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DOMINATION IN THE ZERO-DIVISOR GRAPH OF AN IDEAL OF A NEAR-RING

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Abstract. Let N be a near-ring. In this paper, we associate a graph corresponding to the 3-prime radical $\mathcal{I}$ of N, denoted by $\Gamma_{\mathcal{I}}(N)$. Further we obtain certain topological properties of $Spec(N)$, the spectrum of 3-prime ideals of N and graph theoretic properties of $\Gamma_{\mathcal{I}}(N)$. Using these properties, we discuss dominating sets and connected dominating sets of $\Gamma_{\mathcal{I}}(N)$.

1. Introduction

Throughout this paper, by a near-ring N we always mean a zero-symmetric near-ring with identity 1. For basic definitions in near-rings one may refer [10]. For subsets A, B of N, $(A : B) = \{n \in N : nB \subseteq A\}$. An ideal I of N is said to be a *prime* ideal if $JK \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$ for ideals J and K of N. Let $a, b \in N$. An ideal I of N is 3-*prime* if $aNb \subseteq I$, then either $a \in I$ or $b \in I$. An ideal I of N is 3-*semiprime* if $aNa \subseteq I$, then $a \in I$. An ideal I of N is *completely prime* if $ab \in I$, then either $a \in I$ or $b \in I$. Note that completely prime $\Rightarrow$ 3-prime $\Rightarrow$ prime [14]. Moreover, if N is a commutative ring, then the notions of prime, 3-prime and completely prime are one and the same. The intersection of all proper prime ideals of N is called the *prime radical* of N and denoted by $\mathcal{P}(N)$, the intersection of all proper 3-prime ideals of N is called the 3-*prime radical* of N and denoted by $\mathcal{I}(N)$ and the intersection of all proper completely prime ideals of N is called the *completely prime radical* of N. Let $\mathcal{N}(N)$ denote the set of all nilpotent elements of N. A near-ring N is called 2-*primal* if $\mathcal{P}(N) = \mathcal{N}(N)$. As observed in [5], if N is a zero-symmetric 2-primal near-ring, then the prime radical, the 3-prime radical and the completely prime radical are coincide. A near-ring N is called a *pm-near-ring* if every 3-prime ideal is contained in a unique maximal ideal of N.

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The study on graphs from algebraic structures is an interesting subject for mathematicians since the notion of Cayley graphs from groups [4]. In recent years, many algebraists as well as graph theorists have focused on the zero-divisor graph of rings. In [2], D.F. Anderson and P.S. Livingston introduced the zero-divisor graph of a commutative ring R with identity, denoted by $\Gamma(R)$, as the graph with vertices $Z(R)^* = Z(R)\setminus\{0\}$, the set of nonzero zero-divisors of R, and for distinct vertices x and y are adjacent if and only if $xy = 0$. This concept due to I. Beck [3], who let all the elements of R be vertices of $\Gamma(R)$ and was mainly interested in colorings. S.P. Redmond [11] introduced the zero-divisor graph with respect to an ideal I of R, denoted by $\Gamma_I(R)$, as the graph with vertex set $\{x \in R\setminus I : xy \in I$ for some $y \in R\setminus I\}$, and two distinct vertices x and y are adjacent if and only if $xy \in I$. Later on, the zero-divisor graph and the ideal-based zero-divisor graph were studied in near-rings and one may refer [1, 7]. Subsequently, in [13], authors constructed the zero-divisor graph to an ideal I of a near-ring N, denoted by $\Gamma_I(N)$, as the graph with vertex set $\{x \in N \setminus I : xNy \subseteq I \textit{ or } yNx \subseteq I$ for some $y \in N \setminus I\}$ and two distinct vertices x and y are adjacent if and only if $xNy \subseteq I$ or $yNx \subseteq I$. If I is a totally reflexive ideal of N (i.e, if $aNb \subseteq I$, then $bNa \subseteq I$ for $a, b \in N$), then the vertex set $V(\Gamma_I(N)) = \{x \in N \setminus I : xNy \subseteq I$ for some $y \in N \setminus I\}$. Having constructed $\Gamma_I(N)$ corresponding to a totally reflexive ideal I of N, T. Tamizh Chelvam and S. Nithya [13] proved that Beck's conjecture is true for the class of $\Gamma_I(N)$ and further they characterized all near-rings N for which the graph $\Gamma_I(N)$ is finitely colorable.

Since $\mathcal{I}$ (abbreviation for $\mathcal{I}(N)$) is a 3-prime radical of N, $\mathcal{I}$ is a totally reflexive ideal of N. Due to this, $V(\Gamma_{\mathcal{I}}(N)) = \{x \in N \setminus \mathcal{I} : xNy \subseteq \mathcal{I}$ for some $y \in N \setminus \mathcal{I}\}$ and two distinct vertices x and y are adjacent if and only if $xNy \subseteq \mathcal{I}$. If $\mathcal{I}$ is a 3-prime ideal of N, then the graph $\Gamma_{\mathcal{I}}(N)$ is empty. Hence we consider near-rings N for which $\mathcal{I}$ is not a 3-prime ideal. For an ideal I of N and $x \in N$, the *annihilator* of x is nothing but $(I : Nx) = \{y \in N : yNx \subseteq I\}$. By Proposition 1.42 [10], $(I : Nx)$ is an ideal of N. Since $\mathcal{I}$ is a totally reflexive ideal of N, $(\mathcal{I} : Nx) = \{y \in N : xNy \subseteq \mathcal{I}\}$.

