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COMPONENTWISE LINEAR MODULES OVER A KOSZUL ALGEBRA

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Abstract. In this paper we devote to generalizing some results of componentwise linear modules over a polynomial ring to the ones over a Koszul algebra. Among other things, we show that the i-linear strand of the minimal free resolution of a componentwise linear module is the minimal free resolution of some module which is described explicitly for any $i \in Z$. In addition we present some theorems about when graded modules with linear quotients are componentwise linear.

1. Introduction

Throughout this paper R is a standard graded finitely generated K-algebra over a field K and M is a finitely generated graded R-module. Recall that an (infinite) minimal free resolution of M is an exact sequence of graded R-modules:

$$(*) \qquad \mathrm{F}.(M): \qquad \cdots \longrightarrow F_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

$$\text{where} \qquad F_k = \bigoplus_{j \in Z} R(-j)^{\beta_{k,j}(M)}, \quad k = 0, 1, \cdots,$$

such that all nonzero entries in all matrices $\phi_k, k > 0$ are homogeneous of positive degrees and ϕ_0 is a homogeneous R-linear map. Every graded R-module has a unique minimal free resolution up to isomorphism. In fact, every graded free resolution of M is the direct sum of a minimal free resolution of M and a *trivial complex*. Here a trivial complex is any free resolution of zero module. From a minimal free resolution of M one can obtain many invariants of M including projective dimension, regularity, Hilbert series and etc. For examples£º

$$\mathrm{Proj.dim}(M) = \max\{i | \beta_{i,j}(M) \neq 0 \text{ for some } j\}$$

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$$\text{reg}(M) = \max\{j - i | \beta_{i,j}(M) \neq 0\}$$

If assume further that R is a standard graded polynomial ring $K[x_1, \cdots, x_n]$ over a field K, then the minimal free resolution of a graded R-module has finite length. In this case, one can obtain the Hilbert series and depth of M from its minimal free resolution. The type of M is the K-dimension of $\text{Ext}_R^g(R/\text{m}, M)$ by definition, where $g = \text{depth}M$. It is well-known that the type of M is equivalent to the largest non-vanishing Betti number of M.

Hence it is important to obtain a minimal free resolution of a graded module. However it remains an open question to construct a minimal free resolution for M even when M is a general monomial ideal of a standard graded polynomial ring.

The most simple resolutions are linear resolutions. Recall that a minimal resolution $\text{F}.(M)$ is called (d-)linear if it has the following form:

$$\text{F}.(M): \quad \cdots \to R(-d-i)^{\beta_{i,i+d}(M)} \to \cdots \to R(-d)^{\beta_{0,d}(M)} \to M \to 0.$$

It is equivalent to saying that generators of M are all in the same degree d and all nonzero entries in all matrices $\phi_k, k > 0$ are linear forms. A graded R-module M is called (d-)linear if it admits a (d-)linear minimal free resolution, or equivalently, $\beta_{i,j}(M) = 0$ for all $j \neq i + d$. It is clear that a linear module is generated in a single degree. J. Herzog and T. Hibi generalized this notion by introducing componentwise linear ideal in [7].

Definition 1.1. (Herzog and Hibi [7]). A finitely generated graded R-module M is called *componentwise linear* if for each $i \in Z$, $M_{<i>}$ has a linear resolution, where $M_{<i>}$ is the graded R-submodule of M generated by the degree i component M_i.

When R is a polynomial ring, the class of componentwise linear ideal is rather large: it includes stable monomial ideals, squarefree stable monomial ideals. In particular the generic initial ideal of each graded ideal in a polynomial ring is always componentwise linear ideal. The class of componentwise linear modules over a polynomial ring has been investigated extensively by many authors (see e.g. [7-10].)

We give an example to show that there exist some odd componentwise linear R-modules if R is not a polynomial ring.

Example. Set $R = k[x, y]/(x^2, xy, y^2)$. It is a Koszul algebra of dimension zero. Since $\text{m}^2 = 0$, we see that the maximal graded ideal m consists of linear forms and zero element, it follows that every finitely generated graded R-module is componentwise linear.

Following [1] a standard graded finitely generated K- algebra R is called *Koszul* if $\text{reg}_R(K) = 0$, i.e., K is a 0-linear R- module when it is regarded as the quotient module of R. By [2], we know that a standard graded finitely generated K- algebra R

is Koszul if and only if all finitely generated R-modules have finite regularity. Thus Koszul algebras can be characterized by the Betti diagram (see details in [8, page 69]) of its graded module: R is Koszul if and only if there are only finite nonzero rows in the Betti diagram of every finitely generated graded R-module. It was pointed out in [1] that Koszul algebras are "surprisingly common": they include the coordinate rings of "Sergre-Veronese" embeddings, any algebra with a quadratic straightening law, and any high Veronese subring of an graded ring (See [1,4,5] and its references). Let Δ be a simplicial complex. It was proved in [4] (or see [5]) that the Stanley-Reinser ring $k[\Delta]$ is Koszul if and only if Δ is a flag complex, that is the non-face ideal I_Δ is generated by quadratic monomials.

