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NONDECREASING SOLUTIONS OF A QUADRATIC INTEGRAL EQUATION OF VOLTERRA TYPE

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Abstract. Using the theory of measures of noncompactness and applying a new method, we prove the existence of nondecreasing solutions of a quadratic integral equation of Volterra type in $C(I)$.

1. INTRODUCTION

In this paper, we discuss the following quadratic integral equation of Volterra type

$$x(t) = h(t) + g(t, x(t)) \int_0^t k(t, s) f(s, x(\lambda s)) ds, \qquad t \in I = [0, 1], \tag{1.1}$$

where $f, g : I \times \Re \to \Re$ are given functions, $\lambda \in (0, 1]$.

The study of quadratic integral equation has received much attention over the last thirty years or so. For instance, Cahlon and Eskin [1] prove the existence of positive solutions in the space $C[0, 1]$ and $C^\alpha[0, 1]$ of an integral equation of the Chandrasekhar H-equation with perturbation. Argyros [2] investigates a class of quadratic equations with a nonlinear perturbation. Banaś et al. [3] proves a few existence theorems for some quadratic integral equations. Banaś and Rzepka [4] study the Volterra quadratic integral equation on unbounded interval. Banaś and Sadarangani [5] study the solvability of Volterra-Stieltjes integral equation. In [6-8] the authors proved the existence of nondecreasing solutions of a quadratic integral equation. Dhage [9-10] proves an existence theorem for a certain differential inclusions in Banach algebras. Dhage [11] proves the existence results of some nonlinear functional integral equations. The purpose of this paper is to continue the study of those authors. Using the theory of measures of noncompactness and applying a new method, we prove the existence results of quadratic integral equations of Volterra type.

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The organization of this work is as follows. In section 2, we recall some definitions and theorems about the measure of noncompactness and fixed point theorem. In section 3, we give theorems on the existence of nondecreasing continuous solutions of a quadratic integral equation of Volterra type (1.1). Finally, in section 4, examples are given to show the applications of our results.

2. Preliminaries

Now, we are going to present definitions and basic facts needed further on.

Assume E is a real Banach space with norm $\|\cdot\|$. If X is a nonempty subset of E, we denote by $\bar{X}$ and $ConvX$ the closure and the closed convex of X. Let us denote by Γ_E the family of nonempty bounded subsets of E and by Υ_E its subfamily consisting of all relatively compact sets.

Definition 2.1. [12]. A function $\mu : \Gamma_E \to [0, \infty)$ is said to be a measure of noncompactness in the space E if it satisfies the following conditions:

(1) The family $ker(\mu) = \{X \in \Gamma_E, \mu(X) = 0\}$ is nonempty and $ker(\mu) \subset \Upsilon_E$.

(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(3) $\mu(\bar{X}) = \mu(ConvX) = \mu(X)$.

(4) $\mu(\theta X + (1-\theta)Y) \leq \theta\mu(X) + (1-\theta)\mu(Y)$, $\forall \theta \in [0, 1]$.

(5) If $\{X_n\}$ is a sequence of closed sets from Γ_E such that $X_{n+1} \subset X_n$, for $n = 1, 2, \cdots$, and $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

Remark 2.2. The family $ker\mu$ described above is called the kernel of the measure of noncompactness μ. Further facts concerning measure of noncompactness and their properties may be found in [12-13].

Let us suppose that M is a nonempty subset of a Banach space E and the operator $T : M \to E$ is continuous and transforms bounded sets onto bounded ones. We say that T satisfies the Darbo condition (with constant $k \geq 0$) with respect to a measure of noncompactness μ if for any bounded subset X of M we have

$$\mu(TX) \leq k\mu(X).$$

If T satisfies the Darbo condition with $k < 1$, then it is called a contraction with respect to μ.

Theorem 2.3. [14]. *Let Q be a nonempty, bounded, closed and convex subset of the Banach space E and μ a measure of noncompactness in E. Let $T : Q \to Q$ be a contraction with respect to μ. Then T has a fixed point in the set Q.*

Remark 2.4. Under the assumptions of the above theorem, it can be shown that the set $FixT$ of fixed points of T belonging to Q is a member of $ker\mu$.

