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FIXED POINT THEOREMS FOR NEW GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES AND APPLICATIONS

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Abstract. In this paper, we introduce a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings, nonspreading mappings, hybrid mappings and contractive mappings. Then we prove fixed point theorems for the class of such mappings. Using these results, we prove well-known and new fixed point theorems in a Hilbert space.

1. INTRODUCTION

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is said to be *nonexpansive* [18], *nonspreading* [13], and *hybrid* [19] if

$$\|Tx - Ty\| \le \|x - y\|,$$

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2$$

and

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$, respectively; see also [7], [8] and [20]. These mappings are independent and they are deduced from a firmly nonexpansive mapping in a Hilbert space; see [19]. A mapping $F : C \to H$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \le \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$; see, for instance, Browder [2], Goebel and Kirk [5], and Kohsaka and Takahashi [12]. Recently Kawasaki and Takahashi [10] introduced the following

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nonlinear mapping in a Hilbert space. A mapping T from C into H is said to be *widely generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0$$

for any $x, y \in C$; see also [11].

In this paper, motivated by these mappings, we introduce a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings, nonspreading mappings, hybrid mappings, contractive mappings, contractively nonspreading mappings and contractively hybrid mappings. Then we prove fixed point theorems for the class of such mappings. Using these results, we prove well-known and new fixed point theorems in a Hilbert space.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [18]. Furthermore, in a Hilbert space, we have that

$$2\langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2.2}$$

for all $x, y, z, w \in H$. Let C be a nonempty subset of H and let T be a mapping from C into H. We denote by $F(T)$ the set of fixed points of T.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [17] for the proof of existence of a Banach limit and its other elementary properties. Using means and the Riesz theorem, we can obtain the following result; see [14, 15, 16] and [17].

Lemma 2.1. *Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n \mid n \in \mathbb{N}\}$ such that*

$$\mu_n\langle x_n, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H.$$

3. Fixed Point Theorems

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping T from C into H is called *symmetric generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Tx - Ty\|^2 \ &+\beta(\|x - Ty\|^2 + \|Tx - y\|^2) + \gamma\|x - y\|^2 \\ &+\delta(\|x - Tx\|^2 + \|y - Ty\|^2) \leq 0 \end{aligned} \tag{3.1}$$

for all $x, y \in C$. Such a mapping T is also called *($\alpha, \beta, \gamma, \delta$)-symmetric generalized hybrid*. If $\alpha = 1$, $\beta = \delta = 0$ and $\gamma = -1$ in (3.1), then the mapping T is nonexpansive. If $\alpha = 2$, $\beta = -1$ and $\gamma = \delta = 0$ in (3.1), then the mapping T is nonspreading. Furthermore, if $\alpha = 3$, $\beta = \gamma = -1$ and $\delta = 0$ in (3.1), then the mapping T is hybrid. Recently Kawasaki and Takahashi [10] introduced the following nonlinear mapping in a Hilbert space and they proved a fixed point theorem and a mean convergence theorem for the mappings. Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping T from C into H is said to be *widely generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Tx - Ty\|^2 \ &+\beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &+ \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0 \end{aligned} \tag{3.2}$$

for any $x, y \in C$. Such a mapping T is called *($\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$)-widely generalized hybrid*. Replacing the variables x and y in (3.2), we have that

$$\begin{aligned} \alpha\|Ty - Tx\|^2 \ &+\beta\|y - Tx\|^2 + \gamma\|Ty - x\|^2 + \delta\|y - x\|^2 \\ &+ \max\{\varepsilon\|y - Ty\|^2, \zeta\|x - Tx\|^2\} \leq 0. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we have that

$$\begin{aligned} &2\alpha\|Ty - Tx\|^2 + (\beta + \gamma)(\|y - Tx\|^2 + \|Ty - x\|^2) + 2\delta\|y - x\|^2 \\ &+ \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} + \max\{\varepsilon\|y - Ty\|^2, \zeta\|x - Tx\|^2\} \leq 0. \end{aligned} \tag{3.4}$$

