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RIEMANN-STIELTJES OPERATOR FROM MIXED NORM SPACES TO ZYGMUND-TYPE SPACES ON THE UNIT BALL

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Abstract. In this paper, the authors characterize the boundedness and compactness of the following Riemann-Stieltjes operator

$$L_g(f)(z) = \int_0^1 \mathcal{R}f(tz)g(tz)\,\frac{dt}{t}, z \in B,$$

where $\mathcal{R}f(z)$ is the radial derivative of function f at z, from mixed norm spaces $H(p,\, q,\, \phi)$ to Zygmund-type spaces on the unit ball.

1. Introduction

We begin by fixing notation and some results. Let $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ be points in the complex vector space C^n and $z\overline{w} := \langle z, w\rangle = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}$, where $\overline{w_k}$ is the complex conjugate of w_k. We also write

$$|z| = \sqrt{\langle z, z\rangle} = \sqrt{\sum_{j=1}^{n} |z_j|^2}.$$

We denote by $B = \{z \in C^n : |z| < 1\}$ the open unit ball in C^n. Let S be its boundary of B, and let $H(B)$ denote the class of all holomorphic functions on B. For $f \in H(B)$, let

$$\mathcal{R}f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$$

stand for the radial derivative of f at z ([30, 60]).

The iterated radial derivative operator $\mathcal{R}^m f$ is defined inductively by ([8, 45]):

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$$\mathcal{R}^m f = \mathcal{R}\left(\mathcal{R}^{m-1} f\right), m \in N - \{1\}.$$

A positive continuous function ϕ on $[0, 1)$ is called normal, if there are positive numbers s, t $(0 < s < t)$ and $t_0 \in [0, 1)$ such that (see, for example, [8, 26, 31])

$$\frac{\phi(r)}{(1-r)^s} \text{ is decreasing on } [t_0\, 1) \text{ and } \lim_{r\to 1} \frac{\phi(r)}{(1-r)^s} = 0,$$
$$\frac{\phi(r)}{(1-r)^t} \text{ is increasing on } [t_0\, 1) \text{ and } \lim_{r\to 1} \frac{\phi(r)}{(1-r)^t} = \infty.$$

From now on if we say that a function $\phi : B \to [0, \infty)$ is normal, we will also assume that it is radial, that is, $\phi(z) = \phi(|z|)$, $z \in B$.

For p, $q \in (0, \infty)$, let

$$\|f\|_{p,\,q,\,\phi} = \left(\int_0^1 M_q^p(f, r)\frac{\phi^p(r)}{1-r}dr\right)^{\frac{1}{p}},$$

where

$$M_q(f, r) = \left(\int_S |f(r\zeta)|^q d\sigma(\zeta)\right)^{\frac{1}{q}}, \ 0 \le r < 1.$$

The mixed norm space $H(p,\, q,\, \phi)$ consists of all $f \in H(B)$ such that $\|f\|_{p,\,q,\,\phi} < \infty$. For $1 \le p < \infty$, $H(p, q, \phi)$, equipped with the norm $\|f\|_{p,\,q,\,\phi}$, is a Banach space. When $0 < p < 1$, $\|\cdot\|_{p,q,\phi}$ is a quasinorm on $H(p, q, \phi)$, $H(p, q, \phi)$ is a Fréchet space but not a Banach space. If $0 < p = q < \infty$, then $H(p, p, \phi)$ is the Bergman-type space

$$H(p, p, \phi) = \left\{ f \in H(B) : \int_B |f(z)|^p \frac{\phi^p(|z|)}{1-|z|} dA(z) < \infty \right\},$$

where $dA(z)$ denotes the normalized Lebesgue area measure on the unit ball B such that $A(B) = 1$. Note that if $\phi(r) = (1-r)^{(\alpha+1)/p}$, then $H(p, p, \phi)$ is the weighted Bergman space $A^p_\alpha(B)$ defined for $0 < p < \infty$ and $\alpha > -1$, as the space of all $f \in H(B)$ such that

$$\|f\|^p_{A^p_\alpha} = \int_B |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty.$$

For some results on mixed norm and related spaces, as well as on some operators on them, see, for example, [1, 2, 8, 13, 26, 27, 33, 34, 35, 40, 41, 42, 44, 47, 48, 49, 56, 59] and the references therein.