For our purpose, let us recall the definition of the measure of noncompactness in the space $C(I)$ which will be used in section 3. This measure was introduced in the paper [15].

Let $C(I)$ denote the space of all real functions defined and continuous on the interval $I = [0, 1]$. The space $C(I)$ is furnished with standard norm $\|x\| = max\{|x(t)| : t \in I\}$.

Fix a nonempty and bounded subset X of $C(I)$. For $\varepsilon > 0$, and $x \in X$ denote by $w(x, \varepsilon)$ the modulus of continuity of x defined by

$$w(x, \varepsilon) = sup\{|x(t) - x(s)| : t, s \in I, |t - s| \leq \varepsilon\}.$$

Furthermore, put

$$w(X, \varepsilon) = sup\{w(x, \varepsilon), x \in X\},$$

$$w_0(X) = \lim_{\varepsilon \to 0} w(X, \varepsilon).$$

Next, let us define the following quantities:

$$d(x) = sup\{|x(t) - x(s)| - [x(t) - x(s)] : t, s \in I, s \leq t\},$$

$$d(X) = sup\{d(x) : x \in X\}.$$

Observe that $d(X) = 0$ if and only if all functions belonging to X are nondecreasing on I.

Finally, let

$$\mu(X) = w_0(X) + d(X).$$

It can be showed [15] that the function μ is a measure of noncompactness in the space $C(I)$. Moreover, the kernel $ker\mu$ consists all nonempty and bounded subsets X of $C(I)$ such that functions from X are equicontinuous and nondecreasing on the interval I.

3. Main Results

In this section, by using the measure of noncompactness defined in section 2, we give the existence results of the quadratic integral equation (1.1). Here we list the hypotheses which will be required further on.

(1) $h : I \to \Re^+$ is a continuous and nondecreasing function.

(2) $g : I \times \Re \to \Re$ is a continuous function , there exists a constant $k \geq 0$ such that

$$|g(t, x) - g(t, y)| \leq k|x - y|,$$

for all $t \in I$ and $x, y \in \Re^+$. Moreover, $g : I \times \Re^+ \to \Re^+$.

(3) For arbitrarily $x \in \Re$, $t \to g(t,x)$ is nondecreasing on I, and for arbitrarily $t \in I$, $x \to g(t,x)$ is nondecreasing on $\Re$.

(4) $k : I \times I \to \Re^+$. For each $t \in I$, $k(t,s)$ is measurable on $[0,t]$ and $\bar{k}(t) = esssup|k(t,s)|, 0 \le s \le t$, is bounded on $[0,1]$, let $K = \sup\limits_{0 \le t \le 1} |\bar{k}(t)|$. The map $t \to k_t$ is continuous from $[0,1]$ to $L^\infty[0,1]$, here $k_t(s) = k(t,s)$. Moreover, for arbitrarily $s \in I$, $t \to k(t,s)$ is nondecreasing on I.

(5) $f : I \times \Re \to \Re$ satisfies the Carathéodory type conditions, i.e. $t \to f(t,x)$ is measurable for every $x \in \Re$, $x \to f(t,x)$ is continuous for a.e. $t \in I$. Moreover, $f(t,x) \ge 0$, if $x \ge 0, t \in I$.

(6) There exist a function $L \in L^1(0,1;\Re^+)$ and a nondecreasing continuous function $\Omega : \Re^+ \to \Re^+$such that
$$|f(t,x)| \le L(t)\Omega(|x|),$$
for all $x \in \Re$ and a.e. $t \in I$.

Lemma 3.1. *Under assumptions (2) and (3), we have*
$$d(Gx) \le kd(x)$$
for any function $x \in C(I)$, where $(Gx)(t) = g(t,x(t))$ and k is the same constant as in assumption (2).

Proof. Let us take an arbitrary function $x \in C(I)$ and choose arbitrarily $t_1, t_2 \in I (t_1 < t_2)$.