From $\varepsilon\|x - Tx\|^2,\ \zeta\|y - Ty\|^2 \leq \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\}$, we have that

$$\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 \leq 2\max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\}. \tag{3.5}$$

Similarly, we have that

$$\varepsilon\|y-Ty\|^2+\zeta\|x-Tx\|^2\leq 2\max\{\varepsilon\|y-Ty\|^2,\zeta\|x-Tx\|^2\}. \tag{3.6}$$

Consequently, we have from (3.4), (3.5) and (3.6) that

$$\begin{aligned} 2\alpha\|Ty-Tx\|^2 &+(\beta+\gamma)(\|y-Tx\|^2+\|Ty-x\|^2)+2\delta\|y-x\|^2 \\ &+\frac{1}{2}(\varepsilon+\zeta)(\|x-Tx\|^2+\|y-Ty\|^2)\leq 0. \end{aligned} \tag{3.7}$$

Such a mapping T is symmetric generalized hybrid. We first prove a fixed point theorem for symmetric generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha+2\beta+\gamma\geq 0$, (2) $\alpha+\beta+\delta>0$ *and* (3) $\delta\geq 0$ *hold. Then T has a fixed point if and only if there exists $z\in C$ such that $\{T^nz : n=0,1,\dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha+2\beta+\gamma>0$ on the condition* (1).

Proof. Suppose that T has a fixed point z. Then $\{T^nz : n=0,1,\dots\}=\{z\}$ and hence $\{T^nz : n=0,1,\dots\}$ is bounded. Conversely, suppose that there exists $z\in C$ such that $\{T^nz : n=0,1,\dots\}$ is bounded. Since T is an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid mapping of C into itself, we have that

$$\begin{aligned} \alpha\|Tx-T^{n+1}z\|^2+\beta(\|x-T^{n+1}z\|^2+\|Tx-T^nz\|^2)+\gamma\|x-T^nz\|^2 \\ +\delta(\|x-Tx\|^2+\|T^nz-T^{n+1}z\|^2)\leq 0 \end{aligned}$$

for all $n\in\mathbb{N}\cup\{0\}$ and $x\in C$. Since $\{T^nz\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Since $\mu_n\|Tx-T^nz\|^2=\mu_n\|Tx-T^{n+1}z\|^2$ and $\mu_n\|x-T^{n+1}z\|^2=\mu_n\|x-T^nz\|^2$, we have that

$$\begin{aligned} (\alpha+\beta)\mu_n\|Tx-T^nz\|^2+(\beta+\gamma)\mu_n\|x-T^nz\|^2 \\ +\delta(\|x-Tx\|^2+\mu_n\|T^nz-T^{n+1}z\|^2)\leq 0. \end{aligned}$$

Furthermore, since

$$\mu_n\|Tx-T^nz\|^2=\|Tx-x\|^2+2\mu_n\langle Tx-x,x-T^nz\rangle+\mu_n\|x-T^nz\|^2,$$

we have that

$$\begin{aligned} (\alpha+\beta+\delta)\|Tx-x\|^2+2(\alpha+\beta)\mu_n\langle Tx-x,x-T^nz\rangle \\ +(\alpha+2\beta+\gamma)\mu_n\|x-T^nz\|^2+\delta\mu_n\|T^nz-T^{n+1}z\|^2\leq 0. \end{aligned}$$

From (1) $\alpha+2\beta+\gamma \geq 0$ and (3) $\delta \geq 0$, we have that

$$(\alpha+\beta+\delta)\|Tx-x\|^2+2(\alpha+\beta)\mu_n\langle Tx-x, x-T^nz\rangle \leq 0. \tag{3.8}$$

Since there exists $p \in H$ from Lemma 2.1 such that

$$\mu_n\langle y, T^nz\rangle = \langle y, p\rangle$$

for all $y \in H$, we have from (3.8) that

$$(\alpha+\beta+\delta)\|Tx-x\|^2+2(\alpha+\beta)\langle Tx-x, x-p\rangle \leq 0. \tag{3.9}$$

Since C is closed and convex, we have that

$$p \in \overline{co}\{T^nx : n \in \mathbb{N}\} \subset C.$$

Putting $x=p$, we obtain from (3.9) that

$$(\alpha+\beta+\delta)\|Tp-p\|^2 \leq 0. \tag{3.10}$$

We have from (2) $\alpha+\beta+\delta > 0$ that $\|Tp-p\|^2 \leq 0$. This implies that p is a fixed point in T.