$Spec(N)$, $Max(N)$ and $Min(N)$ denote the set of all proper 3-prime ideals of N, the set of all maximal ideals of N and the set of all minimal 3-prime ideals of N, respectively. For $a \in N$, we define $V(a) = \{P \in Spec(N) : a \in P\}$, $D(a) = \{P \in Spec(N) : a \notin P\} = Spec(N) \setminus V(a)$ and $M(a) = V(a) \cap Max(N)$. Note that $V(a) = V(\langle a\rangle)$ and $D(a) = D(\langle a\rangle)$, where $\langle a\rangle$ is the ideal generated by $a \in N$. Also, for an ideal J of N, $V(J) = \bigcap_{a\in J} V(a)$ and $D(J) = \bigcup_{a\in J} D(a)$. Since the sets $\{V(J) : J$ is an ideal of $N\}$ and $\{D(J) : J$ is an ideal of $N\}$ satisfy the axioms for closed sets and open sets, one can have a topology on $Spec(N)$ and hence $Spec(N)$ is a topological space. Further with respect to this topology $Min(N)$ is a subspace of $Spec(N)$. If N is a zero-symmetric near-ring with identity, then $Max(N) \subseteq Spec(N)$ and so we consider $Max(N)$ as a subspace of $Spec(N)$. Also $\mathcal{B} = \{D(x) : x \in N\}$ is a base of the topological space $Spec(N)$. The operators cl and int denote the closure and the interior in a topological space. For basic definitions of topological space one may refer [8].

Recently, K. Samei [12] studied the relation between properties of a commutative reduced ring R and properties of the graph $\Gamma(R)$ through topological properties of $Spec(R)$. Note that when N is a commutative reduced ring, the 3-prime radical of N is $\{0\}$ and so $\Gamma(N) = \Gamma_{\mathcal{I}}(N)$. In Section 2, we generalize some results proved in [12] for commutative ring to near-rings. In section 3, we construct a dominating set of $\Gamma_{\mathcal{I}}(N)$ through a base of the topological space $Spec(N)$ and on the other way obtain a dense subset in $Spec(N)$ corresponding to every dominating set in $\Gamma_{\mathcal{I}}(N)$. Moreover, we give a topological characterization for the set of all central vertices of $\Gamma_{\mathcal{I}}(N)$ to be a dominating set and the neighbourhood of every vertex in $\Gamma_{\mathcal{I}}(N)$ to be a connected dominating set of $\Gamma_{\mathcal{I}}(N)$.

Let G be a graph with vertex set $V(G)$. Recall that G is *connected* if there is a path between any two distinct vertices of G. The *neighbourhood* of a vertex x in G is the set consisting of all vertices which are adjacent with x. For two vertices x and y of G, the distance $d(x, y)$ to be the length of a shortest path from x to y. The *diameter* of G is $diam(G) = max\,\{d(x, y) : x, y \in V(G)\}$. The *eccentricity* of a vertex x in G is defined as $e(x) = max\,\{d(x, z) : z \in V(G)\}$. The radius of G is the minimum eccentricity among the vertices of G, which is denoted by $rad(G)$. A vertex x in G is a central vertex if $e(x) = rad(G)$. For $S \subseteq V(G)$, the *induced subgraph* H induced by S is the subgraph of G with vertex set S and two vertices are adjacent in H if and only if they are adjacent in G and it is denoted by $\langle S\rangle$. A graph G is *complete* if each pair of distinct vertices is adjacent. For undefined terms in graph theory, we refer to [6].

2. Basic Properties of $\Gamma_{\mathcal{I}}(N)$

The results of this section provide effective criterion for discussing the dominating sets and the connected dominating sets of $\Gamma_{\mathcal{I}}(N)$ in Section 3. One can easily observe the following.

Observation 2.1. Let N be a near-ring and $a \in N$. Then

(i) $V((\mathcal{I} : Na)) = clD(a) = Spec(N) \setminus intV(a)$.

(ii) $a \in V(\Gamma_{\mathcal{I}}(N))$ if and only if $\emptyset \neq clD(a) \neq Spec(N)$.

The following proposition topologically characterizes the concept of distance in $\Gamma_{\mathcal{I}}(N)$. First we need the following Lemma.

Lemma 2.2. ([13, Theorem 2.2]). *Let I be a totally reflexive ideal of a near-ring N. Then $diam(\Gamma_I(N) \leq 3$.*

Proposition 2.3. *Let $\mathcal{I}$ be the 3-prime radical of N and $a, b, c \in V(\Gamma_{\mathcal{I}}(N))$ be distinct elements. Then the following are true.*

(i) *c is adjacent in $\Gamma_{\mathcal{I}}(N)$ to both a and b if and only if $clD(a) \cup clD(b) \subseteq V(c)$;*

(ii) *$d(a, b) = 1$ if and only if $D(a) \cap D(b) = \emptyset$;*

(iii) *$d(a, b) = 2$ if and only if $D(a) \cap D(b) \neq \emptyset$ and $clD(a) \cup clD(b) \neq Spec(N)$;*

(iv) *$d(a, b) = 3$ if and only if $D(a) \cap D(b) \neq \emptyset$ and $clD(a) \cup clD(b) = Spec(N)$.*

Proof.

(i) Note that c is adjacent to both a and b in $\Gamma_{\mathcal{I}}(N)$ if and only if $aNc \subseteq \mathcal{I}$ and $bNc \subseteq \mathcal{I}$ if and only if $D(a) \cap D(c) = D(b) \cap D(c) = \emptyset$ if and only if $D(a) \cup D(b) \subseteq V(c)$ and if and only if $clD(a) \cup clD(b) \subseteq V(c)$.

(ii) Trivial from definitions.

(iii) Suppose $d(a, b) = 2$, then there exists $c \in V(\Gamma_{\mathcal{I}}(N))$ such that c is adjacent to both a and b. By (ii) and (i), $D(a) \cap D(b) \neq \emptyset$ and $clD(a) \cup clD(b) \subseteq V(c)$. Since $c \notin \mathcal{I}$, we have $V(c) \neq Spec(N)$.