If $x(t_2) \ge x(t_1)$, we have
$$\begin{aligned}
&|(Gx)(t_2) - (Gx)(t_1)| - [(Gx)(t_2) - (Gx)(t_1)] \\
&= |g(t_2,x(t_2)) - g(t_1,x(t_1))| - [g(t_2,x(t_2)) - g(t_1,x(t_1))] \\
&= 0 \\
&\le k(|x(t_2) - x(t_1)| - [x(t_2) - x(t_1)]),
\end{aligned}$$
and if $x(t_2) < x(t_1)$, we have
$$\begin{aligned}
&|(Gx)(t_2) - (Gx)(t_1)| - [(Gx)(t_2) - (Gx)(t_1)] \\
&= |g(t_2,x(t_2)) - g(t_1,x(t_1))| - [g(t_2,x(t_2)) - g(t_1,x(t_1))] \\
&\le |g(t_2,x(t_2)) - g(t_2,x(t_1))| + |g(t_2,x(t_1)) - g(t_1,x(t_1))| \\
&\quad - \{[g(t_2,x(t_2)) - g(t_2,x(t_1))] + [g(t_2,x(t_1)) - g(t_1,x(t_1))]\} \\
&= |g(t_2,x(t_2)) - g(t_2,x(t_1))| - [g(t_2,x(t_2)) - g(t_2,x(t_1))] \\
&= 2|g(t_2,x(t_1)) - g(t_2,x(t_2))| \\
&\le 2k|x(t_1) - x(t_2)| \\
&= k(|x(t_2) - x(t_1)| - [x(t_2) - x(t_1)]).
\end{aligned}$$

Therefore, we obtain

$$d(Gx) \leq kd(x).$$

Thus the proof is complete. ■

Now we give the existence results under above hypotheses.

Theorem 3.2. *Under assumptions* (*1*)-(*6*), *equation* (*1.1*) *has at least one nondecreasing solution* $x \in C(I)$ *provided that there exists a constant* R *such that*

$$\int_0^1 L(s)ds < \frac{1}{K(kR+b)} \int_a^R \frac{1}{\Omega(s)}ds, \tag{3.1}$$

where $a = max\{|h(t)| : t \in I\}$, $b = max\{|g(t, 0)| : t \in I\}$.

Proof. Let us consider the operator T defined on the space $C(I)$ by the formula,

$$(Tx)(t) = h(t) + g(t, x(t)) \int_0^t k(t, s)f(s, x(\lambda s))ds.$$

Taking into account assumptions (1)-(6), we infer that the function Tx is continuous on I for any $x \in C(I)$, i.e., the operator T transforms the space $C(I)$ into itself.

In view of assumption (3.1), we infer that there exists a constant $\epsilon > 0$ such that

$$\int_0^1 L(s)ds = A \int_{a+\epsilon}^R \frac{1}{\Omega(s)}ds,$$

where $A = \frac{1}{K(kR+b)}$.

Then there exists a positive integer n such that

$$A \int_{a+\epsilon}^{a+n\epsilon} \frac{1}{\Omega(s)}ds < \int_0^1 L(s)ds \leq A \int_{a+\epsilon}^{a+(n+1)\epsilon} \frac{1}{\Omega(s)}ds.$$

Therefore, there exists a sequence $\{t_0, t_1, t_2, ..., t_n\}$ such that

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1,$$

for which we have

$$\begin{aligned}
\textstyle\int_0^{t_1} L(s)ds &= A \int_{a+\epsilon}^{a+2\epsilon} \frac{1}{\Omega(s)}ds, \\
\textstyle\int_{t_1}^{t_2} L(s)ds &= A \int_{a+2\epsilon}^{a+3\epsilon} \frac{1}{\Omega(s)}ds, \\
\cdots &= \cdots, \\
\textstyle\int_{t_{n-2}}^{t_{n-1}} L(s)ds &= A \int_{a+(n-1)\epsilon}^{a+n)\epsilon} \frac{1}{\Omega(s)}ds, \\
\textstyle\int_{t_{n-1}}^{1} L(s)ds &\leq A \int_{a+n\epsilon}^{a+(n+1)\epsilon} \frac{1}{\Omega(s)}ds.
\end{aligned}$$

If we denote by $W = \{x \in C(I) : x(t) \geq 0 \text{ for } t \in I\}$ and $\|x_i\| = sup\{|x(t)| : t \in [t_{i-1}, t_i]\}$, then $W \subseteq C(I)$. Since $\|x_i\| \leq a + i\epsilon$ for $i = 1, 2, ..., n$, we infer that the set W is a bounded, closed, convex and nonempty subset of the space $C(I)$.