Let μ be a normal function on $[0, 1)$. We say that an $f \in H(B)$ belongs to the space $\mathcal{Z}_\mu = \mathcal{Z}_\mu(B)$, if

$$\sup\left\{\mu(|z|)\left|\mathcal{R}^2 f(z)\right| : z \in B\right\} < \infty.$$

It is easy to check that $\mathcal{Z}_\mu$ becomes a Banach space under the norm

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + \sup\left\{\mu(|z|)\left|\mathcal{R}^2 f(z)\right| : z \in B\right\}.$$

$\mathcal{Z}_\mu$ will be called the Zygmund-type space. Let $\mathcal{Z}_{\mu,0} = \mathcal{Z}_{\mu,0}(B)$ denote the class of holomorphic functions $f \in \mathcal{Z}_\mu$ such that

$$\lim_{|z|\to 1} \mu(|z|)\left|\mathcal{R}^2 f(z)\right| = 0,$$

$\mathcal{Z}_{\mu,0}$ is called the little Zygmund-type space (see [23, 25, 39]). It is easy to see that $\mathcal{Z}_{\mu,0}$ is a closed subspace of $\mathcal{Z}_\mu$. When $\mu(r) = 1 - r^2$, Zygmund-type space $\mathcal{Z}_\mu$ (little Zygmund-type space $\mathcal{Z}_{\mu,0}$) is the classical Zygmund space $\mathcal{Z}$ (little Zygmund-type space $\mathcal{Z}_0$). For some other results on Zygmund-type and related spaces and operators on them, see, for example, [14, 16, 18, 27, 49, 56, 59, 60, 61, 62].

Let $g \in H(B)$. The following Riemann-Stieltjes operator

$$L_g(f)(z) = \int_0^1 \mathcal{R}f(tz)g(tz)\,\frac{dt}{t},\ \ f \in H(B),\ \ z \in B. \tag{1}$$

was recently introduced by S. Li and S. Stević ([10, 12, 13]). This operator is closely related to the extended Cesàro operator

$$T_g(f)(z) = \int_0^1 f(tz)\mathcal{R}g(tz)\,\frac{dt}{t},\ \ f \in H(B),\ \ z \in B.$$

Some characterizations of the boundedness and compactness of the operator L_g between various spaces of holomorphic functions on the unit ball can be found in [3, 15, 19, 21, 24, 37, 47, 64]. Some related integral-type operators in C^n are treated, for example, in [4, 5, 6, 7, 9, 22, 25, 32, 33, 36, 38, 41, 43, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 61, 62, 63].

For related one-dimensional operators, see, for example [11, 14, 16, 17, 18, 20, 27, 28, 42, 65, 66], as well as the related references therein.

The purpose of the paper is to study the boundedness and compactness of the operator L_g from mixed-norm spaces into Zygmund-type spaces. Throughout the paper, the letter C denotes a positive constant which may vary at each occurrence but it is independent of the essential variables.

2. Auxiliary Results

Here we state several auxiliary results most of which will be used in the proof of the main result.

Lemma 1. ([36, 37, 46]). *For every f, $g \in H(B)$ it holds*

$$\mathcal{R}L_g(f)(z) = \mathcal{R}f(z)g(z).$$

Lemma 2. ([45]). *Assume that $m \in N$, $0 < p, q < \infty$, ϕ is normal, $f \in H(p, q, \phi)$. Then there is a positive constant C independent of f such that*

$$|\mathcal{R}^m f(z)| \leq \frac{C|z|}{\phi(|z|)\,(1-|z|^2)^{m+\frac{n}{q}}} \|f\|_{p,\,q,\,\phi},\ z \in B.$$

Lemma 3. ([8]). *Assume that* $0 < p,\ q < \infty$*, for* $\beta > t,\ \omega \in B$ *and*

$$f_\omega(z) = \frac{(1-|\omega|^2)^\beta}{\phi(|\omega|)(1-z\overline{\omega})^{\beta+\frac{n}{q}}},\ z \in B.$$

Then $f_\omega \in H(p,q,\phi)$ *and there is a positive constant* C *independent of* f *such that*

$$\sup_{\omega \in B} \|f_\omega\|_{p,\,q,\,\phi} \leq C.$$

The next Schwartz-type lemma is proved in a standard way (see, e.g. [33, Lemma 3]).