Conversely assume that $D(a) \cap D(b) \neq \emptyset$ and $clD(a) \cup clD(b) \neq Spec(N)$. By (ii), $d(a, b) \neq 1$. Since $clD(a) = V((I : Na))$ and $clD(b) = V(I : Nb)$, there exists $P \in Spec(N)$ with $x, y \notin P$ for some $x \in (\mathcal{I} : Na)$ and $y \in (\mathcal{I} : Nb)$. This implies that $xNy \nsubseteq \mathcal{I}$ and there exists $n \in N$ such that $xny \in (\mathcal{I} : Na)$ and $xny \in (\mathcal{I} : Nb)$. Hence $d(a, b) = 2$.

(iv) By Lemma 2.2, we have $diam(\Gamma_{\mathcal{I}}(N)) \leq 3$. Now proof follows from (ii) and (iii). ■

Lemma 2.4. *Let N be a near-ring. For $\mathcal{F} \subseteq Spec(N)$, the closure of $\mathcal{F}$ is $cl\mathcal{F} = \left\{P' \in Spec(N) : \bigcap\limits_{P \in \mathcal{F}} P \subseteq P'\right\}$.*

Proof. Let $A = \left\{P' \in Spec(N) : \bigcap\limits_{P \in \mathcal{F}} P \subseteq P'\right\}$ and $Q \in A$. Since $\mathcal{B} = \{D(x) : x \in N\}$ is a base for the space $Spec(N)$, it is enough to a $D(x)$ such that $Q \in D(x)$. Clearly $D(x) \cap \mathcal{F} \neq \emptyset$, i.e., $Q \in cl\mathcal{F}$. Suppose $A \subsetneq cl\mathcal{F}$, there is $P_1 \in cl\mathcal{F}$ such that $\bigcap\limits_{P \in \mathcal{F}} P \nsubseteq P_1$. Let $x \in \bigcap\limits_{P \in \mathcal{F}} P \setminus P_1$, then $P_1 \in D(x)$ and $D(x) \cap \mathcal{F} = \emptyset$, a contradiction. Hence $cl\mathcal{F} = A$. ■

From this Lemma 2.4, for every closed subset $\mathcal{F}$ of $Spec(N)$, $\mathcal{F} = V(J)$ where $J = \bigcap\limits_{P \in \mathcal{F}} P$.

Theorem 2.5. *Let $\mathcal{I}$ be the 3-prime radical of a near-ring N. A subset $\mathcal{F}$ of $Spec(N)$ is dense in $Spec(N)$ if and only if $\mathcal{I} = \bigcap\limits_{Q \in \mathcal{F}} Q$.*

Proof. Let $\mathcal{F}$ be dense in $Spec(N)$. Then by Lemma 2.4, $cl\mathcal{F} = \left\{P' \in Spec(N) : \bigcap_{Q\in\mathcal{F}} Q \subseteq P'\right\} = Spec(N)$. Hence $\bigcap_{Q\in\mathcal{F}} Q \subseteq \mathcal{I}$. As $\mathcal{F} \subseteq Spec(N)$, $\mathcal{I} = \bigcap_{Q\in\mathcal{F}} Q$.

Conversely, assume that $\mathcal{I} = \bigcap_{Q\in\mathcal{F}} Q$. Suppose $cl\mathcal{F} = \left\{P' \in Spec(N) : \bigcap_{Q\in\mathcal{F}} Q \subseteq P'\right\} \subsetneq Spec(N)$. Then there exists $P_1 \in Spec(N)$ such that $\bigcap_{Q\in\mathcal{F}} Q \nsubseteq P_1$, i.e., there exists $x \in \bigcap_{Q\in\mathcal{F}} Q \setminus P_1$, a contradiction to the fact that $\mathcal{I} = \bigcap_{Q\in\mathcal{F}} Q$. Hence $\mathcal{F}$ is dense in $Spec(N)$. ■

Note that every maximal ideal is a 3-primal ideal in a zero-symmetric near-ring N with identity 1. This along with Theorem 2.5 give the following corollary.

Corollary 2.6. *Let $\mathcal{I}$ be the* 3*-prime radical of N with $\mathcal{I} = \cap Max(N)$. Then $Max(N)$ is dense in $Spec(N)$.*

Theorem 2.7. *Let N be a near-ring with the* 3*-prime radical $\mathcal{I}$. Then*

(i) *$Spec(N)$ is a compact space,*

(ii) *$Max(N)$ is a compact subspace of $Spec(N)$,*

(iii) *If N is a* 2*-primal pm-near-ring, then $Max(N)$ is Hausdorff. Moreover, if $\mathcal{I} = \cap Max(N)$, then $Spec(N)$ is normal.*

Proof. (i) and (ii) follow from Theorem 2.3(ii) and (iii) in [7].
(iii) Since N is a 2-primal near-ring, all prime radicals are coincide. Now the proof follows from Theorems 2.8 and 2.3(v) in [7]. ■

Theorem 2.8. *Let N be a* 2*-primal pm-near-ring with $\mathcal{I} = \cap Max(N)$. Then $diam(\Gamma_{\mathcal{I}}(N)) = 3$ if and only if there exist at least three distinct maximal ideals in N.*

