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ON UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING FIVE SMALL FUNCTIONS IN SOME ANGULAR DOMAINS

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Abstract. In this paper, we prove that two nonconstant meromorphic functions are identical provided that they share five distinct small functions IM in some angular domains.

1. Introduction and Main Results

It is assumed that the reader is familiar with the Nevanlinna's value distribution theory of meromorphic functions (see [1] or [2]). Let α be a nonconstant meromorphic functions or $\alpha \in \mathbb{C}_\infty (= \mathbb{C} \cup \{\infty\})$. We say that two meromorphic function f and g share α IM (ignoring multiplicities) in $X \subset \mathbb{C}$ provided that in X, $f(z) - \alpha = 0$ if and only if $g(z) - \alpha = 0$, f and g share α CM (counting multiplicities) in X provided that in X, $f(z) - \alpha$ and $g(z) - \alpha$ assume 0 at the same points with the same multiplicities.

R. Nevanlinna [3] proved the following well-known five value theorem.

Theorem A. If two nonconstant meromorphic functions f and g share five distinct values $\alpha_i (i = 1, \cdots, 5)$ IM in $X = \mathbb{C}$, then $f(z) \equiv g(z)$.

After his very work, lots of uniqueness results of meromorphic functions in the complex plane have been obtained, which are introduced systematically in [4]. In [5], J. H. Zheng suggested first time to investigate the uniqueness of meromorphic functions in a precise subset of $\mathbb{C}$ and posed the following question.

Question 1.1. Under what conditions, must two meromorphic functions on $X (\neq \mathbb{C})$ be indentical?

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It is an interesting topic how to extend some important uniqueness results in the complex plane to an angular domain, see [5, 6, 7, 8, 9, 10]. In [5, 6, 9, 10], by using Nevanlinna characteristic for an angular domain, the authors have extended Nevanlinna's five value theorem and four value theorem to some angular domain respectively. Recently, J. H. Zheng [11] proved the following version of five value theorem of meromorphic functions in an angular domain in terms of the Tsuji characteristic.

Theorem B. Let f and g be two nonconstant meromorphic functions in an angular domain $\Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$ $(0 \le \alpha < \beta \le 2\pi)$, and

$$\overline{\lim_{r\to\infty}} \frac{\mathfrak{T}_{\alpha,\beta}(r, f)}{\log r} = \infty.$$

If f and g share five distinct values $a_j (j = 1, \cdots, 5)$ IM in $\Omega(\alpha, \beta)$, then $f \equiv g$.

In this paper, we use $\mathfrak{T}_{\alpha,\beta}(r, f)$ to denote the Tsuji characteristic of f in an angular domain $\Omega(\alpha, \beta)$ and its definition can be found below. The Nevanlinna five value theorem has been extended in [12, 13] to the case of five IM shared small functions. Please see the following result.

Theorem C. Let f and g be two nonconstant meromorphic functions in $\mathbb{C}$, and $\alpha_j (j = 1, \cdots, 5)$ be five distinct small functions with respect to f and g. If f and g share $\alpha_j (j = 1, \cdots, 5)$ IM in $\mathbb{C}$, then $f(z) \equiv g(z)$.

It is natural to hope such an extension of Theorem B would be also available. In order to make our statements understand easily, first of all we introduce some notations and definitions as follows (see [11]).

Let f be a meromorphic function in an angular domain $\Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \le 2\pi$. Define

$$\mathfrak{M}_{\alpha,\beta}(r, f) = \frac{1}{2\pi} \int_{\arcsin(r^{-\omega})}^{\pi - \arcsin(r^{-\omega})} \log^+ \left| f(re^{i(\alpha+\omega^{-1}\theta)} \sin^{\omega^{-1}} \theta)\right| \frac{1}{r^\omega sin^2\theta} d\theta,$$

$$\mathfrak{N}_{\alpha,\beta}(r, f) = \sum_{1<|b_n|<r(\sin(\omega(\beta_n-\alpha)))^{\omega^{-1}}} \Big(\frac{\sin \omega(\beta_n - \alpha)}{|b_n|^\omega} - \frac{1}{r^\omega}\Big) = \omega \int_1^r \frac{\mathfrak{n}_{\alpha,\beta}(t, f)}{t^{\omega+1}} dt,$$