Next suppose that $\alpha+2\beta+\gamma > 0$. Let p_1 and p_2 be fixed points of T. Then we have that

$$\alpha\|Tp_1-Tp_2\|^2+\beta(\|p_1-Tp_2\|^2+\|Tp_1-p_2\|^2)+\gamma\|p_1-p_2\|^2 \\ +\delta(\|p_1-Tp_1\|^2+\|p_2-Tp_2\|^2) \leq 0$$

and hence $(\alpha+2\beta+\gamma)\|p_1-p_2\|^2 \leq 0$. We have from $\alpha+2\beta+\gamma > 0$ that $p_1=p_2$. Therefore a fixed point of T is unique. This completes the proof. ■

As a direct consequence of Theorem 3.1, we obtain the following theorem.

Theorem 3.2. *Let H be a Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha+2\beta+\gamma \geq 0$, (2) $\alpha+\beta+\delta > 0$ *and* (3) $\delta \geq 0$ *hold. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+2\beta+\gamma > 0$ on the condition* (1).

Using Theorem 3.1, we prove the following fixed point theorem. Before proving it, we introduce a more broad class of nonlinear mappings which contains the class of symmetric generalized hybrid mappings. A mapping T from C into H is called *symmetric more generalized hybrid* if there exist $\alpha,\beta,\gamma,\delta,\zeta \in \mathbb{R}$ such that

$$\alpha\|Tx-Ty\|^2+\beta(\|x-Ty\|^2+\|Tx-y\|^2)+\gamma\|x-y\|^2 \\ +\delta(\|x-Tx\|^2+\|y-Ty\|^2)+\zeta\|x-y-(Tx-Ty)\|^2 \leq 0 \tag{3.11}$$

for all $x, y \in C$. Such a mapping T is called *$(\alpha,\beta,\gamma,\delta,\zeta)$-symmetric more generalized hybrid*.

Theorem 3.3. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be an ($\alpha, \beta, \gamma, \delta, \zeta$)-symmetric more generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ *and* (3) $\delta + \zeta \geq 0$ *hold. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition* (1).

Proof. Since $T : C \to C$ is an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping, there exist $\alpha, \beta, \gamma, \delta, \zeta \in \mathbb{R}$ satisfying (3.11). We also have that

$$\begin{aligned} \|x - y - (Tx - Ty)\|^2 &= \|x - Tx\|^2 + \|y - Ty\|^2 \\ &\quad -\|x - Ty\|^2 - \|y - Tx\|^2 + \|x - y\|^2 + \|Tx - Ty\|^2 \end{aligned} \tag{3.12}$$

for all $x, y \in C$. Thus we obtain from (3.11) that

$$\begin{aligned} &(\alpha + \zeta)\|Tx - Ty\|^2 + (\beta - \zeta)(\|x - Ty\|^2 + \|Tx - y\|^2) \\ &\quad + (\gamma + \zeta)\|x - y\|^2 + (\delta + \zeta)(\|x - Tx\|^2 + \|y - Ty\|^2) \leq 0. \end{aligned} \tag{3.13}$$

The conditions (1) $\alpha + 2\beta + \gamma \geq 0$ and (2) $\alpha + \beta + \delta + \zeta > 0$ are equivalent to $(\alpha+\zeta)+2(\beta-\zeta)+(\gamma+\zeta) \geq 0$ and $(\alpha+\zeta)+(\beta-\zeta)+(\delta+\zeta) > 0$, respectively. Furthermore, since (3) $\delta + \zeta \geq 0$ holds, we have the desired result from Theorem 3.1. ■

As a direct consequence of Theorem 3.3, we obtain the following.