Proof. Assume that $diam(\Gamma_{\mathcal{I}}(N)) = 3$, then there exist $a, b, x, y \in V(\Gamma_{\mathcal{I}}(N))$ such that $a{-}x{-}y{-}b$ is a path. Suppose $|Max(N)| = 2$ and let $Max(N) = \{M_1, M_2\}$. As $d(a, b) = 3$, by Proposition 2.3(ii) there exists a 3-prime ideal $P \in D(a) \cap D(b)$. By $P \in D(a)$ and Corollary 2.6, there exists a maximal ideal $M_1 \in D(a)$. Since $aNx \subseteq \mathcal{I}$, $xNy \subseteq \mathcal{I}$ and $\mathcal{I} = M_1 \cap M_2$, $x \in M_1 \setminus M_2$, $y \in M_2 \setminus M_1$. Now $yNb \subseteq \mathcal{I}$ gives that $b \in M_1$. Therefore $M_1 \in D(a) \setminus D(b)$. Similarly, as $P \in D(b)$ we can show that $M_2 \in D(b) \setminus D(a)$. Again $P \in D(anb)$ for some $n \in N$ and $Max(N)$ is dense in $Spec(N)$, we have $D(anb) \cap Max(N) \neq \emptyset$. But $M_i \notin D(anb) \cap Max(N)$ for $i = 1, 2$, a contradiction. Thus $|Max(N)| \geq 3$.

Conversely, suppose that $|Max(N)| \geq 3$ and let M_1, M_2, M_3 be distinct maximal ideals in N. By Theorem 2.7(iii), $Max(N)$ is Hausdorff. Thus there exist $a_i \in N$ such that $M_i \in D(a_i)$, $i = 1, 2, 3$ and $D(a_i)$ are mutually disjoint. Since $a_1 \notin M_1$ and $a_2 \notin M_2$, there exist $a \in M_1$, $b \in M_2$ such that $a + a'_1 = b + a'_2 = 1$ where $a'_1 \in < a_1 >$ and $a'_2 \in < a_2 >$. Thus $M_1 \in V(a) \subseteq D(a_1)$ and $M_2 \in V(b) \subseteq D(a_2)$. Clearly $M_3 \in D(a) \cap D(b)$ and so $D(a) \cap D(b) \neq \emptyset$. Suppose $D(a) \cup D(b) \subsetneq Spec(N)$. Then there exists $P \in Spec(N)$ such that $P \in V(a) \subseteq D(a_1)$ and $P \in V(b) \subseteq D(a_2)$, a contradiction. Therefore $clD(a) \cup clD(b) \supseteq D(a) \cup D(b) = Spec(N)$ and so by Proposition 2.3(iv), $d(a, b) = 3$ which implies $diam(\Gamma_{\mathcal{I}}(N)) = 3$. ■

Theorem 2.9. *Let N be a 2-primal pm-near-ring with $|N| > 4$ and $\cap Max(N) = \langle 0 \rangle$. Then $diam(\Gamma_{\mathcal{I}}(N)) = min\,\{|Max(N)|, 3\}$.*

Proof. Since $\cap Max(N) = \langle 0 \rangle$, $\mathcal{I} = \cap Max(N)$ gives $|Max(N)| > 1$. Suppose that $|Max(N)| = 2$. Let $Max(N) = \{M_1, M_2\}$, then there exist $a_1 \in M_1 \setminus M_2$ and $a_2 \in M_2 \setminus M_1$ such that $a_1 N a_2 \subseteq \mathcal{I}$ and so $a_1, a_2 \in V(\Gamma_{\mathcal{I}}(N))$. Since $|N| > 4$, either M_1 or M_2 contains at least two nonzero elements. If possible, M_1 and M_2 contains only one element, then $|N| = 4$, a contradiction. Without loss of generality assume that there exists nonzero $(a_1 \neq) b_1 \in M_1$. Since $a_2 N b_1 \subseteq \mathcal{I}$ and $b_1 \notin M_2$ which imply $b_1 \in V(\Gamma_{\mathcal{I}}(N))$ and $a_1 N b_1 \nsubseteq M_2$. Therefore $d(a_1, b_1) = 2$ and so $diam(\Gamma_{\mathcal{I}}(N)) = 2$. This along with Theorem 2.8 imply that $diam(\Gamma_{\mathcal{I}}(N)) = min\,\{|Max(N)|, 3\}$. ■

Remark 2.10. In Theorem 2.9, if $|N| = 4$, then the fact is not true. Consider the near-ring of matrices $N = \left\{ \begin{pmatrix} 0 & 0 \\ x & x \end{pmatrix}, x \in \mathbb{Z}_2 \right\}$. Then $diam(\Gamma_{\mathcal{I}}(N)) = 1$. If $|N| < 4$, then the graph $\Gamma_{\mathcal{I}}(N)$ is empty.

Lemma 2.11. *Let N be a 2-primal pm-near-ring with $\mathcal{I} = \cap Max(N)$. For every open subset U of P in $Spec(N)$, there exists $a \in V(\Gamma_{\mathcal{I}}(N))$ such that $P \in intV(a) \subseteq V(a) \subseteq U$. That is, $\{intV(a) : a \in V(\Gamma_{\mathcal{I}}(N))\}$ is a base of the space $Spec(N)$.*

Proof. Let U be a proper open set of $Spec(N)$. Then $\emptyset \neq U^c = Spec(N) \setminus U = V(J)$ for some ideal J of N. By Theorem 2.7(iii), $Spec(N)$ is normal and so there are disjoint open sets U' and U'' in $Spec(N)$ such that $P \in U'$ and $V(J) \subseteq U''$. Since $Spec(N)$ is compact and $V(J)$ is closed, $V(J)$ is compact, so there are $a_i \in N$, $i = 1$ to n such that $V(J) \subseteq \bigcup_{i=1}^{n} D(a_i) = D(J_1) \subseteq U''$, where $J_1 = \sum_{i=1}^{n} < a_i >$. We claim that $J_1 + J = N$. For otherwise, there exists a proper 3-prime ideal Q such that $J_1 + J \subseteq Q$ which gives $Q \in V(J_1)$ and $Q \in V(J) \subseteq D(J_1)$, a contradiction. Thus $J_1 + J = N$, i.e., $a + b = 1$ for some $a \in J_1$ and $b \in J$. Since $U' \cap U'' = \emptyset$, we have $U' \cap D(a) = \emptyset$. Hence $P \in U' \subseteq int\ V(a) \subseteq V(a) \subseteq D(b) \subseteq D(J) = U$. By Observation