where $\omega = \frac{\pi}{\beta-\alpha}$, $b_n = |b_n|e^{i\beta_n}$ are the poles of $f(z)$ in $\Xi(\alpha, \beta, r) = \{z = te^{i\theta} : \alpha < \theta < \beta, 1 < t \le r(sin(\omega(\theta - \alpha)))^{1/\omega}\}$ appearing according to their multiplicities, $\mathfrak{n}_{\alpha,\beta}(r, f)$ denotes the number of poles of $f(z)$ in $\Xi(\alpha, \beta, r)$ counting multiplicities. We also define $\overline{\mathfrak{N}}_{\alpha,\beta}(r, f)$ in the same form of $\mathfrak{N}_{\alpha,\beta}(r, f)$ for distinct poles b_n of f, that is, ignoring their multiplicities. The Tsuji characteristic of meromorphic function f in $\Omega(\alpha, \beta)$ is defined by

$$\mathfrak{T}_{\alpha,\beta}(r, f) = \mathfrak{M}_{\alpha,\beta}(r, f) + \mathfrak{N}_{\alpha,\beta}(r, f).$$

Let f and α be two meromorphic functions in $\Omega(\alpha, \beta)$. α is called a small function with respect to f in $\Omega(\alpha, \beta)$ (in the sense of the Tsuji characteristic) if $\mathfrak{T}_{\alpha,\beta}(r, \alpha) = o(\mathfrak{T}_{\alpha,\beta}(r, f))$ as $r \to \infty$, possibly outside a set E of r of finite linear measure. In the sequel, the term "small function" always means small function in the sense of the Tsuji characteristic.

The following is the question we consider in this paper.

Question 1.2. Do f and g coincide if they share five distinct small functions IM in an angular domain?

Dealing with the above question, we obtain the following results which give an affirmative answer to Question 1.2.

Theorem 1.1. *Let f and g be two nonconstant meromorphic functions in an angular domain $\Omega(\alpha, \beta)$ such that*

$$\overline{\lim_{r\to\infty}} \frac{\mathfrak{T}_{\alpha,\beta}(r, f)}{\log r} = \infty, \tag{1.1}$$

and let α_j $(j = 1, \cdots, 5)$ be five distinct small functions with respect to f and g in $\Omega(\alpha, \beta)$. If f and g share α_j $(j = 1, \cdots, 5)$ IM in $\Omega(\alpha, \beta)$, then $f \equiv g$.

Let $a \in \mathbb{C}_\infty$, we use $n(r, \Omega(\alpha, \beta), f = a)$ to denote the number of zeros of $f - a$ in $\Omega(\alpha, \beta) \bigcap \{z : |z| < r\}$ counting multiplicities. When $a = \infty$, the zeros of $f - a$ mean the poles of f.

Theorem 1.2. *Let f and g be two nonconstant meromorphic functions in an angular domain $\Omega(\alpha, \beta)$ such that for some $\varepsilon > 0$ and for some $a \in \mathbb{C}_\infty$,*

$$\overline{\lim_{r\to\infty}} \frac{\log n(r, \Omega(\alpha + \varepsilon, \beta - \varepsilon), f = a)}{\log r} > \frac{\pi}{\beta - \alpha}. \tag{1.2}$$

Let $\alpha_j (j = 1, \cdots, 5)$ be five distinct small functions with respect to f and g in $\Omega(\alpha, \beta)$. If f and g share α_j $(j = 1, \cdots, 5)$ IM in $\Omega(\alpha, \beta)$, then $f \equiv g$.

Remark 1.1. It is well know that every meromorphic function of order ρ $(0 < \rho \leq +\infty)$ must have at least one direction $\arg z = \theta (0 \leq \theta < 2\pi)$ such that for sufficiently small $\varepsilon > 0$,

$$\overline{\lim_{r\to\infty}} \frac{\log n(r, \Omega(\theta - \varepsilon, \theta + \varepsilon), f = a)}{\log r} = \rho$$

holds for all $a \in \mathbb{C}_\infty$ with at most two exceptional values. So when f is of order $\rho \in (1/2, \infty]$, the angular domain satisfying (1.2) must exist.