For any $x \in W$, we have

$$\begin{aligned}
|Tx(t)| &= |h(t) + g(t, x(t)) \int_0^t k(t,s) f(s, x(\lambda s)) ds| \\
&\leq a + (|g(t, x(t)) - g(t, 0)| + |g(t, 0)|)| \int_0^t k(t,s) f(s, x(\lambda s)) ds| \\
&\leq a + K(k|x(t)| + b) \int_0^t L(s) \Omega(|x(\lambda s)|) ds \\
&\leq a + K(k(a + n\epsilon) + b) \int_0^t L(s) \Omega(|x(\lambda s)|) ds \\
&\leq a + K(kR + b) \int_0^t L(s) \Omega(|x(\lambda s)|) ds,
\end{aligned}$$

and

$$\begin{aligned}
\|Tx\|_i &= sup\{|(Tx)(t)| : t \in [t_{i-1}, t_i]\} \\
&\leq sup\{a + K(kR + b) \int_0^t L(s) \Omega(|x(\lambda s)|) ds : t \in [t_{i-1}, t_i]\} \\
&\leq a + K(kR + b) \int_0^{t_i} L(s) \Omega(|x(\lambda s)|) ds \\
&\leq a + K(kR + b) [\int_0^{t_1} L(s) \Omega(|x(\lambda s)|) ds + \int_{t_1}^{t_2} L(s) \Omega(|x(\lambda s)|) ds \\
&\quad + \cdots + \int_{t_{i-1}}^{t_i} L(s) \Omega(|x(\lambda s)|) ds] \\
&\leq a + K(kR + b) [\int_0^{t_1} L(s) ds \Omega(a + \epsilon) + \int_{t_1}^{t_2} L(s) ds \Omega(a + 2\epsilon) \\
&\quad + \cdots + \int_{t_{i-1}}^{t_i} L(s) ds \Omega(a + i\epsilon)] \\
&\leq a + K(kR + b) A [\int_{a+\epsilon}^{a+2\epsilon} \frac{1}{\Omega(s)} ds \Omega(a + \epsilon) + \int_{a+2\epsilon}^{a+3\epsilon} \frac{1}{\Omega(s)} ds \Omega(a + 2\epsilon) \\
&\quad + \cdots + \int_{a+i\epsilon}^{a+(i+1)\epsilon} \frac{1}{\Omega(s)} ds \Omega(a + i\epsilon)] \\
&\leq a + K(kR + b) A i \epsilon \\
&\leq a + i\epsilon,
\end{aligned}$$

which implies that $T : W \to W$ is a bounded operator.

Next, let us take a nonempty subset $X \subseteq W$. Fix $x \in X$, then for any arbitrarily choose $t_1, t_2 \in I (t_1 < t_2)$, we have