In this paper we mainly investigate componentwise linear modules over a Koszul algebra. In Section 2, we give some properties of such modules. For example, we show that $\mathrm{Soc}(M)$ is always componentwise linear for any graded module M (Proposition 2.1.). On the other hand, quotient modules of a componentwise linear module are usually not componentwise linear (Theorem 2.7). In addition, we generalize a result of componentwise linear ideal ([7, Proposition 1.3] or [8, Proposition 8.2.13]) of a polynomial ring to the one of componentwise linear module over a Koszul algebra (Theorem 2.11). Note that the original proof cannot be applied to our case since we cannot use the Koszul complex to compute the Betti numbers in the case of a Koszul algebra. Also, we show in Theorem 2.16 that the i-linear strand of the minimal free resolution of a componentwise linear module is the minimal free resolution of some module which is described explicitly. In the final section, we first show a graded module over a Koszul algebra which has linear quotients with respect to a (not necessarily minimal) system of generators ordered by their degrees is componentwise linear (Theorem 3.1). We also show that under some assumptions a graded module is componentwise linear if and only if it has linear quotients (Corollary 3.3). Finally we generalize a result of [8] by showing that a graded module over a polynomial ring which has linear quotients with respect to a minimal system of generators is componentwise linear.

We fix some notations. Throughout this paper R is a standard graded finitely generated K-algebra over a field K, m is the maximal graded ideal of R and M is a finitely generated graded R-module. We use $d(M), D(M)$ to denote the minimal degree and the maximal degree of generators of M respectively. That is:

$$d(M) = \min\{d \in Z | M_d \neq 0\}, \quad D(M) = \max\{d \in Z | (M/\mathrm{m}M)_d \neq 0\}.$$

Clearly $d(M) \leq D(M) \leq \mathrm{reg}\ (M)$ for any graded module M.

We record two known results for later use.

Lemma 1.2. ([3]). *A graded R-module M is linear if and only if* $\mathrm{reg}\ (M) = d(M) = D(M)$.

The following result can be proved by using Tor functor directly.

Lemma 1.3. (see e.g. [12]). *Let* $0 \to L \to M \to N \to 0$ *be a short exact sequence of finitely generated graded R-modules. Then*

(1) $\operatorname{reg}(N) \le \max\{\operatorname{reg}(M), \operatorname{reg}(L) - 1\}$;
(2) $\operatorname{reg}(M) \le \max\{\operatorname{reg}(L), \operatorname{reg}(N)\}$;
(3) $\operatorname{reg}(L) \le \max\{\operatorname{reg}(M), \operatorname{reg}(N) + 1\}$.

2. Componentwise Linear Modules

In this section we assume further R is a Koszul algebra. We will obtain some properties of componentwise linear modules over a Koszul algebra. Among other things, we show that the graded Betti numbers of a componentwise linear module can be determined by the Betti numbers of its components and the i-linear strand of the minimal free resolution of a componentwise linear module is acyclic for any $i \in Z$.

Proposition 2.1. *For any finitely generated graded R-module M,* $\operatorname{Soc}(M)$ *is componentwise linear. Moreover if we assume further R is a standard graded polynomial ring and M has finite length, then M is componentwise linear if and only if* $\operatorname{Soc}(M) = M$.

Proof. Let $d \in Z$ and set $N = (\operatorname{Soc}(M))_{<d>}$. Then $d(N) = D(N) = d$. On the other hand since $\mathrm{m}N = 0$, $\operatorname{reg}(N) \le \max\{r | N_r \ne 0\} = d$ by [1, Theorem 1] and so $\operatorname{reg}(N) = d(N) = D(N)$, which implies N is linear by Lemma 1.2.

Assume further R is a standard graded polynomial ring and M has finite length. If $M \ne \operatorname{Soc}(M)$, then exists $d \in Z$ such that $\mathrm{m}M_{<d>} \ne 0$. Note that $\operatorname{reg}(M_{<d>}) > d = D(M_{<d>})$ by [1, Lemma 1.3(c)], we have $M_{<d>}$ is not linear and so M is not componentwise linear. ■

Proposition 2.2. *If M is componentwise linear, then so is* $\mathrm{m}M$. *In particular, if M is linear, then so is* $\mathrm{m}M$.