Theorem 3.4. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be an ($\alpha, \beta, \gamma, \delta, \zeta$)-symmetric more generalized hybrid mapping from C into itself such that the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ *and* (3) $\delta + \zeta \geq 0$ *hold. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition* (1).

The following theorem is an extension of Theorem 3.4.

Theorem 3.5. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be an ($\alpha, \beta, \gamma, \delta, \zeta$)-symmetric more generalized hybrid mapping from C into itself which satisfies the conditions* (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ *and* (3) *there exists $\lambda \in [0, 1)$ such that $(\alpha + \beta)\lambda + \delta + \zeta \geq 0$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition* (1).

Proof. Since T is an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself, we obtain that

$$\begin{aligned} &\alpha\|Tx - T^{n+1}z\|^2 + \beta(\|x - T^{n+1}z\|^2 + \|Tx - T^n z\|^2) + \gamma\|x - T^n z\|^2 \\ &+\delta(\|x - Tx\|^2 + \|T^n z - T^{n+1}z\|^2) + \zeta\|(x - Tx) - (T^n z - T^{n+1}z)\|^2 \leq 0 \end{aligned}$$

for any $n \in \mathbb{N} \cup \{0\}$ and for any $x \in C$.

Let $\lambda \in [0,1) \cap \{\lambda : (\alpha+\beta)\lambda+\zeta+\eta \geq 0\}$ and define $S = (1-\lambda)T + \lambda I$. Since C is convex, S is a mapping from C into itself. Since C is bounded, $\{S^n z : n = 0, 1, \ldots\}$ is bounded for any $z \in C$. Since $\lambda \neq 1$, we obtain that $F(S) = F(T)$. Moreover, from $T = \frac{1}{1-\lambda}S - \frac{\lambda}{1-\lambda}I$ and (2.1), we have that

$$
\begin{aligned}
&\alpha \left\| \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x \right) - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y \right) \right\|^2 \\
&\quad + \beta \left\| x - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y \right) \right\|^2 + \beta \left\| \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x \right) - y \right\|^2 \\
&\quad + \gamma \|x-y\|^2 \\
&\quad + \delta \left\| x - \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x \right) \right\|^2 + \delta \left\| y - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y \right) \right\|^2 \\
&\quad + \zeta \left\| \left(x - \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x \right) \right) - \left(y - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y \right) \right) \right\|^2 \\
&= \alpha \left\| \frac{1}{1-\lambda}(Sx - Sy) - \frac{\lambda}{1-\lambda}(x-y) \right\|^2 \\
&\quad + \beta \left\| \frac{1}{1-\lambda}(x - Sy) - \frac{\lambda}{1-\lambda}(x-y) \right\|^2 \\
&+ \beta \left\| \frac{1}{1-\lambda}(Sx - y) - \frac{\lambda}{1-\lambda}(x-y) \right\|^2 + \gamma \|x-y\|^2 \\
&\quad + \delta \left\| \frac{1}{1-\lambda}(x - Sx) \right\|^2 + \delta \left\| \frac{1}{1-\lambda}(y - Sy) \right\|^2 \\
&\quad + \zeta \left\| \frac{1}{1-\lambda}(x - Sx) - \frac{1}{1-\lambda}(y - Sy) \right\|^2 \\
&= \frac{\alpha}{1-\lambda}\|Sx - Sy\|^2 + \frac{\beta}{1-\lambda}\|x - Sy\|^2 \\
&\quad + \frac{\beta}{1-\lambda}\|Sx - y\|^2 + \left(-\frac{\lambda}{1-\lambda}(\alpha + 2\beta) + \gamma \right) \|x-y\|^2 \\
&\quad + \frac{\delta + \beta\lambda}{(1-\lambda)^2}\|x - Sx\|^2 + \frac{\delta + \beta\lambda}{(1-\lambda)^2}\|y - Sy\|^2 \\
&\quad + \frac{\zeta + \alpha\lambda}{(1-\lambda)^2}\|(x - Sx) - (y - Sy)\|^2 \leq 0.
\end{aligned}
$$