$$
\begin{aligned}
&|(Tx)(t_2)-(Tx)(t_1)| \leq |h(t_2)-h(t_1)| \\
&\quad +|g(t_2,x(t_2))\int_0^{t_2} k(t_2,s)f(s,x(\lambda s))ds - g(t_1,x(t_1))\int_0^{t_1} k(t_1,s)f(s,x(\lambda s))ds| \\
&\leq |h(t_2)-h(t_1)| + |g(t_2,x(t_2))-g(t_1,x(t_2))||\int_0^{t_2} k(t_2,s)f(s,x(\lambda s))ds| \\
&\quad +|g(t_1,x(t_2))-g(t_1,x(t_1))||\int_0^{t_2} k(t_2,s)f(s,x(\lambda s))ds| \\
&\quad +|g(t_1,x(t_1))||\int_0^{t_2} k(t_2,s)f(s,x(\lambda s))ds - \int_0^{t_1} k(t_2,s)f(s,x(\lambda s))ds| \\
&\quad +|g(t_1,x(t_1))||\int_0^{t_1} k(t_2,s)f(s,x(\lambda s))ds - \int_0^{t_1} k(t_1,s)f(s,x(\lambda s))ds| \\
&\leq |h(t_2)-h(t_1)| + |g(t_2,x(t_2))-g(t_1,x(t_2))|\int_0^{t_2} |k(t_2,s)f(s,x(\lambda s))|ds \\
&\quad +k|x(t_2)-x(t_1)|\int_0^{t_2} |k(t_2,s)f(s,x(\lambda s))|ds \\
&\quad +|g(t_1,x(t_1))|\int_{t_1}^{t_2} |k(t_2,s)f(s,x(\lambda s))|ds \\
&\quad +|g(t_1,x(t_1))||\int_0^{t_1} k(t_2,s)-k(t_1,s)f(s,x(\lambda s))ds| \\
&\leq |h(t_2)-h(t_1)| + |g(t_2,x(t_2))-g(t_1,x(t_2))|K\int_0^1 L(s)ds\Omega(R) \\
&\quad +k|x(t_2)-x(t_1)|K\int_0^1 L(s)\Omega(|x(\lambda s)|)ds \\
&\quad +(kR+b)K\int_{t_1}^{t_2} L(s)ds\Omega(R) \\
&\quad +(kR+b)|k(t_2,\cdot)-k(t_1,\cdot)|_{L^\infty}\int_0^1 L(s)ds\Omega(R),
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 L(s)\Omega(|x(\lambda s)|)ds &= \int_0^{t_1} L(s)\Omega(|x(\lambda s)|)ds + \int_{t_1}^{t_2} L(s)\Omega(|x(\lambda s)|)ds \\
&\quad +.... + \int_{t_{n-1}}^1 L(s)\Omega(|x(\lambda s)|)ds \\
&\leq \int_0^{t_1} L(s)ds\Omega(a+\epsilon) + \int_{t_1}^{t_2} L(s)ds\Omega(a+2\epsilon) \\
&\quad +.... + \int_{t_{n-1}}^1 L(s)ds\Omega(a+n\epsilon) \\
&\leq A\int_{a+\epsilon}^{a+2\epsilon} \frac{1}{\Omega(s)}ds\Omega(a+\epsilon) + A\int_{a+2\epsilon}^{a+3\epsilon} \frac{1}{\Omega(s)}ds\Omega(a+2\epsilon)
\end{aligned}
$$

$$
\begin{aligned}
&+ .... + A \int_{a+n\epsilon}^{a+(n+1)\epsilon} \frac{1}{\Omega(s)} ds \Omega(a+n\epsilon) \\
&\leq An\epsilon.
\end{aligned}
$$

In view of assumptions (1)-(6) and keeping in mind the fact that g is uniformly continuous on the set $I \times [-R, R]$, we have

$$w_0(TX) \leq kKAn\epsilon w_0(X).$$

Further, taking into account the assumptions of our theorem and lemma 3.1, we obtain

$$
\begin{aligned}
&|(Tx)(t_2) - (Tx)(t_1)| - [(Tx)(t_2) - (Tx)(t_1)] \leq |h(t_2) - h(t_1)| - [h(t_2) - h(t_1)] \\
&\quad + |g(t_2, x(t_2)) \int_0^{t_2} k(t_2, s) f(s, x(\lambda s)) ds - g(t_1, x(t_1)) \int_0^{t_1} k(t_1, s) f(s, x(\lambda s)) ds| \\
&\quad - [g(t_2, x(t_2)) \int_0^{t_2} k(t_2, s) f(s, x(\lambda s)) ds - g(t_1, x(t_1)) \int_0^{t_1} k(t_1, s) f(s, x(\lambda s)) ds] \\
&\leq |g(t_2, x(t_2)) - g(t_1, x(t_1))| | \int_0^{t_2} k(t_2, s) f(s, x(\lambda s)) ds| \\
&\quad + |g(t_1, x(t_1))| | \int_0^{t_2} k(t_2, s) f(s, x(\lambda s)) ds - \int_0^{t_1} k(t_2, s) f(s, x(\lambda s)) ds| \\
&\quad + |g(t_1, x(t_1))| | \int_0^{t_1} k(t_2, s) f(s, x(\lambda s)) ds - \int_0^{t_1} k(t_1, s) f(s, x(\lambda s)) ds| \\
&\quad - [g(t_2, x(t_2)) - g(t_1, x(t_1))] \int_0^{t_2} k(t_2, s) f(s, x(s)) ds \\
&\quad - g(t_1, x(t_1)) [\int_0^{t_2} k(t_2, s) f(s, x(\lambda s)) ds - \int_0^{t_1} k(t_2, s) f(s, x(\lambda s)) ds] \\
&\quad - g(t_1, x(t_1)) [\int_0^{t_1} k(t_2, s) f(s, x(\lambda s)) ds - \int_0^{t_1} k(t_1, s) f(s, x(\lambda s)) ds] \\
&\leq \{|g(t_2, x(t_2)) - g(t_1, x(t_1))| - [g(t_2, x(t_2)) - g(t_1, x(t_1))]\} \\
&\qquad \int_0^{t_2} k(t_2, s) f(s, x(\lambda s)) ds \\
&\leq d(Gx) K \int_0^1 L(s) \Omega(|x(\lambda s)|) ds \\
&\leq kK \int_0^1 L(s) \Omega(|x(\lambda s)|) ds d(x) \\
&\leq kKAn\epsilon d(x).
\end{aligned}
$$