Remark 1.2. For the case that α_j $(j = 1, \cdots, 5)$ are five distinct complex number, the result in Theorem 1.2 has been obtained in [6].

Theorem 1.3. *Let f and g be two nonconstant meromorphic functions in $\mathbb{C}$, $\Omega(\alpha, \beta)$ be an angular domain satisfying*

$$\overline{\lim_{\varepsilon\to 0^+}}\ \overline{\lim_{r\to+\infty}} \frac{\log T_0(r, \Omega(\alpha+\varepsilon, \beta-\varepsilon), f)}{\log r} > \frac{\pi}{\beta-\alpha}. \tag{1.3}$$

Let α_j $(j = 1, \cdots, 5)$ be five distinct small functions with respect to f and g in $\Omega(\alpha, \beta)$. If f and g share α_j $(j = 1, \cdots, 5)$ IM in $\Omega(\alpha, \beta)$, then $f \equiv g$, where $T_0(r, \Omega(\alpha, \beta), f)$ denote the Ahlfors-Shimizu characteristic of f in $\Omega(\alpha, \beta)$.

Remark 1.3. For the case that α_j $(j = 1, \cdots, 5)$ are five distinct complex number, the result in Theorem 1.3 has been obtained in [10].

2. Lemmas

Lemma 2.1. (see [11]). *Let f be a nonconstant meromorphic function in $\Omega(\alpha, \beta)$. Then for every $a \in \mathbb{C}$ we have*

$$\mathfrak{T}_{\alpha,\beta}\Big(r, \frac{1}{f-a}\Big) = \mathfrak{T}_{\alpha,\beta}(r, f) + O(1).$$

Lemma 2.2. (see [11]). *Let f be a nonconstant meromorphic function in $\Omega(\alpha, \beta)$ and k be a positive integer. Then for $0 < r < R$, we have*

$$\mathfrak{M}_{\alpha,\beta}\Big(r, \frac{f^{(k)}}{f}\Big) \le c\Big\{\log^+ \mathfrak{T}_{\alpha,\beta}(R, f) + \log\frac{R}{R-r} + 1\Big\},$$

where $c > 0$ is a constant. Furthermore, we have

$$\mathfrak{M}_{\alpha,\beta}\Big(r, \frac{f^{(k)}}{f}\Big) = O\Big\{\log r + \log^+ \mathfrak{T}_{\alpha,\beta}(r, f)\Big\}$$

as $r \to \infty$, possibly outside a set E of r of finite linear measure.

For the sake of convenience, we use $Q_{\alpha,\beta}(r, f)$ to denote any quantity satisfying

$$Q_{\alpha,\beta}(r, f) = O\Big\{\log r + \log^+ \mathfrak{T}_{\alpha,\beta}(r, f)\Big\}$$

as $r \to \infty$, possibly outside a set E of r of finite linear measure.

Lemma 2.3. (see [11]). *Let f be a nonconstant meromorphic function in $\Omega(\alpha, \beta)$ and $b_j \in \mathbb{C}_\infty (j = 1, \cdots, q)$ be $q(\ge 3)$ distinct complex number. Then*

$$(q-2)\mathfrak{T}_{\alpha,\beta}(r, f) \le \sum_{j=1}^{q} \overline{\mathfrak{N}}_{\alpha,\beta}\Big(r, \frac{1}{f-b_j}\Big) + Q_{\alpha,\beta}(r, f).$$

Lemma 2.4. (see [11]). *Let f be a nonconstant meromorphic function in $\Omega(\alpha,\beta)$. Then for all irreducible rational function $R(z,f)$ in f with coefficients meromorphic and small with respect to f in $\Omega(\alpha,\beta)$, we have*

$$\mathfrak{T}_{\alpha,\beta}(r,R(z,f)) = d\mathfrak{T}_{\alpha,\beta}(r,f) + o(\mathfrak{T}_{\alpha,\beta}(r,f))$$

as $r\to\infty$, possibly outside a set E of r of finite linear measure, where d denotes the degree of $R(z,f)$ in f.