Proof. For any $d \in Z$, we see that

$$M_{<d>} = M_d \oplus \mathrm{m}_1 M_d \oplus \mathrm{m}_2 M_d \oplus \cdots, \qquad (\mathrm{m}M)_{<d+1>} = \mathrm{m}_1 M_d \oplus \mathrm{m}_2 M_d \oplus \cdots.$$

Hence we can obtain the following short exact sequence

$$0 \to (\mathrm{m}M)_{<d+1>} \to M_{<d>} \to M_d \to 0.$$

Here we regard M_d as a graded R-module with $\mathrm{m}M_d = 0$. Thus $M_d \cong K(-d)^n$ for some n and reg $(M_d) = d$. It follows that $\operatorname{reg}((\mathrm{m}M)_{<d+1>}) \le d + 1$ by Lemma 1.3, implying $(\mathrm{m}M)_{<d+1>}$ is $(d+1)$-linear, as required. ■

Corollary 2.3. *Let M be a finitely generated graded R-module generated in degrees* $d_1 < d_2 < \cdots < d_r$. *If* $M_{<d_i>}$ *is linear for each* $i = 1, \cdots, r$, *then M is componentwise linear.*

Proof. Fix $d \in Z$. If $d < d_1$, then $M_{<d>} = 0$ and it is linear. If $d \geq d_1$, we let $k = \max\{i|d_i \leq d\}$; then $M_{<d>} = \mathrm{m}^{(d-d_k)}M_{<d_k>}$ is linear by Proposition 2.2. ■

Assume that M^1, M^2, M are finitely generated graded R-modules such that $M \cong M^1 \oplus M^2$. Then: (1) M^1, M^2 are d-linear if and only if M is d-linear; (2) $M_{<d>} = M^1_{<d>} \oplus M^2_{<d>}$ for any $d \in Z$. Based on those facts, we have:

Proposition 2.4. *Let M be the direct sum of finitely generated graded R-modules $M^i, i = 1, \cdots, k$. Then M is componentwise linear if and only if M^i is componentwise linear for each $i = 1, \cdots, k$.*

For convenience, we use $\mathfrak{M}(R)$ for the category of finitely generated graded R-modules whose morphisms are the homogeneous R-module homomorphisms of degree 0. Let $f : M \longrightarrow N$ be a morphism in $\mathfrak{M}(R)$. Then it induces a morphism $f_{<d>} : M_{<d>} \rightarrow N_{<d>}$ naturally. Hence we can regard "$_{<d>}$" as a functor from $\mathfrak{M}(R)$ to $\mathfrak{M}(R)$. It is routine to check that if f is surjective (resp. injective), then $f_{<d>}$ is surjective (resp. injective). However "$_{<d>}$" is not an exact functor in general.

Lemma 2.5. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence in $\mathfrak{M}(R)$ and $d \in Z$. Then*

(1) *the sequence $0 \rightarrow L_{\geq d} \cap M_{<d>} \rightarrow M_{<d>} \rightarrow N_{<d>} \rightarrow 0$ is exact.*

(2) *the induced complex $0 \rightarrow L_{<d>} \rightarrow M_{<d>} \rightarrow N_{<d>} \rightarrow 0$ is exact if and only if $L_{d+i} \cap \mathrm{m}_i M_d = \mathrm{m}_i L_d$ for each $i > 0$.*

Proof. By straightforward check.

Lemma 2.6. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence in $\mathfrak{M}(R)$ satisfying $d(L) = D(L) = d(M)$. Then for any $d \in Z$, the induced complexes $0 \rightarrow L_{<d>} \rightarrow M_{<d>} \rightarrow N_{<d>} \rightarrow 0$ and $0 \rightarrow \mathrm{m}L_{<d>} \rightarrow \mathrm{m}M_{<d>} \rightarrow \mathrm{m}N_{<d>} \rightarrow 0$ are exact.*

Proof. Denote $d(L)$ by k. If $d < k$, then $L_{<d>} = M_{<d>} = N_{<d>} = 0$ and we are done. If $d \geq k$, then $L_{d+i} = \mathrm{m}_i L_d$ for each $i > 0$ and hence the first sequence is exact by Lemma 2.5. The second sequence is a restriction of the first one, hence it is exact. ■

We give a necessary condition for which a quotient module of a componentwise linear module is componentwise linear.

Theorem 2.7. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence in $\mathfrak{M}(R)$. Assume M and N are componentwise linear and $L \cap M_{<d(M)>} \neq 0$. Then $d(M) \in \{d(L), d(L) - 1\}$.*

Proof. It is clear that $d(M) \leq d(L)$. Set $k = d(M)$. Since

$$0 \to L \cap M_{<k>} \to M_{<k>} \to N_{<k>} \to 0$$

is exact and since $\text{reg}(M_{<k>}) = \text{reg}(N_{<k>}) = k$, we obtain $\text{reg}(L \cap M_{<k>}) \leq k+1$. It follows that $d(L) \leq d(L \cap M_{<k>}) \leq \text{reg}(L \cap M_{<k>}) \leq k+1$, as required. ■

Use this result, we can give a complete description of cyclic graded R-module which is componentwise linear.

Example 2.8. If M is a nonzero cyclic graded R-module which is componentwise linear, then there exist $a \in Z$ and an 1-linear ideal I such that $M \cong (R/I)[-a]$. In the case when R is a standard graded polynomial ring, every ideal generated by 1-linear forms is 1-linear, hence we obtain that a cyclic graded R-module is componentwise linear if and only if $M \cong (R/I)[-a]$ for some $d \in Z$ and some ideal I generated by linear forms.