This estimate implies

$$d(TX) \leq kKAn\epsilon d(X). \tag{3.2}$$

Consequently, we get

$$\begin{aligned}\mu(TX) &= w_0(TX) + d(TX)\\ &\leq kKAn\epsilon(w_0(X) + d(X))\\ &< (k\frac{R-a}{kR+b})(w_0(X) + d(X))\\ &\leq (1 + \frac{-b-ka}{kR+b})(w_0(X) + d(X))\\ &\leq (1 + \frac{-b-ka}{kR+b})\mu(X),\end{aligned}$$

which implies T is a contraction with respect to μ on W.

Thus, applying fixed point theorem, we infer that there exists a function $x \in W$ that is a solution of equation (1.1). Moreover, in view of Remark 2.4 and the description of the kernel of noncompactness μ, we deduce that all solutions of the equation (1.1) are nondecreasing on I. This completes the proof. ■

Theorem 3.3. *Under assumptions (1)-(6), equation (1.1) has at least one nondecreasing solution $x \in C(I)$ provided there exists a constant R satisfied*

$$a + K(kR + b)\Omega(R)\int_0^1 L(s))ds \leq R. \tag{3.3}$$

Proof. In view of (3.3), we have

$$\int_0^1 L(s))ds \leq \frac{R-a}{K(kR+b)\Omega(R)} < \frac{1}{K(kR+b)}\int_a^R \frac{1}{\Omega(s)}ds.$$

Then, applying Theorem 3.2 we obtain the desired assertion. ■

4. Examples

Example 4.1. Consider the following quadratic integral equation

$$x(t) = 1 + \frac{1}{3}arctanx(t)\int_0^t x(\lambda s)ds, \qquad t \in I. \tag{4.1}$$

Obviously this equation is a particular case of equation (1.1), where $h(t) = 1$, $g(t, x(t)) = \frac{1}{3}arctanx(t)$, $k(t,s) = 1$, $f(s, x(t)) = x(t)$.

We know there exists a constant $R = e$ such that

$$1 < \frac{1}{\frac{1}{3}e}\int_1^e \frac{1}{s}ds = \frac{3}{e}.$$

So, by Theorem 3.2, we conclude that equation (4.1) has at least one nondecreasing solution.

Remark 4.2. For the above equation, we can not obtain a constat R such that

$$1+\frac{1}{3}R^2 \leq R.$$

By using Theorem 3.3, we do not know whether or not the equation (4.1) has a solution. Thus, Theorem 3.2 is more general than the Theorem 3.3.

Example 4.3. Consider the following differential equation

$$\begin{cases} (\frac{x(t)}{g(t,x(t))})' = f(t,x(t)), \qquad a.e. t \in I, \\ x(0)=0, \end{cases} \tag{4.2}$$

where g satisfies assumptions (2), (3) and $g(t,x) \neq 0$ for all $t \in I$ and $x \in \Re$, f satisfies assumptions (5) and (6).

Then equation (4.2) can be regarded as the following quadratic integral equation

$$x(t) = g(t,x(t)) \int_0^t f(s,x(s))ds, \qquad t \in I=[0,1]. \tag{4.3}$$

If there exists a constant R such that

$$\int_0^1 L(s)ds < \frac{1}{kR+b}\int_0^R \frac{1}{\Omega(s)}ds.$$

Then applying Theorem 3.2, we can prove that the equation (4.2) has at least one nondecreasing solution in $C(I)$.

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