Lemma 4. *Assume* φ *is a holomorphic self-map of* B*,* ϕ *is normal,* $0 < p$*,* $q < \infty$ *and* $g \in H(B)$*. Then* $L_g : H(p,\,q,\,\phi) \to \mathcal{Z}_\mu$ *is compact if and only if* $L_g : H(p,\,q,\,\phi) \to \mathcal{Z}_\mu$ *is bounded and for any bounded sequence* $\{f_n\}$ *in* $H(p,q,\phi)$ *which converges to zero uniformly on compact subsets of* B *as* $n \to \infty$*, we have* $\|L_g(f_n)\|_{\mathcal{Z}_\mu} \to 0, n \to \infty$.

Lemma 5. ([23, 62]). *A closed set* K *in* $\mathcal{Z}_{\mu,0}$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f \in K} \mu(|z|) \left|\mathcal{R}^2 f(z)\right| = 0.$$

3. The Boundedness and Compactness of $L_g : H(p,\,q,\,\phi) \to \mathcal{Z}_\mu\ (\mathcal{Z}_{\mu,0})$

In this section we formulate and prove our main result. Assume that $g \in H(B)$, ϕ and μ are normal.

Theorem 1 Assume that $0 < p,\ q < \infty$. Then $L_g : H(p,\,q,\,\phi) \to \mathcal{Z}_\mu$ is bounded if and only if

$$\sup_{z \in B} \frac{\mu(|z|)|zg(z)|}{\phi(|z|)\,(1-|z|^2)^{2+\frac{n}{q}}} < \infty, \tag{2}$$

and

$$\sup_{z \in B} \frac{\mu(|z|)|z\mathcal{R}g(z)|}{\phi(|z|)\,(1-|z|^2)^{1+\frac{n}{q}}} < \infty. \tag{3}$$

Proof. First assume that conditions (2) and (3) hold. For any $f \in H(p, q, \phi)$, by Lemmas 1 and 2, we have

$$\begin{aligned}
&\mu(|z|)\left|\mathcal{R}^2\left(L_g(f)\right)(z)\right| \\
&= \mu(|z|)\left|\mathcal{R}\left(\mathcal{R}f(z)g(z)\right)\right| \\
&= \mu(|z|)\left|\mathcal{R}^2 f(z)g(z)+\mathcal{R}f(z)\mathcal{R}g(z)\right| \\
&\leq C\|f\|_{p,q,\phi}\left(\frac{\mu(|z|)|zg(z)|}{\phi(|z|)\left(1-|z|^2\right)^{2+\frac{n}{q}}}+\frac{\mu(|z|)|z\mathcal{R}g(z)|}{\phi(|z|)\left(1-|z|^2\right)^{1+\frac{n}{q}}}\right).
\end{aligned}$$

From this along with the fact $(L_g(f))(0) = 0$, it follows the operator L_g: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded.

Conversely, assume that the operator L_g: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded. Then for any $f \in H(p, q, \phi)$, there is a positive constant C independent of f such that $\|L_g(f)\|_{\mathcal{Z}_\mu} \leq C\|f\|_{p,q,\phi}$. For a fixed $\omega \in B$, set

$$\begin{aligned}
f_\omega(z) = &\left(t+2+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)}\frac{1}{(1-z\overline{w})^{t+1+\frac{n}{q}}} \\
&-\left(t+1+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)}\frac{1}{(1-z\overline{w})^{t+2+\frac{n}{q}}},\ z\in B,
\end{aligned} \tag{4}$$

then

$$\begin{aligned}
\mathcal{R}f_\omega(z) = &\left(t+2+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)}\sum_{j=1}^{n} z_j\frac{\partial}{\partial z_j}\frac{1}{(1-z\overline{w})^{t+1+\frac{n}{q}}} \\
&-\left(t+1+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)}\sum_{j=1}^{n} z_j\frac{\partial}{\partial z_j}\frac{1}{(1-z\overline{w})^{t+2+\frac{n}{q}}} \\
= &\left(t+2+\frac{n}{q}\right)\left(t+1+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)}\frac{z\overline{w}}{(1-z\overline{w})^{t+2+\frac{n}{q}}} \\
&-\left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)}\frac{z\overline{w}}{(1-z\overline{w})^{t+3+\frac{n}{q}}},
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\mathcal{R}^2 f_\omega(z) = &\left(t+2+\frac{n}{q}\right)\left(t+1+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)}\mathcal{R}\frac{z\overline{w}}{(1-z\overline{w})^{t+2+\frac{n}{q}}} \\
&-\left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)}\mathcal{R}\frac{z\overline{w}}{(1-z\overline{w})^{t+3+\frac{n}{q}}} \\
= &\left(t+2+\frac{n}{q}\right)\left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)}\frac{(z\overline{w})^2}{(1-z\overline{w})^{t+3+\frac{n}{q}}}
\end{aligned} \tag{6}$$