Proposition 2.9. *Let M be a finitely generated graded R-module with $d(M) = d$. Then M is componentwise linear if and only if both $M/M_{<d>}$ and $M_{<d>}$ are componentwise linear. Moreover in this case we have*

$$\beta_i(M_{<j>}) - \beta_i((\mathrm{m}M)_{<j-1>}) = \beta_i(N_{<j>}) - \beta_i((\mathrm{m}N)_{<j-1>}) + \beta_{i,i+j}(M_{<d>}),$$

for each i, j, where $N = M/M_{<d>}$.

Proof. By Lemma 2.6 we obtain the short exact sequence

$$(1) \qquad 0 \to (M_{<d>})_{<j>} \to M_{<j>} \to N_{<j>} \to 0$$

for each j. If M is componentwise linear, then $(M_{<d>})_{<j>}$ and $M_{<j>}$ are linear graded modules with the same regularity j, and thus $\text{reg}(N_{<j>}) \leq j$ by Lemma 1.3. It follows that $N_{<j>}$ is linear. The converse statement can be proved similarly.

Assume M is componentwise linear. Since only possible nonzero graded Betti numbers of three modules in (1) are

$$\beta_{i,i+j}(M_{<j>}), \beta_{i,i+j}((M_{<d>})_{<j>}), \beta_{i,i+j}(N_{<j>}), i \in Z,$$

one can use the long exact sequence obtained by acting Tor functor on (1) to obtain

$$(2) \qquad \beta_i(M_{<j>}) = \beta_i((M_{<d>})_{<j>}) + \beta_i(N_{<j>}).$$

Similarly, from the short exact sequence $0 \to \mathrm{m}(M_{<d>})_{<j-1>} \to \mathrm{m}M_{<j-1>} \to \mathrm{m}(M/M_{<d>})_{<j-1>} \to 0$ (see Lemma 2.6) one can obtain that

$$(3) \qquad \beta_i(\mathrm{m}M_{<j-1>}) = \beta_i(\mathrm{m}(M_{<d>})_{<j-1>}) + \beta_i(\mathrm{m}N_{<j-1>}).$$

Note that: if $j < d$, then $(M_{<d>})_{<j>} = \mathrm{m}(M_{<d>})_{<j-1>} = 0$; if $j = d$, then $(M_{<d>})_{<j>} = M_{<d>}, \mathrm{m}(M_{<d>})_{<j-1>} = 0$; if $j > d$, then $(M_{<d>})_{<j>} = \mathrm{m}(M_{<d>})_{<j-1>}$, and since $\beta_i(M_{<d>}) = \beta_{i,i+d}(M_{<d>})$, it is not difficult to see that

$$(4) \qquad \beta_{i,i+j}(M_{<d>}) = \beta_i((M_{<d>})_{<j>}) - \beta_i(\mathrm{m}(M_{<d>})_{<j-1>}).$$

Now the desired equality follows by combining (2), (3) and (4). ■

If A is a subset of Z, we use $M_{<A>}$ to denote the graded R-submodule of M generated by the subset $\cup_{i\in A} M_i$.

Corollary 2.10. *Let M be a componentwise linear graded R-module. Then $M/M_{<\leq d>}$ and $M_{<\leq d>}$ are componentwise linear for any $d \in Z$.*

Proof. Assume that M is generated in degrees $d_1 < d_2 < \cdots < d_n$. Then $N := M/M_{<d_1>}$ is componentwise linear by Proposition 2.9. Since $d(N) = d_2$ and $N_{<d_2>} = M_{<d_1,d_2>}/M_{<d_1>}$, we obtain $M/M_{<d_1,d_2>} \cong N/N_{<d_2>}$ and $M_{<d_1,d_2>}$ are componentwise linear by Proposition 2.9. It follows by induction that $M/M_{<d_1,\cdots,d_k>}$ is componentwise linear for any $1 \leq k \leq n$. Now the assertion follows by the fact $M_{<\leq d>} = M_{<d_1,\cdots,d_k>}$, where k is the largest i such that $d_i \leq d$. ■

Theorem 2.11. *Let M be a componentwise linear R-module, where R is a Koszul algebra. Then $\beta_{i,i+j}(M) = \beta_i(M_{<j>}) - \beta_i(\mathrm{m}M_{<j-1>})$ for all i, j.*

Proof. We use induction on $t := D(M) - d(M)$. If $t = 0$, then M is linear and so $\beta_{i,i+j}(M) = 0$ if $j \neq d(M)$. If $j \leq d(M) - 1$, then $M_{<j>} = \mathrm{m}M_{<j-1>} = 0$ and so $\beta_i(M_{<j>}) - \beta_i(\mathrm{m}M_{<j-1>}) = 0 = \beta_{i,i+j}(M)$; if $j = d(M)$, then $M_{<j>} = M$ and $\mathrm{m}M_{j-1} = 0$, which implies $\beta_i(M_{<j>}) - \beta_i(\mathrm{m}M_{<j-1>}) = \beta_i(M) = \beta_{i,i+j}(M)$; if $j \geq d(M) + 1$, then $M_{<j>} = \mathrm{m}M_{<j-1>}$ and so $\beta_i(M_{<j>}) - \beta_i(\mathrm{m}M_{<j-1>}) = 0 = \beta_{i,i+j}(M)$. This proves the equality holds when $t = 0$.