Lemma 2.5. *Let f be a nonconstant meromorphic function in $\Omega(\alpha,\beta)$, and $\beta_j (j = 1,2,3)$ be small functions with respect to f in $\Omega(\alpha,\beta)$. Then*

$$\mathfrak{T}_{\alpha,\beta}(r,f) \le \sum_{j=1}^{3} \overline{\mathfrak{N}}_{\alpha,\beta}\Big(r,\frac{1}{f-\beta_j}\Big) + Q_{\alpha,\beta}(r,f) + o(\mathfrak{T}_{\alpha,\beta}(r,f)).$$

Proof. Set

$$F(z) = \frac{f(z)-\beta_1(z)}{f(z)-\beta_2(z)}\cdot\frac{\beta_3(z)-\beta_2(z)}{\beta_3(z)-\beta_1(z)}.$$

Then combining Lemmas 2.3 and 2.4, we obtain the result. ∎

Lemma 2.6. (see [14]). *Let $g:(0,\infty)\to R$ and $h:(0,\infty)\to R$ be monotone nondecreasing functions such that $g(r)\le h(r)$ outside of an exceptional set E of finite linear measure. Then for any $\alpha>1$, there exists r_0 such that $g(r)\le h(\alpha r)$ for all $r>r_0$.*

Let f be meromorphic function in $\Omega(\alpha,\beta)$, we define $N(r,\Omega(\alpha,\beta),f=a)$ as

$$N(r,\Omega(\alpha,\beta),f=a) = \int_1^r \frac{n(t,\Omega(\alpha,\beta),f=a)}{t}dt.$$

Especially, when $a=\infty$, we denote it as $N(r,\Omega(\alpha,\beta),f)$. We also define $\overline{N}(r,\Omega(\alpha,\beta), f=a)$ in the same form of $N(r,\Omega(\alpha,\beta),f=a)$ for distinct zeros of $f-a$, that is, ignoring their multiplicities.

Lemma 2.7. (see [11]). *Let f be a nonconstant meromorphic function in $\Omega(\alpha,\beta)$. Then for $\varepsilon>0$, we have*

$$\mathfrak{N}_{\alpha,\beta}(r,f) \ge \omega c^{\omega}\frac{N(cr,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f)}{r^{\omega}} + \omega^2 c^{\omega}\int_1^{cr}\frac{N(t,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f)}{t^{\omega+1}}dt,$$

where $\omega=\frac{\pi}{\beta-\alpha}$, $0<c<1$ is a constant depending on ε.

Lemma 2.8. (see [15]). *Let f be meromorphic in $\mathbb{C}$. Then for any three distinct points a_1,a_2,a_3 on $\mathbb{C}_\infty$ and any small $\varepsilon>0$, we have*

$$T_0(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f) \le 3\sum_{i=1}^{3}\overline{N}(2r,\Omega(\alpha,\beta),f=a_i) + O\big((\log r)^2\big).$$

Lemma 2.9. *Let f be a meromorphic function in $\mathbb{C}$, and $\Omega(\alpha,\beta)$ be an angular domain. If*

$$\overline{\lim_{\varepsilon\to 0^+}}\ \overline{\lim_{r\to+\infty}} \frac{\log T_0(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f)}{\log r}=\lambda, \tag{2.1}$$

where $0<\lambda\le\infty$, then

$$\overline{\lim_{\varepsilon\to 0^+}}\ \overline{\lim_{r\to+\infty}} \frac{\log \overline{N}(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f=a)}{\log r}\ge\lambda \tag{2.2}$$

for all $a\in\mathbb{C}_\infty$ with at most two exceptional values.