$$- \left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\left(t+3+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)}\frac{(z\overline{w})^2}{(1-z\overline{w})^{t+4+\frac{n}{q}}}$$
$$+ \left(t+2+\frac{n}{q}\right)\left(t+1+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)}\frac{z\overline{w}}{(1-z\overline{w})^{t+2+\frac{n}{q}}}$$
$$- \left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)}\frac{z\overline{w}}{(1-z\overline{w})^{t+3+\frac{n}{q}}}.$$

By Lemma 3, $f_\omega \in H(p,q,\phi)$ and $\sup_{\omega\in B}\|f_\omega\|_{p,q,\phi} \le C$. By applying (5) and (6), we get

$$(7)\quad \mathcal{R}f_\omega(\omega)=0,\quad \mathcal{R}^2 f_\omega(\omega) = -\left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\frac{|\omega|^4}{\phi(|\omega|)(1-|\omega|^2)^{2+\frac{n}{q}}},$$

thus for any $\omega \in B$, we get

$$(8)\quad \begin{aligned} &\left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\frac{\mu(|\omega|)|g(\omega)||\omega|^4}{\phi(|\omega|)(1-|\omega|^2)^{2+\frac{n}{q}}} \\ &= \mu(|\omega|)\left|\mathcal{R}^2 f_\omega(\omega)g(\omega)+\mathcal{R}f_\omega(\omega)\mathcal{R}g(\omega)\right| \\ &\le \|L_g(f_\omega)\|_{\mathcal{Z}_\mu} \le C\|L_g\|_{H(p,q,\phi)\to\mathcal{Z}_\mu}. \end{aligned}$$

Let $r\in(0,1)$, from (8) we get

$$(9)\quad \begin{aligned} &\sup_{r<|\omega|<1}\frac{\mu(|\omega|)|\omega g(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{2+\frac{n}{q}}} \\ &\le \frac{C}{r^3}\sup_{r<|\omega|<1}\mu(|\omega|)\left|\mathcal{R}^2 f_\omega(\omega)g(\omega)+\mathcal{R}f_\omega(\omega)\mathcal{R}g(\omega)\right| \\ &\le C\|L_g\|_{H(p,q,\phi)\to\mathcal{Z}_\mu}. \end{aligned}$$

Using the fact

$$\sup_{|\omega|\le r}\frac{\mu(|\omega|)|\omega g(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{2+\frac{n}{q}}} \le C\sup_{|\omega|\le r}\mu(|\omega|)|g(\omega)| \le C,$$

and inequality (9), we get that (2) holds.

To prove (3), set

$$(10)\quad \begin{aligned} h_\omega(z) = &\left(t+3+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)}\frac{1}{(1-z\overline{w})^{t+1+\frac{n}{q}}} \\ &-\left(t+1+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)}\frac{1}{(1-z\overline{w})^{t+2+\frac{n}{q}}},\ z\in B. \end{aligned}$$