If $t > 0$, then the short exact sequence

$$0 \to M_{<d>} \to M \to M/M_{<d>} \to 0,$$

induces the following exact sequence

$$\cdots \to \mathrm{Tor}^R_{i+1}(K, M/M_{<d>})_{i+1+j-1} \to \mathrm{Tor}^R_i(K, M_{<d>})_{i+j} \to \mathrm{Tor}^R_i(K, M)_{i+j}$$
$$\to \mathrm{Tor}^R_i(K, M/M_{<d>})_{i+j} \to \mathrm{Tor}^R_{i-1}(K, M_{<d>})_{i-1+j+1} \to \cdots$$

By Proposition 2.9, $M/M_{<d>}$ is componentwise linear. Note that $D(M/M_{<d>}) = D(M)$ and $d(M/M_{<d>}) > d(M)$, we have $D(M/M_{<d>}) - d(M/M_{<d>}) < t$. By induction, $\mathrm{reg}(M/M_{<d>}) = D(M)$ (see Corollary 2.12). Set $D = D(M), d = d(M)$. There are five cases to consider.

Case 1. If $j > D$, then $\mathrm{Tor}_i^R(K, M_{<d>})_{i+j} = \mathrm{Tor}_i^R(K, M/M_{<d>})_{i+j} = 0$ and so $\mathrm{Tor}_i^R(K, M)_{i+j} = 0$. Hence $\beta_{i,i+j}(M) = \beta_i(M_{<j>}) - \beta_i(\mathrm{m}M_{<j-1>}) = 0$.

Case 2. If $j = D$, $\beta_{i,i+j}(M) = \beta_{i,i+j}(M/M_{<d>}) = \beta_i(M_{<j>}) - \beta_i(\mathrm{m}M_{<j-1>})$ by Proposition 2.9 and induction.

Case 3. If $j < d$, then $\beta_{i,i+j}(M) = 0 = \beta_i(M_{<j>}) - \beta_i(\mathrm{m}M_{<j-1>})$.

Case 4. If $j = d$, then $\mathrm{Tor}_{i+1}^R(K, M/M_{<d>})_{i+1+j-1} = \mathrm{Tor}_{i-1}^R(K, M_{<d>})_{i-1+j+1} = 0$ it follows that $\beta_{i,i+j}(M) = \beta_i(M_{<d>}) + \beta_{i,i+j}(M/M_{<d>}) = \beta_i(M_{<j>}) - \beta_i(\mathrm{m}M_{<j-1>})$ by Proposition 2.9.

Case 5. If $d < j < D$, then $\mathrm{Tor}_i^R(K, M_{<d>})_{i+j} = \mathrm{Tor}_{i-1}^R(K, M_{<d>})_{i-1+j+1} = 0$ which implies $\beta_{i,i+j}(M) = \beta_{i,i+j}(M/M_{<d>}) = \beta_i(M_{<j>}) - \beta_i(\mathrm{m}M_{<j-1>})$ by Proposition 2.9 again. ■

Corollary 2.12. *If M is componentwise linear, then* $\mathrm{reg}(M) = D(M)$.

Proof. Note that for any $j > D(M)$, we have $M_{<j>} = \mathrm{m}M_{<j-1>}$£¬ which implies $\beta_{i,i+j}(M) = 0$ by Theorem 2.11. It follows that $\mathrm{reg}(M) \leq D(M)$, as required. ■

We give an example to show that the converse statement of Corollary 2.12 is not true. For this we need a lemma.

Lemma 2.13. *Assume M is minimally generated by homogeneous elements $f_1, \cdots, f_k$. Let $d \in Z$ and $N = M/M_{\geq d}$. Then N is minimally generated by $\{\overline{f}_i | \deg\ f_i \leq d-1\}$, where $\overline{f}_i$ is the image of f_i in N.*

Proof. Clearly N is generated by $\{\overline{f}_i | \deg\ f_i \leq d-1\}$. Assume $\{\overline{f}_i | \deg\ f_i \leq d-1\} = \overline{f}_1, \cdots, \overline{f}_t$, where $t \leq s$. If we omit f_i in $\overline{f}_1, \cdots, \overline{f}_t$. Then there exist homogeneous elements $a_i \in R$ such that $\overline{f}_i = \sum_{1 \leq j \leq t, j \neq i} a_j \overline{f}_j$ and $\deg(a_j \overline{f}_j) = \deg(\overline{f}_i)$ for each $j \neq i$, which implies $f := f_i - \sum_{1 \leq j \leq t, j \neq i} a_i f_i \in M_{\geq d}$. Since $\deg(f) \leq d-1$, we have $f = 0$, a contradiction. ■

Example 2.14. Let $M = R \oplus R(-5)$. Then M is componentwise linear by Proposition 2.4 with $d(M) = 0$ and $D(M) = 5$. Put $N = M/M_{\geq 6}$. We see that N has finite length and so $\mathrm{reg}(N) = D(N) = 5$ by [1, Theorem 1, Lemma 3(c)] and Lemma 2.13. However, N is not componentwise linear by Theorem 2.7.