2.1, $a \in V(\Gamma_{\mathcal{I}}(N))$. Suppose $U = Spec(N)$. Since $|Max(N)| > 1$, there exists a maximal ideal M containing c such that $c \notin P$. Then $P \in D(c) \neq Spec(N)$. Hence there exists $a \in V(\Gamma_{\mathcal{I}}(N))$ such that $P \in int\ V(a) \subseteq V(a) \subseteq D(b) \subseteq D(c) \subset U$. ∎

In view of Lemma 2.11, we observe that the following remarks.

Remark 2.12. For every nonempty open subset U of $Spec(N)$, by Lemma 2.11, there exists $b \in N$ such that $\emptyset \neq D(b) \neq Spec(N)$ and $D(b) \subseteq U$. Choose $P_1 \in D(b)$ and $P_2 \in V(b)$. Since $Spec(N)$ is normal, there exist $c_1, c_2 \in N$ such that $P_1 \in D(c_1) \subseteq D(b)$, $P_2 \in D(c_2)$ and $D(c_1) \cap D(c_2) = \emptyset$. Therefore $c_1 N c_2 \subseteq \mathcal{I}$. Hence for every nonempty open subset U of $Spec(N)$, there exists $c_1 \in V(\Gamma_{\mathcal{I}}(N))$ such that $D(c_1) \subseteq U$.

If N is a 2-primal pm-near-ring, then by Theorem 2.7(ii) and (iii), $Max(N)$ is a compact Hausdorff space and by Theorem 3.26 in [9], $Max(N)$ is normal. By the argument similar to the proof of Lemma 2.11, $\{intM(a) : a \in V(\Gamma_{\mathcal{I}}(N))\}$ is a basis of $Max(N)$.

Proposition 2.13. *Let $\mathcal{I}$ be the 3-prime radical of N and $a \in V(\Gamma_{\mathcal{I}}(N))$. If $e(a) = 1$, then $|Min(N)| = 2$.*

Proof. We claim that $P_1 = \mathcal{I} \cup \{a\}$ and $P_2 = (\mathcal{I} : Na)$ are the only minimal 3-primal ideals of N. Let $x_1, x_2 \in P_1$. Since $e(a) = 1$, for every $y \in P_2$ $(x_1 - x_2)Ny \subseteq \mathcal{I}$ which yields $x_1 - x_2 \in P_1$. If $x \in P_1$, then $xNy + \mathcal{I} = \mathcal{I}$ for every $y \in P_2$ and so $(n + x - n)Ny \subseteq \mathcal{I}$ for every $n \in N$, i.e., $n + x - n \in P_1$. Thus P_1 is a normal subgroup of N. Let $x \in P_1$ and $n, n' \in N$, then $xnNy \subseteq \mathcal{I}$ which gives $P_1 N \subseteq P_1$ and since $xNy + \mathcal{I} = \mathcal{I}$, $(n(n' + x) - nn')Ny \subseteq \mathcal{I}$, i.e., $n(n' + x) - nn' \in P_1$. Hence P_1 is an ideal of N. Assume that $x_1 N x_2 \subseteq P_1$, $x_1, x_2 \in N$, then $x_1 N x_2 N y \subseteq \mathcal{I}$ for every $y \in P_2$.

Case 1. Let $x_1 N x_2 \subseteq \mathcal{I}$. Suppose that both $x_1, x_2 \notin P_1$, then $x_1, x_2 \in V(\Gamma_{\mathcal{I}}(N))$. As $e(a) = 1$ gives $aNx_1 \subseteq \mathcal{I}$ and $aNx_2 \subseteq \mathcal{I}$, so $(a + x_2)Nx_1 \subseteq \mathcal{I}$. Thus $a + x_2 \in V(\Gamma_{\mathcal{I}}(N))$ such that $d(a, a + x_2) = 2$, a contradiction.

Case 2. Suppose $x_1 n x_2 = a \notin \mathcal{I}$ for some $n \in N$. From this, $x_1 Na \nsubseteq \mathcal{I}$ and $x_2 Na \nsubseteq \mathcal{I}$. Since every $y \in P_2$, $aNy \subseteq \mathcal{I}$, $x_1 n x_2 N y \subseteq \mathcal{I}$. If $x_2 N y \subseteq \mathcal{I}$, then $d(a, x_2) =$, a contradiction. Also, if $x_2 N y \nsubseteq \mathcal{I}$, then $x_1 N x_2 N y \subseteq \mathcal{I}$ implies $x_1 \in V(\Gamma_{\mathcal{I}}(N))$. Since P_2 is an ideal, $x_2 N y N a \subseteq \mathcal{I}$, so $d(a, x_1) = 2$, again a contradiction. Also P_1 is a minimal 3-prime ideal of N.

Since P_2 is an ideal, it remains to prove that P_2 is 3-prime. Let $x_1 N x_2 \subseteq P_2$, then $x_1 N x_2 N a \subseteq \mathcal{I}$. If $x_2 N a \subseteq \mathcal{I}$, then $x_2 \in P_2$. Otherwise, there exists $n \in N$ such that $x_2 n a = a$, as $x_2 N a \subseteq P_1$. Hence $x_1 \in P_2$. Therefore P_2 is a minimal 3-prime ideal of N.