A straightforward calculation shows that

$$
\begin{aligned}
\mathcal{R}h_\omega(z) \;=\; & \left(t+3+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)}\sum_{j=1}^{n} z_j\frac{\partial}{\partial z_j}\frac{1}{(1-z\overline{w})^{t+1+\frac{n}{q}}}\\
& -\left(t+1+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)}\sum_{j=1}^{n} z_j\frac{\partial}{\partial z_j}\frac{1}{(1-z\overline{w})^{t+2+\frac{n}{q}}}\\
=\; & \left(t+3+\frac{n}{q}\right)\left(t+1+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)}\frac{z\overline{w}}{(1-z\overline{w})^{t+2+\frac{n}{q}}}\\
& -\left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)}\frac{z\overline{w}}{(1-z\overline{w})^{t+3+\frac{n}{q}}},
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
& \mathcal{R}^2h_\omega(z)\\
=\; & \left(t+3+\frac{n}{q}\right)\left(t+1+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)}\mathcal{R}\frac{z\overline{w}}{(1-z\overline{w})^{t+2+\frac{n}{q}}}\\
& -\left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)}\mathcal{R}\frac{z\overline{w}}{(1-z\overline{w})^{t+3+\frac{n}{q}}}\\
=\; & \left(t+3+\frac{n}{q}\right)\left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)}\frac{(z\overline{w})^2}{(1-z\overline{w})^{t+3+\frac{n}{q}}}\\
& -\left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\left(t+3+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)}\frac{(z\overline{w})^2}{(1-z\overline{w})^{t+4+\frac{n}{q}}}\\
& +\left(t+3+\frac{n}{q}\right)\left(t+1+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)}\frac{z\overline{w}}{(1-z\overline{w})^{t+2+\frac{n}{q}}}\\
& -\left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)}\frac{z\overline{w}}{(1-z\overline{w})^{t+3+\frac{n}{q}}}.
\end{aligned}
\tag{12}
$$

By Lemma 3, we have $h_\omega \in H(p,q,\phi)$ and $\sup_{\omega\in B}\|h_\omega\|_{p,q,\phi} \le C$. By using (11) and (12), we get

$$
\mathcal{R}h_\omega(\omega) = \mathcal{R}^2h_\omega(\omega) = \left(t+1+\frac{n}{q}\right)\frac{|\omega|^2}{\phi(|\omega|)(1-|\omega|^2)^{1+\frac{n}{q}}},
\tag{13}
$$

from (13) and (2) we have

$$
\begin{aligned}
& \left(t+1+\tfrac{n}{q}\right)\tfrac{\mu(|\omega|)|\mathcal{R}g(\omega)|\omega|^2}{\phi(|\omega|)(1-|\omega|^2)^{1+\frac{n}{q}}}\\
=\; & \mu(|\omega|)\left|\mathcal{R}h_\omega(\omega)\mathcal{R}g(\omega)\right|\\
\le\; & \|L_g(h_\omega)\|_{\mathcal{Z}_\mu} + \mu(|\omega|)\left|\mathcal{R}^2h_\omega(\omega)g(\omega)\right|\\
=\; & \|L_g(h_\omega)\|_{\mathcal{Z}_\mu} + \left(t+1+\frac{n}{q}\right)\frac{\mu(|\omega|)|g(\omega)||\omega|^2}{\phi(|\omega|)(1-|\omega|^2)^{1+\frac{n}{q}}}\\
\le\; & C\|L_g\|_{H(p,q,\phi)\to\mathcal{Z}_\mu} + \left(t+1+\frac{n}{q}\right)\frac{\mu(|\omega|)|g(\omega)|\omega|}{\phi(|\omega|)(1-|\omega|)^{2+\frac{n}{q}}}\\
\le\; & C.
\end{aligned}
\tag{14}
$$

Let $r \in (0,1)$, by using (14) we get

$$\sup_{r<|\omega|<1} \frac{\mu(|\omega|)|\omega \mathcal{R}g(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{1+\frac{n}{q}}} \le C. \tag{15}$$

Note that

$$\sup_{|\omega|\le r} \frac{\mu(|\omega|)|\omega \mathcal{R}g(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{1+\frac{n}{q}}} \le C. \tag{16}$$

From (15) and (16), we get that (3) holds.

Theorem 2. *Assume that* $0 < p,\, q < \infty$. *Then* $L_g : H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{\mu(|z|)|zg(z)|}{\phi(|z|)\,(1-|z|^2)^{2+\frac{n}{q}}} = 0, \tag{17}$$

and

$$\lim_{|z|\to 1} \frac{\mu(|z|)|z\mathcal{R}g(z)|}{\phi(|z|)\,(1-|z|^2)^{1+\frac{n}{q}}} = 0. \tag{18}$$