The following proposition can be regarded as a partial converse statement of Corollary 2.12.

Proposition 2.15. *Let M be finitely generated graded R-module. Then M is componentwise linear if and only if* $\mathrm{reg}(M_{<\leq t>}) = D(M_{<\leq t>})$ *for all $t \in Z$.*

Proof. It is clear that if M is componentwise linear then $\operatorname{reg}(M_{<\leq t>}) = D(M_{<\leq t>})$ for all $t \in Z$. Conversely assume $\operatorname{reg}(M_{<\leq t>}) = D(M_{<\leq t>})$ for all $t \in Z$. We use induction on $D(M) - d(M)$. If $D(M) = d(M)$, then we are done. If $D(M) - d(M) > 0$, we consider the short exact sequence $0 \to M_{<d>} \to M \to M/M_{<d>} \to 0$, where $d = d(M)$. It is clear that $M_{<d>}$ has a d-linear resolution since $\operatorname{reg}(M_{<d>}) \leq d$. By Lemmas 2.6 and 1.3, we have $\operatorname{reg}((M/M_{<d>})_{<t>}) \leq \operatorname{reg}(M_{<t>}) \leq t$ for any $t \in Z$ and it follows that $M/M_{<d>}$ is componentwise linear by induction hypothesis. Hence M is componentwise linear by Proposition 2.9. ∎

Let $\mathrm{F}.(M)$ be the minimal free resolution of M as in **Introduction**. For any integer i, the *i-linear strand* of $\mathrm{F}.(M)$ is defined to be the complex

$$\mathrm{F}.^{<i>}(M) : \cdots \to F_k^{<i>} \xrightarrow{\phi_k^{<i>}} F_{k-1}^{<i-1>} \xrightarrow{\phi_k^{<i-1>}} \cdots \xrightarrow{\phi_k^{<1>}} F_0 \longrightarrow 0$$

where $F_k^{<i>} = R(-k-i)^{\beta_{k,k+i}(M)}$ and the map $\phi_k^{<i>} : F_k^{<i>} \to F_k^{<i-1>}$ is the corresponding component of $\phi_k : F_k = \bigoplus_j F_k^{<j>} \longrightarrow F_{k-1} = \bigoplus_j F_{k-1}^{<j>}$. In general, $\mathrm{F}.^{<i>}(M)$ is not acyclic. Motivated by [13, Proposition 4.9] we obtain the following result.

Theorem 2.16. *Let M be a componentwise linear graded R-module generated in degrees $d_1 < d_2 < \cdots < d_r$. Then*

(1) $\mathrm{F}.^{<d>}(M)$ *vanishes if $d \notin \{d_1, \cdots, d_r\}$;*

(2) $\mathrm{F}.^{<d_i>}(M)$ *is the minimal free resolution of $(M/M_{<d_1,\cdots,d_{i-1}>})_{<d_i>}$ for any $1 \leq i \leq r$.*

Proof. We use induction on r. The case when $r = 1$ is clear. Assume $r > 1$ and let $N = M_{<d_1,\cdots,d_{r-1}>}$. Then N is componentwise linear by Corollary 2.10. By induction hypothesis, we have $\mathrm{F}.^{<d>}(N)$ vanishes if $d \notin \{d_1, \cdots, d_{r-1}\}$ and $\mathrm{F}.^{<d_i>}(N)$ is the minimal free resolution of $(N/N_{<d_1,\cdots,d_{i-1}>})_{<d_i>}$ for $1 \leq i \leq r-1$. Consider the short exact sequence $0 \to N \to M \to M/N \to 0$. Since M/N has a d_r-linear resolution, we obtain that $\mathrm{F}.^{<d>}(M) = \mathrm{F}.^{<d>}(N)$ if $d < d_r$ and $\mathrm{F}.^{<d>}(M) = \mathrm{F}.^{<d>}(M/N)$ if $d \geq d_r$. Note that $(N/N_{<d_1,\cdots,d_i>})_{<d>} = (M/M_{<d_1,\cdots,d_i>})_{<d>}$ if $i < r$ and $d < d_r$, the result follows. ∎

We observe that if $\mathrm{m}M = 0$, then $(M/M_{<d_1,\cdots,d_{i-1}}>)_{<d_i>} = M_{<d_i>}$ for $1 \leq i \leq r$. By this fact, we obtain the following result immediately.