If $P \in Min(N) \setminus \{P_1, P_2\}$, then $a \notin P$ and there is some $b \in P_2$ such that $b \notin P$. It is clear that $aNb \subseteq \mathcal{I}$, a contradiction to the fact that $a, b \notin P$. ∎

Remark 2.14. The converse of the Proposition 2.13 is not true. Consider the near-ring $N = \mathbb{Z}_3 \times \mathbb{Z}_5$, then the graph $\Gamma_{\mathcal{I}}(N)$ is $K_{2,4}$ and N has exactly two minimal 3-prime ideals, but no one vertex in $\Gamma_{\mathcal{I}}(N)$ has eccentricity one.

Proposition 2.15. *Let N be a 2-primal pm-near-ring with $\mathcal{I} = \cap Max(N)$, $a \in V(\Gamma_{\mathcal{I}}(N))$ and $e(a) \neq 1$. Then*

(i) $e(a) = 2$ *if and only if* $|clD(a)| = 1$,

(ii) $e(a) = 3$ *if and only if* $|clD(a)| > 1$.
In particular, $e(a) = min\{|clD(a)| + 1, 3\}$.

Proof. (i) Assume that $e(a) = 2$. Suppose $|clD(a)| > 1$. Clearly $D(a) \neq \emptyset$, then there is a maximal ideal, say M in $D(a)$. Now we prove that $D(a)$ contains at least two distinct maximal ideals. For otherwise, it contains only one maximal ideal M. Since $|clD(a)| > 1$, there is a 3-prime ideal $(M \neq)Q \in clD(a)$. Therefore there exists $x \in M \setminus Q$ such that $aNx \subseteq \cap Max(N) = \mathcal{I}$ which is a contradiction to the fact that $D(a) \cap D(x) \neq \emptyset$. Hence there are maximal ideals M, M' in $D(a)$. Let $b \in M' \setminus M$, then $aNb \nsubseteq M$ and so $M \in D(anb)$ for some $n \in N$. By Lemma 2.11, there exists $c \in N$ such that $M \in intV(c) \subseteq D(anb) \subseteq clD(a)$, consequently, $clD(a) \cup clD(c) = Spec(N)$ and $M' \in D(a) \cap clD(c)$ gives that $D(a) \cap D(c) \neq \emptyset$. Then by Proposition 2.3(iv), $d(a, c) = 3$, a contradiction.

Conversely assume that $|clD(a)| = 1$, then there is $P \in Spec(N)$ such that $D(a) = clD(a) = \{P\}$. On the contrary, suppose that $d(a, b) = 3$ for some $b \in V(\Gamma_{\mathcal{I}}(N))$. Again by Proposition 2.3(iv), we have $D(a) \cup clD(b) = clD(a) \cup clD(b) = Spec(N)$. This implies that $clD(b) = V(a)$. Therefore $D(a) \cap D(b) = \emptyset$, a contradiction. This shows that $e(a) = 2$.

(ii) Proof follows from the hypothesis and (i). ∎

3. Dominating Sets in $\Gamma_{\mathcal{I}}(N)$

A subset D of $V(\Gamma_{\mathcal{I}}(N))$ is called a *dominating set* if for every $v \in V(\Gamma_{\mathcal{I}}(N)) - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the cardinality of the smallest possible dominating set in G. A dominating set D is called a *connected dominating set* if the induced subgraph $< D >$ is connected. The connected domination number $\gamma_c(G)$ is the cardinality of the smallest possible connected dominating set. The following theorem exposes a close connection between $\Gamma_{\mathcal{I}}(N)$ and the topological space $Spec(N)$.

Theorem 3.1. *Let N be a 2-primal pm-near-ring with $\mathcal{I} = \cap Max(N)$. Then*

(i) *For every dominating set of* $\Gamma_{\mathcal{I}}(N)$, *there exists a dense subset in* $Spec(N)$.

(ii) *For every base for the open sets of the space $Spec(N)$, there exists a dominating set in $\Gamma_{\mathcal{I}}(N)$.*

Proof. (i) Suppose D is a dominating set. For every $a \in D$, there exists $b \in V(\Gamma_{\mathcal{I}}(N))$ such that $aNb \subseteq \mathcal{I}$. Since $Max(N)$ is dense, $\emptyset \neq D(b) \cap Max(N) \subseteq M(a)$ and $Max(N) \setminus M(a) \neq \emptyset$. Then we take $M_a \in intM(a)$ and $M'_a \in Max(N) \setminus M(a)$. First we show that the set $\mathcal{A} = \{M_a : a \in D\} \cup \{M'_a : a \in D\}$ is a dense subset of $Max(N)$. By Remark 2.12(ii) $\{intM(c) : c \in V(\Gamma_{\mathcal{I}}(N))\}$ is a basis for $Max(N)$. Therefore it is sufficient to prove that for every $c \in V(\Gamma_{\mathcal{I}}(N))$, $\mathcal{A} \cap intM(c) \neq \emptyset$. Let $c \in V(\Gamma_{\mathcal{I}}(N))$. If $c \in D$ implies that $M_c \in \mathcal{A} \cap intM(c)$. Otherwise, since D is a dominating set, there exists $d \in D$ such that $cNd \subseteq \mathcal{I}$. Thus $M'_d \in Max(N) \setminus M(d) \subseteq intM(c)$ and so $M'_d \in \mathcal{A} \cap intM(c)$. This shows that $\mathcal{A}$ is a dense subset in $Max(N)$ this along with $Max(N)$ is dense in $Spec(N)$ lead to $\mathcal{A}$ is dense in $Spec(N)$.
(ii) Let $\mathcal{B} = \{B_\lambda : \lambda \in \Lambda\}$ be a base for the open sets of the space $Spec(N)$. By Remark 2.12(i), for every $B_\lambda \in \mathcal{B}$, there exists $a_\lambda \in V(\Gamma_{\mathcal{I}}(N))$ such that $D(a_\lambda) \subseteq B_\lambda$. We claim that $D = \{a_\lambda : \lambda \in \Lambda\}$ is a dominating set. Let $b \in V(\Gamma_{\mathcal{I}}(N))$. Then there exists $B_\lambda \in \mathcal{B}$ such that $B_\lambda \subseteq intV(b)$. Therefore $D(a_\lambda) \subseteq intV(b)$, i.e., $a_\lambda Nb \subseteq \mathcal{I}$ and consequently D is a dominating set. ∎