Therefore S is an $\left(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, -\frac{\lambda}{1-\lambda}(\alpha + 2\beta) + \gamma, \frac{\delta+\beta\lambda}{(1-\lambda)^2}, \frac{\zeta+\alpha\lambda}{(1-\lambda)^2} \right)$-symmetric more generalized hybrid mapping. Furthermore, we obtain that

$$\frac{\alpha}{1-\lambda}+\frac{2\beta}{1-\lambda}+\frac{\gamma}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+2\beta)+\gamma=\alpha+2\beta+\gamma\geq 0,$$
$$\frac{\alpha}{1-\lambda}+\frac{\beta}{1-\lambda}+\frac{\delta+\beta\lambda}{(1-\lambda)^2}+\frac{\zeta+\alpha\lambda}{(1-\lambda)^2}=\frac{\alpha+\beta+\delta+\zeta}{(1-\lambda)^2}>0,$$
$$\frac{\delta+\beta\lambda}{(1-\lambda)^2}+\frac{\zeta+\alpha\lambda}{(1-\lambda)^2}=\frac{(\alpha+\beta)\lambda+\delta+\zeta}{(1-\lambda)^2}\geq 0.$$

Therefore by Theorem 3.4 we obtain $F(S)\neq\emptyset$.

Next suppose that $\alpha+2\beta+\gamma>0$. Let p_1 and p_2 be fixed points of T. Then

$$\begin{aligned}&\alpha\|Tp_1-Tp_2\|^2+\beta(\|p_1-Tp_2\|^2+\|Tp_1-p_2\|^2)+\gamma\|p_1-p_2\|^2\\&+\delta(\|p_1-Tp_1\|^2+\|p_2-Tp_2\|^2)+\zeta\|(p_1-Tp_1)-(p_2-Tp_2)\|^2\\=&(\alpha+2\beta+\gamma)\|p_1-p_2\|^2\leq 0\end{aligned}$$

and hence $p_1=p_2$. Therefore a fixed point of T is unique. ∎

For the case $\beta+\delta=0$ in Theorem 3.5, we have the following theorem.

Theorem 3.6. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be an $(\alpha,-\beta,\gamma,\beta,\zeta)$-symmetric more generalized hybrid mapping from C into itself, i.e., there exist $\alpha,\beta,\gamma,\zeta\in\mathbb{R}$ such that*

$$\begin{aligned}&\alpha\|Tx-Ty\|^2+\beta(\|x-Ty\|^2+\|Tx-y\|^2)+\gamma\|x-y\|^2\\&-\beta(\|x-Tx\|^2+\|y-Ty\|^2)+\zeta\|x-y-(Tx-Ty)\|^2\leq 0\end{aligned}\tag{3.14}$$

for all $x,y\in C$. Furthermore, suppose that T satisfies the following conditions: (1) $\alpha+2\beta+\gamma\geq 0$, (2) $\alpha+\zeta>0$ *and* (3) *there exists* $\lambda\in[0,1)$ *such that* $(\alpha+\beta)\lambda-\beta+\zeta\geq 0$. *Then T has a fixed point. In particular, a fixed point of T is unique in the case of* $\alpha+2\beta+\gamma>0$ *on the condition* (1).

4. Applications

In this section, we prove well-known and new fixed point theorems in a Hilbert space by using fixed point theorems obtained in Section 3.

Let H be a Hilbert space and let C be a nonempty subset of H. Then $T:C\to H$ is called a *widely strict pseudo-contraction* if there exists $r\in\mathbb{R}$ with $r<1$ such that

$$\|Tx-Ty\|^2\leq\|x-y\|^2+r\|(I-T)x-(I-T)y\|^2,\quad\forall x,y\in C.$$

We call such T a *widely r-strict pseudo-contraction*. If $0\leq r<1$, then T is a *strict pseudo-contraction*; see [4]. Furthermore, if $r=0$, then T is nonexpansive. Conversely, let $S:C\to H$ be a nonexpansive mapping and define $T:C\to H$ by

$T = \frac{1}{1+n}S + \frac{n}{1+n}I$ for all $x \in C$ and $n \in \mathbb{N}$. Then T is a widely $(-n)$-strict pseudo-contraction. In fact, from the definition of T, it follows that $S = (1+n)T - nI$. Since S is nonexpansive, we have that for any $x, y \in C$,

$$\|(1+n)Tx - nx - ((1+n)Ty - ny)\|^2 \leq \|x - y\|^2$$

and hence

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - n\|(I - T)x - (I - T)y\|^2.$$

Using Theorem 3.3, we first prove the following fixed point theorem.