Corollary 2.17. *Let M be a finitely generated graded R-module with $\mathrm{m}M = 0$. Then $\mathrm{F}.^{<d>}(M)$ is the minimal free resolution of $M_{<d>}$ for any $d \in Z$.*

3. The Module with Linear Quotients

In this section we will investigate the graded modules with linear quotients. The first two results involve graded modules over any Koszul algebra (see Theorem 3.1 and Corollary 3.3). As an application we give an example of a non-componentwise linear module over a proper Koszul algebra in Example 3.4. The last result (Theorem 3.7) involves graded modules over polynomial ring, which is a generalization of [8, Theorem 8.2.15].

Theorem 3.1. *Let M be a graded R-module (not necessarily minimally) generated by homogeneous elements $f_1, f_2, \cdots, f_n$. Suppose* $\deg\ f_1 \leq \deg\ f_2 \cdots \leq\ \deg f_n$ *and* $(f_1, \cdots, f_{i-1}) : f_i$ *is an 1-linear ideal for $i = 1, \cdots, n$. Then M is componentwise linear.*

Proof. We use induction on n. If $n = 1$, then $M \cong (R/I)[-\deg\ f_1]$, where I is 1-linear, implying M is $(\deg\ f_1)$-linear.

If $n > 1$, we set $d = d(M)$ and $k = \max\{i | \deg\ f_i = d\}$. Then $M_{<d>} = (f_1, \cdots, f_k)$, and it is d-linear by [6, Lemma 2.16]. On the other hand $M/M_{<d>} = (\overline{f}_{k+1}, \cdots, \overline{f}_n)$ is componentwise linear by induction, where $\overline{f}_i$ is the image of f_i in $M/M_{<d>}$. Now the assertion follows from Proposition 2.9. ■

Example 3.2: Let $S = k[x, y, z]$ be a standard graded polynomial ring over a field k, and let $R = S/(xz)$. Then R is a Koszul algebra by [4]. Assume that I is the ideal of R generated by $\overline{x}^2, \overline{xy}^2, \overline{y}^3$. Then the successive colon ideals of I are $(\overline{z}), (\overline{x}, \overline{z}), (\overline{x})$.

Since $\operatorname{reg}_R(R/(\overline{z})) \leq \operatorname{reg}_S(R/(\overline{z})) = \operatorname{reg}_S(S/(z)) = 0$ by [1, Theorem 1], we have $(\overline{z})$ is an 1-linear ideal of R. Similarly $(\overline{x}, \overline{z}), (\overline{x})$ are 1-linear ideals of R. Hence I is a componentwise linear R-module by Theorem 3.1. ■

Corollary 3.3. *Let M be a graded R-module (not necessarily minimally) generated by homogeneous elements $f_1, f_2, \cdots, f_n$ with* $\deg f_1 < \cdots < \deg f_r$. *Then M is componentwise linear if and only if $(f_1, \cdots, f_{i-1}) : f_i$ is an 1-linear ideal for $i = 1, \cdots, n$.*

Proof. In view of Theorem 3.1, we only need to prove the "only if" part. Assume that M is componentwise linear and denote $\deg(f_i)$ by d_i, $i = 1, \cdots, r$. Since $M_{<d_1>} = (f_1) \cong (R/I_1)[-d_1]$, where $I = 0 : f_1$, and since $M_{<d_1>}$ is linear, we have I_1 is an 1-linear ideal by Example 2.8. Set $I_i = (f_1, \cdots, f_{i-1}) : f_i$. Since $M_{<d_1,d_2>}/M_{<d_1>} \cong (R/I_2)[-d_2]$ and since $M_{<d_1,d_2>}/M_{<d_1>}$ is componentwise linear by Corollary 2.10, we obtain I_2 is 1-linear by Example 2.8. Now the result follows by induction. ■

Example 3.4. Let $S = k[w, x, y, z]$ be a standard graded polynomial ring over a field k and let $I = (w, xy, z^3)$ be an ideal of S. Since $(w, xy) : z^3 = (w, xy)$ is not linear, we have I is not componentwise linear by Corollary 3.3.

Remark. As is pointed out in Example 2.8: if we assume further R is a standard graded polynomial ring, then "an 1-linear ideal" can be changed into "an ideal generated by linear forms."

Hereafter we assume that R is a standard graded polynomial ring over a field K. We will give a generalization of [8, Theorem 8.2.15]. Our proof is in the same vein as the one of [8, Theorem 8.2.15].

Lemma 3.5. *Let M be a finitely generated graded R-module which is minimally generated by homogeneous elements $f_1, f_2, \cdots, f_s$. Assume $N = (f_1, f_2, \cdots, f_s)$ is componentwise linear and $N : f_s$ is generated by linear forms. Then for any $d \leq \deg(f_s)$, $M_{<d>}$ is linear.*

Proof. Set $k = \deg(f_s)$. Since if $d < k$, then $M_{<d>} = N_{<d>}$ is linear, we only need to prove $M_{<k>}$ is linear. Assume $N : f_s = (l_1, \cdots, l_r)$, where each l_i is a linear form. We claim that $N_{<k>} : f_s = (l_1, \cdots, l_r)$. Clearly, $N_{<k>} : f_s \subset (l_1, \cdots, l_r)$. Conversely, for each i we have $l_i f_s \in N$. Then $l_i f_s = a_1 f_1 + \cdots + a_{s-1} f_{s-1}$ with each a_i homogeneous. Since M is minimally generated by $f_1, f_2, \cdots, f_s$, it follows that each nonzero a_i are of positive degree. Since $\deg(l_i f_s) = k + 1$, we see that if $a_i \neq 0$ then $\deg(f_i) \leq k$, thus $l_i f_s \in (N_{\leq k})_{k+1}$. Note that $(N_{\leq k})_{k+1} = (N_{<k>})_{k+1}$, we obtain $l_i f_s \in N_{<k>}$, which implies $N_{<k>} : f_s = (l_1, \cdots, l_r)$.