In a topological space X, a point x of X is said to be an *isolated point* of X if the one point set $\{x\}$ is open in X. $\mathcal{P}_0(N)$, $\mathcal{M}_0(N)$ and $\mathcal{I}_0(N)$ denote the sets of isolated points of the spaces $Spec(N)$, $Max(N)$ and $Min(N)$, respectively. The following lemma shows that these isolated points sets are coincide in a pm-near-ring N with $\mathcal{I} = \cap Max(N)$.

Lemma 3.2. *Let N be a pm-near-ring with $\mathcal{I} = \cap Max(N)$. Then $\mathcal{P}_0(N) = \mathcal{M}_0(N) = \mathcal{I}_0(N)$.*

Proof. First we show that $\mathcal{P}_0(N) = \mathcal{M}_0(N)$. Suppose $\{M\}$ is open in $Max(N)$, then $D(a) \cap Max(N) = \{M\}$ for some $a \in N$. It follows that $a \in \bigcap_{M' \in Max(N)\setminus\{M\}} M'$. Therefore $< a > M \subseteq \cap Max(N) = \mathcal{I}$. Since every $P \in Spec(N)$ is prime, $\langle a \rangle \subseteq P$ or $M \subseteq P$ and so $D(a) = \{M\}$, i.e., $M \in \mathcal{P}_0(N)$. The opposite inclusion is trivial. Now it is sufficient to show that $\mathcal{M}_0(N) = \mathcal{I}_0(N)$. Let $P' \in \mathcal{I}_0(N)$ such that $\{P'\} = D(b) \cap Min(N)$, $b \in N$. Then $P' \subseteq M'$ for a unique maximal ideal M' and so $b \in \bigcap_{M \in Max(N)\setminus\{M'\}} M \setminus P'$. This implies that $\bigcap_{M \in Max(N)\setminus\{M'\}} M \neq \mathcal{I}$, i.e., there exists $c \notin \mathcal{I}$ and $c \in \bigcap_{M \in Max(N)\setminus\{M'\}} M$. Therefore $c \notin M'$ and hence M' is an isolated point of $Max(N)$, so $M' \in \mathcal{M}_0(N) = \mathcal{P}_0(N)$ and consequently $P' = M' \in \mathcal{M}_0(N)$. Since $\mathcal{M}_0(N) = \mathcal{P}_0(N)$, $\mathcal{M}_0(N) \subseteq \mathcal{I}_0(N)$. ∎

Theorem 3.3. *Let N be a 2-primal pm-near-ring with $\mathcal{I} = \cap Max(N)$ and $|Min(N)| > 2$. Then the set of central vertices of $\Gamma_{\mathcal{I}}(N)$ is a dominating set if and only if the set of isolated points of $Spec(N)$ is dense in $Spec(N)$.*

Proof. Let D be the set of central vertices of $\Gamma_{\mathcal{I}}(N)$. Since $diam(\Gamma_{\mathcal{I}}(N)) \leq 3$ and Proposition 2.13, $e(a) = 2$ for every $a \in D$. Then by Proposition 2.15(i), we have $D = \{a \in V(\Gamma_{\mathcal{I}}(N)) : |clD(a)| = 1\}$. Now we claim that $Y = \{P_a : D(a) = \{P_a\}, a \in D\}$ is a dense subset of $Spec(N)$. Let U be a nonempty open set which does not contain any isolated points. Since $Max(N)$ is dense in $Spec(N)$, there exists $M \in U \cap Max(N)$. By Lemma 3.2, $|U \cap Max(N)| > 1$, so there are distinct maximal ideals $M, M' \in U$. Clearly $(\mathcal{I} : Na_0) \nsubseteq M$, $a_0 \in D$, otherwise, $M \in V((\mathcal{I} : Na_0)) = clD(a_0)$, a contradiction. Then there exists $y \in (\mathcal{I} : Na_0) \setminus M$ and $x \in M' \setminus M$ such that $b = xny \in P_{a_0} \cap M' \setminus M$ for some $n \in N$ and there is $b' \in N$ such that $M \in D(b') \subseteq U$. Therefore by Lemma 2.11, there exists $c \in N$ and $n' \in N$ such that $M \in intV(c) \subseteq D(bn'b') \subseteq U$, consequently $P_{a_0}, M' \in cl\ D(c)$, i.e., $c \in V(\Gamma_{\mathcal{I}}(N)) \setminus D$. Since D is a dominating set, there exists $a \in D$ such that $aNc \subseteq \mathcal{I}$. Hence $P_a \in D(a) \subseteq intV(c) \subseteq U$, i.e., U contains an isolated point of $Spec(N)$. This leads to $U \cap Y \neq \emptyset$, i.e., Y is dense in $Spec(N)$.
Conversely, let $Y = \{P_\lambda : \lambda \in \Lambda\}$ be the set of isolated points of $Spec(N)$. Consider $D = \{a_\lambda : D(a_\lambda) = \{P_\lambda\}\}$, then $e(a_\lambda) = 2$ for every $\lambda \in \Lambda$ and so every element of D is a central vertex of $\Gamma_{\mathcal{I}}(N)$. Suppose that $b \in V(\Gamma_{\mathcal{I}}(N)) \setminus D$. Since Y is dense in $Spec(N)$, then there exists $P_\lambda \in intV(b) \cap Y$. Therefore $D(a_\lambda) \subseteq intV(b)$ which implies that $a_\lambda Nb \subseteq \mathcal{I}$, i.e., D is a dominating set. ■