Theorem 4.1. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be a widely strict pseudo-contraction from C into itself, i.e., there exists $r \in \mathbb{R}$ with $r < 1$ such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + r\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{4.1}$$

Then T has a fixed point in C.

Proof. We first assume that $r \leq 0$. We have from (4.8) that for any $x, y \in C$,

$$\|Tx - Ty\|^2 - \|x - y\|^2 - r\|(I - T)x - (I - T)y\|^2 \leq 0. \tag{4.2}$$

Then T is a $(1, 0, -1, 0, -r)$-symmetric more generalized hybrid mapping. Furthermore, (1) $\alpha + 2\beta + \gamma = 1 - 1 \geq 0$, (2) $\alpha + \beta + \delta + \zeta = 1 - r > 0$ and (3) $\delta + \zeta = -r \geq 0$ in Theorem 3.3 are satisfied. Thus T has a fixed point from Theorem 3.3. Assume that $0 \leq r < 1$ and define a mapping T as follows:

$$Sx = \lambda x + (1 - \lambda)Tx, \quad \forall x \in C,$$

where $r \leq \lambda < 1$. Then S is a mapping from C into itself and $F(S) = F(T)$. From $Sx = \lambda x + (1 - \lambda)Tx$, we also have that

$$Tx = \frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x.$$

Thus we obtain from (4.8) and (2.1) that

$$\begin{aligned}
0 &\geq \|\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x - (\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y)\|^2 \\
&\quad - \|x - y\|^2 - r\|x - y - \{\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x - (\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y)\}\|^2 \\
&= \|\frac{1}{1-\lambda}(Sx - Sy) - \frac{\lambda}{1-\lambda}(x - y)\|^2 \\
&\quad - \|x - y\|^2 - r\|\frac{1}{1-\lambda}(x - y) - \frac{1}{1-\lambda}(Sx - Sy)\|^2 \\
&= \frac{1}{1-\lambda}\|Sx - Sy\|^2 - \frac{\lambda}{1-\lambda}\|x - y\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{1-\lambda}\cdot\frac{\lambda}{1-\lambda}\|x-y-(Sx-Sy)\|^2-\|x-y\|^2\\
&-\frac{r}{(1-\lambda)^2}\|x-y-(Sx-Sy)\|^2\\
=&\frac{1}{1-\lambda}\|Sx-Sy\|^2-\frac{1}{1-\lambda}\|x-y\|^2+\frac{\lambda-r}{(1-\lambda)^2}\|x-y-(Sx-Sy)\|^2.
\end{aligned}
$$

Then S is $(\frac{1}{1-\lambda},0,-\frac{1}{1-\lambda},0,\frac{\lambda-r}{(1-\lambda)^2})$-symmetric more generalized hybrid. From

$$
\frac{1}{1-\lambda}+2\cdot 0-\frac{1}{1-\lambda}=0,\quad \frac{1}{1-\lambda}+\frac{\lambda-r}{(1-\lambda)^2}>0\quad\text{and}\quad \frac{\lambda-r}{(1-\lambda)^2}\geq 0,
$$

(1) $\alpha+2\beta+\gamma\geq 0$, (2) $\alpha+\beta+\delta+\zeta>0$ and (3) $\delta+\zeta\geq 0$ in Theorem 3.3 are satisfied. Thus S has a fixed point in C from Theorem 3.3 and hence T has a fixed point. This completes the proof. ■