Since $M_{<k>} = N_{<k>} + (f_s)$, we have $M_{<k>}/N_{<k>} \cong (S/I)[-k]$, where $I = (l_1, \cdots, l_r)$ and $\operatorname{reg}(M_{<k>}/N_{<k>}) = k$. Since $\operatorname{reg}(N_{<k>}) = k$, it follows that $\operatorname{reg} M_{<k>} = k$ by Lemma 1.3 and so $M_{<k>}$ is $k-$linear, as required. ∎

Lemma 3.6. *Let M be a finitely generated graded R-module which is minimally generated by homogeneous elements $f_1, f_2, \cdots, f_s$ and let $N = (f_1, f_2, \cdots, f_{s-1})$ Assume $N : f_s$ is generated by linear forms $l_1, \cdots, l_r$ and $l_1, \cdots, l_r, l_{r+1}, \cdots, l_n$ forms a basis of R_1. Put $M' = (f_1, f_2, \cdots, f_{s-1}, l_{r+1} f_s, \cdots, l_n f_s)$, a submodule of M. Then*

(1) *M' is minimally generated by $f_1, f_2, \cdots, f_{s-1}, l_{r+1} f_s, \cdots, l_n f_s$.*

(2) *the colon ideals $(N + l_{r+1} f_s + \cdots + l_{r+i-1} f_s) : l_{r+i} f_s$, $i = 1, \cdots, n - r$, are all generated by linear forms.*

Proof. (1) It is clear we cannot omit any f_i. If we omit $l_{r+i} f_s$ for some i, then $l_{r+i} f_s = a_1 l_{r+1} f_s + \cdots + a_{i-1} l_{r+i-1} f_s + a_{i+1} l_{r+i+1} f_s + \cdots + a_{n-r} l_n f_s + g$ with $a_i \in S$ and $g \in N$. It follows that $l_{r+i} - a_1 l_{r+1} - \cdots - a_{i-1} l_{r+i-1} - a_{i+1} l_{r+i+1} + \cdots + a_{n-r} l_n \in (l_1, \cdots, l_r)$, and hence $l_{r+i} \in (l_1, \cdots, \widehat{l_{r+i}}, \cdots, l_n)$, a contradiction.

(2) Let $h \in (N + l_{r+1} f_s + \cdots + l_{r+i-1} f_s) : l_{r+i} f_s$, where $i = 1, \cdots, n$. Then

$$h l_{r+i} f_s = g + a_1 l_{r+1} f_s + \cdots + a_{i-1} l_{r+i-1} f_s$$

for some $g \in N, a_j \in S$, which implies $hl_{r+i}+a_1l_{r+1}+\cdots+a_{i-1}l_{r+i-1} \in (l_1, \cdots, l_r)$. Since $l_1, \cdots, l_{r+i}$ forms a regular sequence, we obtain $h \in (l_1, \cdots, l_r, \cdots, l_{r+i-1})$, and so $(N + l_{r+1}f_s + \cdots + l_{r+i-1}f_s) : l_{r+i}f_s = (l_1, \cdots, l_r, \cdots, l_{r+i-1})$. ■

Theorem 3.7. *Let M be a graded R-module minimally generated by homogeneous elements $f_1, f_2, \cdots, f_s$. Suppose $(f_1, \cdots, f_{i-1}) : f_i$ is generated by linear forms for $i = 1, \cdots, n$. Then M is componentwise linear.*

Proof. By induction we can assume $N := (f_1, \cdots, f_{s-1})$ is componentwise linear. To show M is componentwise linear we use induction on $D(N) - k$, where $k = \deg(f_s)$.

If $D(N) - k \leq 0$, then the result follows from Lemma 3.5 and Corollary 2.3. Now assume $D(N) > k$. Since $M_{<d>}$ is linear for $d \leq k$ by Lemma 3.5, we only need to prove $M_{<j>}$ is linear for $j \geq k+1$. Note that $M_{<j>} = M'_{<j>}$ (see Lemma 3.6) for $j \geq k+1$ and since M' is componentwise linear by induction, the result follows. ■

Remark. It was pointed out in [8, Page 142] the condition in Theorem 3.7 that M has linear quotients with respect to a *minimal* system of homogeneous generators cannot be omitted. Hence Theorem 3.7 cannot imply Theorem 3.1 even when R is a polynomial ring.

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