Proof. If the conclusion doesn't hold, then there exist at least three distinct values a_1,a_2,a_3 such that for sufficiently small $\varepsilon>0$,

$$\overline{\lim_{r\to+\infty}} \frac{\log \overline{N}(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f=a_i)}{\log r}<\lambda \quad (i=1,2,3) \tag{2.3}$$

Set

$$\mu_i=\overline{\lim_{r\to+\infty}} \frac{\log \overline{N}(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f=a_i)}{\log r}. \quad (i=1,2,3)$$

Let σ be a real number such that $\max\limits_{1\le i\le 3}\{\mu_i\}<\sigma<\lambda$. Then by (2.3), for sufficiently large r,

$$\overline{N}(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f=a_i)<r^\sigma. \quad (i=1,2,3) \tag{2.4}$$

Combining Lemma 2.8 and (2.4), we have

$$\begin{aligned} &T_0(r,\Omega(\alpha+2\varepsilon,\beta-2\varepsilon),f)\\ &\le 3\sum_{i=1}^{3}\overline{N}(2r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f=a_i)+O\big((\log r)^2\big)\\ &< Mr^\sigma, \end{aligned} \tag{2.5}$$

where M is a positive number. By (2.5) we have

$$\overline{\lim_{r\to+\infty}} \frac{\log T_0(r,\Omega(\alpha+2\varepsilon,\beta-2\varepsilon),f)}{\log r}\le\sigma.$$

Hence

$$\overline{\lim_{\varepsilon\to 0^+}}\ \overline{\lim_{r\to+\infty}} \frac{\log T_0(r,\Omega(\alpha+\varepsilon,\beta-\varepsilon),f)}{\log r}\le\sigma<\lambda,$$

which contradicts (2.1). Lemma 2.9 is thus completely proved. ■

Lemma 2.10. *Let f be a meromorphic function in $\mathbb{C}$, and $\Omega(\alpha, \beta)$ be an angular domain. If*

$$\varlimsup_{\varepsilon \to 0^+} \varlimsup_{r \to +\infty} \frac{\log T_0(r, \Omega(\alpha+\varepsilon, \beta-\varepsilon), f)}{\log r} > \frac{\pi}{\beta-\alpha},$$

then

$$\varlimsup_{r \to \infty} \frac{\mathfrak{T}_{\alpha,\beta}(r, f)}{\log r} = \infty.$$

Proof. Set

$$\lambda = \varlimsup_{\varepsilon \to 0^+} \varlimsup_{r \to +\infty} \frac{\log T_0(r, \Omega(\alpha+\varepsilon, \beta-\varepsilon), f)}{\log r}. \tag{2.6}$$

By Lemma 2.9 and (2.6), there exists some $a \in \mathbb{C}_\infty$ such that

$$\varlimsup_{\varepsilon \to 0^+} \varlimsup_{r \to +\infty} \frac{\log \overline{N}(r, \Omega(\alpha+\varepsilon, \beta-\varepsilon), f=a)}{\log r} \geq \lambda. \tag{2.7}$$

By (2.7), for any given $\varepsilon_1 (0 < 2\varepsilon_1 < \lambda - \frac{\pi}{\beta-\alpha})$, there exists at least some $\varepsilon_0 (> 0)$ sufficiently small such that

$$\varlimsup_{r \to +\infty} \frac{\log \overline{N}(r, \Omega(\alpha+\varepsilon_0, \beta-\varepsilon_0), f=a)}{\log r} \geq \lambda - \varepsilon_1. \tag{2.8}$$

Let μ be a real number such that $\frac{\pi}{\beta-\alpha} < \mu < \lambda - \varepsilon_1$. Then by (2.8), there exists a sequence $r_n \to \infty$ such that

$$\overline{N}(r_n, \Omega(\alpha+\varepsilon_0, \beta-\varepsilon_0), f=a) > r_n^{\mu} \tag{2.9}$$

holds for r_n sufficiently large. By Lemma 2.7 and (2.9), we get

$$\mathfrak{N}_{\alpha,\beta}(r_n, f=a) \geq \frac{\pi c^{\mu+\frac{\pi}{\beta-\alpha}}}{\beta-\alpha} r_n^{\mu-\frac{\pi}{\beta-\alpha}}. \tag{2.10}$$