Let H be a Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. For $u\in H$ and $s,t\in(0,1)$, we define the following mapping:

$$
Sx=tx+(1-t)\big(su+(1-s)Tx\big)
$$

for all $x\in C$. We call such S a *TWY* mapping generated by u,T,s,t. Since $Sx=tx+s(1-t)u+(1-t)(1-s)Tx$, we have that for any $x,y\in C$,

$$
\begin{aligned}
\|Sx-Sy\|^2&=\|t(x-y)+(1-t)(1-s)(Tx-Ty)\|^2\\
&=t^2\|x-y\|^2+(1-t)^2(1-s)^2\|Tx-Ty\|^2\\
&\quad+2t(1-t)(1-s)\langle x-y,Tx-Ty\rangle\\
&=t^2\|x-y\|^2+(1-t)^2(1-s)^2\|Tx-Ty\|^2\\
&\quad+t(1-t)(1-s)(\|x-Ty\|^2+\|y-Tx\|^2\\
&\quad-\|x-Tx\|^2-\|y-Ty\|^2)\\
&=t^2\|x-y\|^2+(1-t)^2(1-s)^2\|Tx-Ty\|^2\\
&\quad+t(1-t)(1-s)(\|x-Ty\|^2+\|y-Tx\|^2)\\
&\quad-t(1-t)(1-s)(\|x-Tx\|^2+\|y-Ty\|^2)
\end{aligned}\tag{4.3}
$$

Similarly, we have that

$$
\begin{aligned}
&\|x-Sy\|^2+\|y-Sx\|^2\\
=&\,s(1-t)^2(\|u-x\|^2+\|u-y\|^2)\\
&-s(1-s)(1-t)^2(\|u-Tx\|^2+\|u-Ty\|^2)\\
&-t(1-t)(1-s)(\|x-Tx\|^2+\|y-Ty\|^2)\\
&+(1-t)(1-s)(\|x-Ty\|^2+\|y-Tx\|^2)+2t\|x-y\|^2,
\end{aligned}\tag{4.4}
$$

$$\begin{aligned} &\|x-Sx\|^2+\|y-Sy\|^2 \\ &= s(1-t)^2(\|u-x\|^2+\|u-y\|^2) \\ &\quad - s(1-s)(1-t)^2(\|u-Tx\|^2+\|u-Ty\|^2) \\ &\quad + (1-s)(1-t)^2(\|x-Tx\|^2+\|y-Ty\|^2). \end{aligned} \tag{4.5}$$

We also have that

$$\begin{aligned} &\|x-y-Sx-Sy)\|^2 \\ &= (1-s)(1-t)^2(\|x-Tx\|^2+\|y-Ty\|^2) \\ &\quad - (1-s)(1-t)^2(\|x-Ty\|^2+\|y-Tx\|^2) \\ &\quad + (1-t)^2\|x-y\|^2+(1-t)^2(1-s)^2\|Tx-Ty\|^2. \end{aligned} \tag{4.6}$$

Using (4.4) and (4.5), we have that

$$\begin{aligned} &\|x-Sx\|^2+\|y-Sy\|^2-\|x-Sy\|^2-\|y-Sx\|^2 \\ &= (1-s)(1-t)(\|x-Tx\|^2+\|y-Ty\|^2 \\ &\quad - \|x-Ty\|^2-\|y-Tx\|^2) - 2t\|x-y\|^2. \end{aligned} \tag{4.7}$$

Using (4.3) and (4.7), we have the following theorem.

Theorem 4.2. *Let H be a real Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be a widely strict pseudo-contraction from C into itself, i.e., there exists $r \in \mathbb{R}$ with $r < 1$ such that*

$$\|Tx-Ty\|^2 \le \|x-y\|^2 + r\|(I-T)x-(I-T)y\|^2, \quad \forall x,y \in C. \tag{4.8}$$

Let $u \in C$ and $s \in (0,1)$. Define a mapping $U : C \to C$ as follows:

$$Ux = su + (1-s)Tx, \quad \forall x \in C.$$

Then U has a unique fixed point in C.

