience, we assume (without loss of generality) that $x_{n_i} \rightharpoonup z$. Consequently, we derive from the above conclusions that

$$y_{n_i} \rightharpoonup z, \quad u_{n_i} \rightharpoonup z, \quad Ax_{n_i} \rightharpoonup z \quad \text{and} \quad z_{n_i} \rightharpoonup Az. \tag{4.26}$$

By the demi-closed principle of the nonexpansive mappings S and T (see Lemma 2.6), we deduce $z \in Fix(T)$ and $Az \in Fix(S)$ (according to (4.24) and (4.22), respectively). Note that $u_{n_i} = P_C y_{n_i} \in C$ and $z_{n_i} = P_Q Ax_{n_i} \in Q$. From (4.26), we deduce $z \in C$ and $Az \in Q$. To this end, we deduce $z \in C \cap Fix(T)$ and $Az \in Q \cap Fix(S)$. That is to say, $z \in \Gamma$. Therefore,

$$\begin{aligned} \limsup_{n\to\infty}\langle(\sigma f - B)p, x_n - p\rangle &= \lim_{i\to\infty}\langle(\sigma f - B)p, x_{n_i} - p\rangle \\ &= \lim_{i\to\infty}\langle(\sigma f - B)p, z - p\rangle \\ &\leq 0. \end{aligned} \tag{4.27}$$

Finally, we prove $x_n \to p$. From (3.1),we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \langle \alpha_n(\sigma f(x_n) - Bp) + \beta_n(x_n - p) \\
&\quad + ((1-\beta_n)I - \alpha_n B)(Tu_n - p), x_{n+1} - p\rangle \\
&= \alpha_n\langle \sigma f(x_n) - Bp, x_{n+1} - p\rangle + \beta_n\langle x_n - p, x_{n+1} - p\rangle \\
&\quad + \langle((1-\beta_n)I - \alpha_n B)(Tu_n - p), x_{n+1} - p\rangle \\
&\le \alpha_n\sigma\langle f(x_n) - f(p), x_{n+1} - p\rangle + \alpha_n\langle \sigma f(p) - Bp, x_{n+1} - p\rangle \\
&\quad + \beta_n\|x_n - p\|\|x_{n+1} - p\| + (1 - \beta_n - \alpha_n\gamma)\|Tu_n - p\|\|x_{n+1} - p\| \\
&\le [1 - (\gamma - \sigma\rho)\alpha_n]\|x_n - p\|\|x_{n+1} - p\| + \alpha_n\langle \sigma f(p) - Bp, x_{n+1} - p\rangle \\
&\le \frac{1 - (\gamma - \sigma\rho)\alpha_n}{2}\|x_n - p\|^2 + \frac{1}{2}\|x_{n+1} - p\|^2 \\
&\quad + \alpha_n\langle \sigma f(p) - Bp, x_{n+1} - p\rangle.
\end{aligned}$$

It follows that

$$\|x_{n+1}-p\|^2 \le [1-(\gamma-\sigma\rho)\alpha_n]\|x_n-p\|^2 + 2\alpha_n\langle \sigma f(p)-Bp, x_{n+1}-p\rangle. \tag{4.28}$$

Applying Lemma 2.7 and (4.27) to (4.28), we deduce $x_n \to p$. The proof is completed. ∎

In (3.1), if take $T = I$ and $S = I$, then we have

Algorithm 4.3. Taking $x_0 \in H_1$ arbitrarily, we define a sequence $\{x_n\}$ by the following:

$$\begin{cases} v_n = P_C(x_n - \delta A^*(I - P_Q)Ax_n), \\ x_{n+1} = \alpha_n\sigma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)v_n, \end{cases} \tag{4.29}$$

for all $n \in \mathbb{N}$.

Corollary 4.4. *Suppose the solution set Γ' of the split feasibility problem (1.1) is nonempty. Assume the following conditions hold:*

$(A1)$: $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

$(A2)$: $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

$(A3)$: $\sigma\rho < \gamma$.

Then the sequence $\{x_n\}$ generated by algorithm (4.29) converges strongly to $p = Proj_{\Gamma}(\sigma f + I - B)p$ which solves the following VI:

$$\langle(\sigma f - B)x, y - x\rangle \le 0, \forall y \in \Gamma'.$$

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