Then by Lemma 2.1 and (2.10), we prove the conclusion. ∎

3. Proofs of Theorems

Proof of Theorem 1.1. Suppose that $f \not\equiv g$. Set

$$L(w) = \frac{w-\alpha_1}{w-\alpha_2} \cdot \frac{\alpha_3-\alpha_2}{\alpha_3-\alpha_1}. \tag{3.1}$$

Let $F(z) = L(f(z)), G(z) = L(g(z)), \beta_j = L(\alpha_j),\ (j = 1, \cdots, 5)$. By (3.1) and Lemma 2.4, we get $\beta_1 = 0, \beta_2 = \infty, \beta_3 = 1$, and $\beta_1, \cdots, \beta_5$ are small functions with respect to F and G. By the assumption of Theorem 1.1 and (3.1), we know that F and G share $0, 1, \infty$ IM. Then combining Lemmas 2.1, 2.3, we get

$$\begin{aligned}\mathfrak{T}_{\alpha,\beta}(r,F) &\leq \overline{\mathfrak{N}}_{\alpha,\beta}\Big(r,\frac{1}{F}\Big)+\overline{\mathfrak{N}}_{\alpha,\beta}\Big(r,\frac{1}{F-1}\Big)+\overline{\mathfrak{N}}_{\alpha,\beta}(r,F)+Q_{\alpha,\beta}(r,F)\\ &\leq \overline{\mathfrak{N}}_{\alpha,\beta}\Big(r,\frac{1}{G}\Big)+\overline{\mathfrak{N}}_{\alpha,\beta}\Big(r,\frac{1}{G-1}\Big)+\overline{\mathfrak{N}}_{\alpha,\beta}(r,G)+Q_{\alpha,\beta}(r,F)\\ &\leq 3\mathfrak{T}_{\alpha,\beta}(r,G)+Q_{\alpha,\beta}(r,F).\end{aligned} \tag{3.2}$$

Similarly, we have

$$\mathfrak{T}_{\alpha,\beta}(r,G) \leq 3\mathfrak{T}_{\alpha,\beta}(r,F)+Q_{\alpha,\beta}(r,G). \tag{3.3}$$

Hence by (3.2) and (3.3), we get

$$Q_{\alpha,\beta}(r,F)=Q_{\alpha,\beta}(r,G). \tag{3.4}$$

We claim that at least three among $\overline{\mathfrak{N}}_{\alpha,\beta}\Big(r,\frac{1}{F-\beta_j}\Big)$ $(j=1,\cdots,5)$ are not equal to $Q_{\alpha,\beta}(r,F)+o(\mathfrak{T}_{\alpha,\beta}(r,F))$. Otherwise, by Lemma 2.5, we get

$$\mathfrak{T}_{\alpha,\beta}(r,F) \leq Q_{\alpha,\beta}(r,F)+o(\mathfrak{T}_{\alpha,\beta}(r,F)). \tag{3.5}$$

By (3.5) and Lemma 2.6, we get

$$\varlimsup_{r\to\infty}\frac{\mathfrak{T}_{\alpha,\beta}(r,f)}{\log r}<\infty,$$

which contradicts (1.1).

Without loss of generality, we assume that

$$\overline{\mathfrak{N}}_{\alpha,\beta}\Big(r,\frac{1}{F-\beta_5}\Big) \neq Q_{\alpha,\beta}(r,F)+o(\mathfrak{T}_{\alpha,\beta}(r,F)). \tag{3.6}$$

Now we use the method of [13] and [16] to complete the proof. Set

$$H=\frac{F'(\beta_4'G-\beta_4G')(F-G)}{F(F-1)G(G-\beta_4)}-\frac{G'(\beta_4'F-\beta_4F')(F-G)}{G(G-1)F(F-\beta_4)}. \tag{3.7}$$

Then by (3.7), we get

$$H=\frac{(F-G)H_1}{F(F-1)(F-\beta_4)G(G-1)(G-\beta_4)}, \tag{3.8}$$

where

$$\begin{aligned}H_1&=F'(\beta_4'G-\beta_4G')(G-1)(F-\beta_4)-G'(\beta_4'F-\beta_4F')(F-1)(G-\beta_4)\\ &=\beta_4'FF'G^2-\beta_4'FF'G-\beta_4(\beta_4-1)FF'G'-\beta_4\beta_4'F'G^2+\beta_4\beta_4'F'G\\ &\quad-\beta_4'F^2GG'+\beta_4'FGG'+\beta_4(\beta_4-1)F'GG'+\beta_4\beta_4'F^2G'-\beta_4\beta_4'FG'.\end{aligned} \tag{3.9}$$

Noting that $f \not\equiv g$, by (3.1), we have

$$F \not\equiv G. \tag{3.10}$$

We discuss the following two cases.