Proof. First assume that $L_g : H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is compact. Let $\{z_k\}$ be a sequence in B such that $|z_k| \to 1$ as $k \to \infty$. Set

$$f_k(z) = f_{z_k}(z),\ k \in N,$$

f_ω here is defined in (4). Then $f_k \in H(p,\, q,\, \phi)$, $\sup\limits_{k\in N} \|f_k\|_{p,\,q,\,\phi} \le C$, and $\{f_k\}$ converges to 0 uniformly on compact subsets of B, using the compactness of $L_g : H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ and Lemma 4, we get $\lim\limits_{k\to\infty} \|L_g(f_k)\|_{\mathcal{Z}_\mu} = 0$. By (13) we have

$$\mathcal{R}f_k(z_k) = 0,\ \mathcal{R}^2 f_k(z_k) = -\left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\frac{|z_k|^4}{\phi(|z_k|)(1-|z_k|^2)^{2+\frac{n}{q}}},$$

so

$$\begin{aligned}
&\left(t+1+\frac{n}{q}\right)\left(t+2+\frac{n}{q}\right)\frac{\mu(|z_k|)|g(z_k)||z_k|^4}{\phi(|z_k|)(1-|z_k|^2)^{2+\frac{n}{q}}} \\
&= \mu(|z_k|)\left|\mathcal{R}^2 f_k(z_k)g(z_k) + \mathcal{R}f_k(z_k)\mathcal{R}g(z_k)\right| \\
&\le \|L_g(f_k)\|_{\mathcal{Z}_\mu},
\end{aligned}$$

hence

$$\lim_{k\to\infty} \frac{\mu(|z_k|)|z_k g(z_k)|}{\phi(|z_k|)(1-|z_k|^2)^{2+\frac{n}{q}}} = 0, \tag{19}$$

from which (17) holds.

Set

$$h_k(z) = h_{z_k}(z), z \in B,$$

h_ω here is defined in (10), then $h_k \in H(p, q, \phi)$, $\sup\limits_{k\in N} \|h_k\|_{p,q,\phi} \le C$, and $\{h_k\}$ converges to 0 uniformly on compact subsets of B. From (14) we get

$$\mathcal{R}h_k(z_k) = \mathcal{R}^2 h_k(z_k) = \left(t + 1 + \frac{n}{q}\right) \frac{|z_k|^2}{\phi(|z_k|)\,(1 - |z_k|^2)^{1+\frac{n}{q}}}. \tag{20}$$

By Lemma 4 and (19), we have

$$\begin{aligned} &\left(t + 1 + \frac{n}{q}\right) \frac{\mu(|z_k|)|\mathcal{R}g(z_k)||z_k|^2}{\phi(|z_k|)(1 - |z_k|^2)^{1+\frac{n}{q}}} \\ &\le C\|L_g(h_k)\|_{\mathcal{Z}_\mu} + \left(t + 1 + \frac{n}{q}\right) \frac{\mu(|z_k|)|g(z_k)||z_k|^2}{\phi(|z_k|)(1 - |z_k|^2)^{1+\frac{n}{q}}} \\ &\le C\|L_g(h_k)\|_{\mathcal{Z}_\mu} + \left(t + 1 + \frac{n}{q}\right) \frac{\mu(|z_k|)|g(z_k)||z_k|}{\phi(|z_k|)(1 - |z_k|^2)^{2+\frac{n}{q}}} \to 0,\ k \to \infty, \end{aligned}$$

from which (18) holds.

Conversely, suppose (17) and (18) hold. Then it is easy to see that (2) and (3) hold. By Theorem 1, we get $L_g : H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded and for any $\varepsilon > 0$, $\exists \delta \in (0, 1)$ such that for $\delta < |z| < 1$

$$\frac{\mu(|z|)|zg(z)|}{\phi(|z|)\,(1 - |z|^2)^{2+\frac{n}{q}}} < \varepsilon, \tag{21}$$

and

$$\frac{\mu(|z|)|z\mathcal{R}g(z)|}{\phi(|z|)\,(1 - |z|^2)^{1+\frac{n}{q}}} < \varepsilon. \tag{22}$$