Proof. Since T is a widely r-strict pseudo-contraction from C into itself, we have that for any $x, y \in C$,

$$\|Tx-Ty\|^2 - \|x-y\|^2 - r\|x-y-(Tx-Ty)\|^2 \le 0.$$

If $r \le 0$, then T is a nonexpansive mapping. Therefore U is a contractive mapping. Using the fixed point theorem for contractive mappings, we have that U has a unique fixed point in C. Let $0 < r < 1$. Since

$$\begin{aligned} \|x-y-(Tx-Ty)\|^2 &= \|x-Tx\|^2+\|y-Ty\|^2 \\ &- \|x-Ty\|^2-\|y-Tx\|^2+\|x-y\|^2+\|Tx-Ty\|^2, \end{aligned}$$

we have that

$$\begin{aligned}&(1-r)\|Tx-Ty\|^2-(1+r)\|x-y\|^2\\ &-r(\|x-Tx\|^2+\|y-Ty\|^2-\|x-Ty\|^2-\|y-Tx\|^2)\le 0.\end{aligned}$$

For u, T and $s, r \in (0,1)$, define a TWY mapping S as follows:

$$Sx = rx + (1-r)\big(su + (1-s)Tx\big), \quad \forall x \in C.$$

Then we have from (4.3) that

$$\begin{aligned}&\frac{1}{(1-r)(1-s)^2}\|Sx-Sy\|^2-\frac{r^2}{(1-r)(1-s)^2}\|x-y\|^2\\ &+\frac{r}{1-s}(\|x-Tx\|^2+\|y-Ty\|^2-\|x-Ty\|^2-\|y-Tx\|^2)\\ &-(1+r)\|x-y\|^2\\ &-r(\|x-Tx\|^2+\|y-Ty\|^2-\|x-Ty\|^2-\|y-Tx\|^2)\le 0.\end{aligned}$$

We have from (4.7) that

$$\begin{aligned}&\frac{1}{(1-r)(1-s)^2}\|Sx-Sy\|^2-\frac{r^2}{(1-r)(1-s)^2}\|x-y\|^2\\ &+\frac{r}{(1-r)(1-s)^2}(\|x-Sx\|^2+\|y-Sy\|^2-\|x-Sy\|^2-\|y-Sx\|^2)\\ &+\frac{2r^2}{(1-r)(1-s)^2}\|x-y\|^2-(1+r)\|x-y\|^2\\ &-\frac{r}{(1-r)(1-s)}(\|x-Sx\|^2+\|y-Sy\|^2-\|x-Sy\|^2-\|y-Sx\|^2)\\ &-\frac{2r^2}{(1-r)(1-s)}\|x-y\|^2\le 0\end{aligned}$$

and hence

$$\begin{aligned}&\frac{1}{(1-r)(1-s)^2}\|Sx-Sy\|^2\\ &-\frac{rs}{(1-r)(1-s)^2}(\|x-Sy\|^2+\|y-Sx\|^2)\\ &+\left(\frac{r^2}{(1-r)(1-s)^2}-\frac{1-s+r^2(1+s)}{(1-r)(1-s)}\right)\|x-y\|^2\\ &+\frac{rs}{(1-r)(1-s)^2}(\|x-Sx\|^2+\|y-Sy\|^2)\le 0.\end{aligned}$$

For this inequality, we apply Theorem 3.2. We first obtain that

$$\frac{1}{(1-r)(1-s)^2} - \frac{2rs}{(1-r)(1-s)^2} + \frac{r^2}{(1-r)(1-s)^2} - \frac{1-s+r^2(1+s)}{(1-r)(1-s)}$$
$$= \frac{s(1+r)(2-s(1-r))}{(1-r)(1-s)^2} > 0.$$

Furthermore, we have that

$$\tfrac{1}{(1-r)(1-s)^2} - \tfrac{rs}{(1-r)(1-s)^2} + \frac{rs}{(1-r)(1-s)^2} = \frac{1}{(1-r)(1-s)^2} > 0,$$
$$\frac{rs}{(1-r)(1-s)^2} \geq 0.$$

Thus S has a unique fixed point z in C from Theorem 3.2. Since z is a fixed point of S, we have $z = rz + (1-r)(su + (1-s)Tz$. From $1 - r \neq 0$, we have that

$$z = su + (1-s)Tz.$$

This completes the proof. ∎

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