Case 1. $H \equiv 0$. By (3.7) and (3.10), we get

$$\frac{F'(\beta_4' G - \beta_4 G')}{(F-1)(G-\beta_4)} \equiv \frac{G'(\beta_4' F - \beta_4 F')}{(G-1)(F-\beta_4)}. \tag{3.11}$$

If β_4 is a constant, then by $\beta_4 \neq 1$ and (3.11), we get $F \equiv G$, which contradicts (3.10). So β_4 is not a constant. By (3.11), we get

$$\frac{F'(\beta_4' G - \beta_4 G')}{G'(\beta_4' F - \beta_4 F')} - 1 \equiv \frac{(F-1)(G-\beta_4)}{(G-1)(F-\beta_4)} - 1.$$

Hence we get

$$\frac{F' - G'}{F - G} \equiv \frac{(1-\beta_4)G'(\beta_4' F - \beta_4 F')}{\beta_4' G(G-1)(F-\beta_4)} + \frac{G'}{G}. \tag{3.12}$$

By (3.6), we know that there is a point z_0 such that z_0 is a common zero of $F - \beta_5$ and $G - \beta_5$, but is not a zero or a pole of $\beta_4, \beta_4', \beta_5, \beta_5 - 1, \beta_5 - \beta_4$. It is obvious that z_0 is a pole of the left side of (3.12), and not a pole of the right side of (3.12), which is a contradiction.

Case 2. $H \not\equiv 0$. By (3.7), we get

$$\begin{aligned} H = {} & \frac{F'}{F-1} \cdot \frac{\beta_4' G - \beta_4 G'}{G(G-\beta_4)} - \Big(\frac{F'}{F-1} - \frac{F'}{F}\Big) \cdot \frac{\beta_4' G - \beta_4 G'}{G-\beta_4} \\ & - \Big(\frac{G'}{G-1} - \frac{G'}{G}\Big) \cdot \frac{\beta_4' F - \beta_4 F'}{F-\beta_4} + \frac{G'}{G-1} \cdot \frac{\beta_4' F - \beta_4 F'}{F(F-\beta_4)}. \end{aligned} \tag{3.13}$$

Since

$$\frac{\beta_4' G - \beta_4 G'}{G(G-\beta_4)} = \frac{G'}{G} - \frac{G' - \beta_4'}{G - \beta_4}, \quad \frac{\beta_4' G - \beta_4 G'}{G-\beta_4} = \beta_4' - \frac{\beta_4(G' - \beta_4')}{G-\beta_4}, \tag{3.14}$$

then by Lemma 2.2 and (3.4), we get

$$\begin{aligned} \mathfrak{M}_{\alpha,\beta}\Big(r, \frac{\beta_4' G - \beta_4 G'}{G(G-\beta_4)}\Big) & \le \mathfrak{M}_{\alpha,\beta}\Big(r, \frac{G'}{G}\Big) + \mathfrak{M}_{\alpha,\beta}\Big(r, \frac{G' - \beta_4'}{G-\beta_4}\Big) \\ & = Q_{\alpha,\beta}(r, F) + o(\mathfrak{T}_{\alpha,\beta}(r, F)), \end{aligned} \tag{3.15}$$

$$\mathfrak{M}_{\alpha,\beta}\Big(r,\frac{\beta_4'G-\beta_4G'}{G-\beta_4}\Big)\le\mathfrak{M}_{\alpha,\beta}(r,\beta_4')+\mathfrak{M}_{\alpha,\beta}\Big(r,\frac{\beta_4(G'-\beta_4')}{G-\beta_4}\Big)$$
$$=Q_{\alpha,\beta}(r,F)+o(\mathfrak{T}_{\alpha,\beta}(r,F)). \tag{3.16}$$