Set $a_k \in H(p, q, \phi)$, $\sup\limits_{k\in N} \|a_k\|_{p,q,\phi} \le C$, and $\{a_k\}$ converges to 0 uniformly on compact subsets of B, by Lemmas 1 and 2, the Cauchy inequality, (21) and (22), we have for sufficiently large k

$$\begin{aligned} \|L_g(a_k)\|_{\mathcal{Z}_\mu} &= |L_g(a_k)(0)| + \sup_{z\in B} \mu(|z|) \left|\mathcal{R}^2(L_g(a_k))(z)\right| \\ &= \sup_{z\in B} \mu(|z|) \left|\mathcal{R}\left(a_k(z)g(z)\right)\right| \\ &\le \sup_{\{z\in B:|z|\le\delta\}} \mu(|z|) \left|\mathcal{R}^2 a_k(z)g(z) + \mathcal{R}a_k(z)\mathcal{R}g(z)\right| \\ &\quad + \sup_{\{z\in B:|z|>\delta\}} \mu(|z|) \left|\mathcal{R}^2 a_k(z)g(z) + \mathcal{R}a_k(z)\mathcal{R}g(z)\right| \end{aligned}$$

$$
\begin{aligned}
&\le \varepsilon \sup_{\{z\in B:|z|\le\delta\}} \mu(|z|)(|g(z)|+|\mathcal{R}g(z)|)\\
&\quad + \sup_{\{z\in B:|z|>\delta\}} \mu(|z|)\left|\mathcal{R}^2 a_k(z)g(z)+\mathcal{R}a_k(z)\mathcal{R}g(z)\right|\\
&\le C\varepsilon + L \sup_{\{z\in B:|z|>\delta\}} \left(\frac{\mu(|z|)|zg(z)|}{\phi(|z|)\,(1-|z|^2)^{2+\frac{n}{q}}}+\frac{\mu(|z|)|z\mathcal{R}g(z)|}{\phi(|z|)\,(1-|z|^2)^{1+\frac{n}{q}}}\right)\\
&< (C+2L)\varepsilon,
\end{aligned}
$$

hence

$$\lim_{k\to\infty}\|L_g(a_k)\|_{\mathcal{Z}_\mu}=0.$$

It follows from Lemma 4 that $L_g : H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is compact.

Theorem 3. *Assume that* $0 < p\ ,q < \infty$. *Then the following statements are equivalent:*

(a) $L_g : H(p,\, q,\, \phi) \to \mathcal{Z}_{\mu,0}$ *is compact;*

(b) $L_g : H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ *is compact.*

Proof. $(a) \Rightarrow (b)$ This implication is obvious.

$(b) \Rightarrow (a)$. Assume that $L_g : H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is compact, by Theorem 2, for any $f \in H(p,\, q,\, \phi)$

$$
\begin{aligned}
&\mu(|z|)\left|\mathcal{R}^2\left(L_g(f)\right)(z)\right|\\
&= \mu(|z|)\left|\mathcal{R}\left(\mathcal{R}f(z)g(z)\right)\right|\\
&= \mu(|z|)\left|\mathcal{R}^2 f(z)g(z)+\mathcal{R}f(z)\mathcal{R}g(z)\right|\\
&\le C\|f\|_{p,q,\phi}\left(\frac{\mu(|z|)|zg(z)|}{\phi(|z|)\,(1-|z|^2)^{2+\frac{n}{q}}}+\frac{\mu(|z|)|z\mathcal{R}g(z)|}{\phi(|z|)\,(1-|z|^2)^{1+\frac{n}{q}}}\right)\\
&\quad \to 0,\ |z|\to 1,
\end{aligned}
\tag{23}
$$

we see that $L_g(f) \in \mathcal{Z}_{\mu,0}$. Since $L_g : H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is bounded, we have $L_g : H(p,\, q,\, \phi) \to \mathcal{Z}_{\mu,0}$ is bounded. Hence the set

$$L_g\{f\in H(p,\, q,\, \phi) : \|f\|_{p,q,\phi}\le 1\}$$

is bounded in $\mathcal{Z}_{\mu,0}$. By Lemma 5, we wish to show

$$\lim_{|z|\to 1}\ \sup_{\|f\|_{p,q,\phi}\le 1} \mu(|z|)\left|\mathcal{R}^2(L_g(f))(z)\right| = 0. \tag{24}$$

In fact, since $L_g : H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is compact, by Theorem 2, (17) and (18) hold. Combining with (17) and (18) and (23) we see that $\lim\limits_{|z|\to 1}\ \sup\limits_{\|f\|_{p,q,\phi}\le 1} \mu(|z|)$ $\left|\mathcal{R}^2(L_g(f))(z)\right| = 0$, which is what we wanted to prove. It follows that $L_g : H(p,\, q,\, \phi) \to \mathcal{Z}_{\mu,0}$ is compact.

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