Combining (3.13), (3.15) and (3.16), we get

$$\mathfrak{M}_{\alpha,\beta}(r,H)=Q_{\alpha,\beta}(r,F)+o(\mathfrak{T}_{\alpha,\beta}(r,F)). \tag{3.17}$$

Next we estimate $\mathfrak{N}_{\alpha,\beta}(r,H)$. By (3.7), we know that the poles of H only possibly occur from the zeros of $F,G,F-1,G-1,F-\beta_4$ and $G-\beta_4$, the poles of F,G and β_4. Let E_0 be the set of all zeros, 1-points and poles of β_4. We discuss the following four subcases.

Subcase 1. Suppose that z_1 is a zero of F with multiplicity m_1 and G with multiplicity n_1, but $z_1\notin E_0$. Then by (3.9), we know that z_1 is a zero of H_1 with multiplicity at least m_1+n_1-1. Noting that z_1 is a zero of $F-G$ with multiplicity $\min\{m_1,n_1\}$, by (3.8), we deduce that z_1 is not a pole of H.

Subcase 2. Suppose that z_2 is a pole of F with multiplicity m_2 and G with multiplicity n_2, but $z_2\notin E_0$. Then by (3.9), we know that z_2 is a pole of H_1 with multiplicity at most $2m_2+2n_2+1$. Noting that z_2 is a pole of $F-G$ with multiplicity at most $\max\{m_2,n_2\}$, by (3.8), we deduce that z_2 is not a pole of H.

Subcase 3. Suppose that z_3 is a zero of $F-1$ with multiplicity m_3 and $G-1$ with multiplicity n_3, but $z_3\notin E_0$. Noting that z_3 is a zero of $F-G$ with multiplicity $\min\{m_3,n_3\}$, a simple pole of $\frac{F'}{F-1}$ and $\frac{G'}{G-1}$, by (3.7), we deduce that z_3 is not a pole of H.

Subcase 4. Suppose that z_4 is a zero of $F-\beta_4$ with multiplicity m_4 and $G-\beta_4$ with multiplicity n_4, but $z_4\notin E_0$. By (3.14), we know that z_4 is a simple pole of $\frac{\beta_4'G-\beta_4G'}{G(G-\beta_4)}$ and $\frac{\beta_4'F-\beta_4F'}{F(F-\beta_4)}$. Noting that z_4 is a zero of $F-G$, by (3.7), we deduce that z_4 is not a pole of H.

From the above, we get

$$\mathfrak{N}_{\alpha,\beta}(r,H)=o(\mathfrak{T}_{\alpha,\beta}(r,F)). \tag{3.18}$$

Thus by (3.17) and (3.18), we get

$$\mathfrak{T}_{\alpha,\beta}(r,H)=Q_{\alpha,\beta}(r,F)+o(\mathfrak{T}_{\alpha,\beta}(r,F)). \tag{3.19}$$

Since F and G share β_5 IM, by (3.7) and (3.19), we get

$$\overline{\mathfrak{N}}_{\alpha,\beta}\Big(r,\frac{1}{F-\beta_5}\Big)\le\overline{\mathfrak{N}}_{\alpha,\beta}\Big(r,\frac{1}{H}\Big)\le Q_{\alpha,\beta}(r,F)+o(\mathfrak{T}_{\alpha,\beta}(r,F)),$$

which contradicts (3.6). Theorem 1.1 is completely proved. ∎

Proof of Theorem 1.2. By Lemma 2.7, (1.2) implies (1.1). So combining Theorem 1.1 we get the conclusion of Theorem 1.2. ■

Proof of Theorem 1.3. By (1.3) and Lemma 2.10, we know that f satisfies (1.1). Hence by Theorem 1.1 we obtain that $f \equiv g$ in $\Omega(\alpha, \beta)$. Then by the identity principle we prove that $f \equiv g$ in $\mathbb{C}$. ■

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