Proposition 3.4. *Let N be a 2-primal pm-near-ring with $\mathcal{I} = \cap Max(N)$. If $Spec(N)$ has an isolated point, then there exists $a \in N$ such that the neighbourhood $N(a)$ of a in $\Gamma_{\mathcal{I}}(N)$ is a dominating set.*

Proof. Let P be an isolated point in $Spec(N)$. Then there exists $a \in N$ such that $\{P\} = D(a)$ and so $|clD(a)| = 1$. If $e(a) = 1$, then clearly $N(a)$ is a dominating set. Otherwise, since $|clD(a)| = 1$, Proposition 2.15(ii) implies that $e(a) = 2$. Suppose there is a vertex $b \notin N(a)$ which is not dominated by any $c \in N(a)$. As $diam(\Gamma_{\mathcal{I}}(N)) \leq 3$, $d(b, c) = 2$ or 3 and hence $d(a, b) > 2$, a contradiction. ■

Remark 3.5. From the Proposition 3.4, $\gamma(\Gamma_{\mathcal{I}}(N)) \leq |N(a)|$ and the bound is sharp. For example, consider the near-ring $N = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then the corresponding graph $\Gamma_{\mathcal{I}}(N)$ as given in Figure 1. Here, $D((1,0,0)) = \{\{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2\}$, then $\{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is an isolated point and hence the neighbourhood set $N((1,0,0)) = \{(0,1,1),(0,0,1)(0,1,0)\}$ is a minimum dominating set.

Theorem 3.6. *Let $\mathcal{I}$ be the 3-prime radical of a near-ring N and $diam(\Gamma_{\mathcal{I}}(N)) = 2$. Then the following are equivalent.*

(i) *For every $x \in V(\Gamma_{\mathcal{I}}(N))$, the neighbourhood $N(x)$ in $\Gamma_{\mathcal{I}}(N)$ of x induces a connected subgraph of $\Gamma_{\mathcal{I}}(N)$ and hence it is a connected dominating set.*

(ii) For every pair of distinct $a, b \in V(\Gamma_{\mathcal{I}}(N))$, there exists $c \in N$ such that $clD(a) \cup clD(b) \subseteq V(c)$.

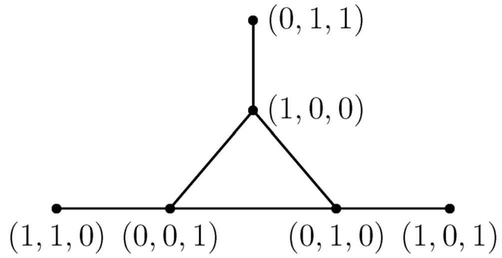

Figure 1.

Proof. (i)⇒(ii) Let $a, b \in V(\Gamma_{\mathcal{I}}(N))$. If $aNb \nsubseteq \mathcal{I}$, since $diam(\Gamma_{\mathcal{I}}(N)) = 2$, the result follows from Proposition 2.3(i). So it is enough to discuss the case that $aNb \subseteq \mathcal{I}$. Again by diameter of $\Gamma_{\mathcal{I}}(N)$, there exists $c' \in V(\Gamma_{\mathcal{I}}(N)) \setminus \{a, b\}$ such that $c' \in N(a)$ or $c' \in N(b)$. Without loss of generality, $c' \in N(a)$. Since induced subgraph of $N(a)$ is connected, there is a path lies between c' and b. Then there exists $c \in N(a)$ such that $c \in N(b)$. Therefore by Proposition 2.3(i), $clD(a) \cup clD(b) \subseteq V(c)$.
(ii)⇒(i) Let $x_1, x_2 \in N(x)$. If $x_1 N x_2 \subseteq \mathcal{I}$, then $x_1 - x_2$ is a path. Otherwise, there exists $n \in N$ such that $x_1 n x_2 \notin \mathcal{I}$. Consider x, x_1 and x, x_2, by our assumption and proposition 2.3(i), there exist $y_1, y_2 \in N$ such that $y_1 \in N(x) \cap N(x_1)$ and $y_2 \in N(x) \cap N(x_2)$. Then $x_1 - y_1 - x_1 n x_2 - y_2 - x_2$ is a path in the induced subgraph of $N(x)$. Thus $N(x)$ induces a connected subgraph of $\Gamma_{\mathcal{I}}(N)$ and since $diam(\Gamma_{\mathcal{I}}(N)) = 2$, for every $x \in V(\Gamma_{\mathcal{I}}(N))$, $N(x)$ is a dominating set. ■

Proposition 3.7. *Let $\mathcal{I}$ be the 3-prime radical of N such that for every $P \in Spec(N)$, $\bigcap_{Q \in Spec(N) \setminus \{P\}} Q \neq \mathcal{I}$. Then $\gamma_c(\Gamma_{\mathcal{I}}(N)) \leq |Spec(N)|$.*

Proof. For every $P \in Spec(N)$, take $a_P \in \bigcap_{Q \in Spec(N) \setminus \{P\}} Q \setminus \mathcal{I}$. We show that the set $D = \{a_P : P \in Spec(N)\}$ is a connected dominating set of $V(\Gamma_{\mathcal{I}}(N))$. Suppose $b \in V(\Gamma_{\mathcal{I}}(N)) \setminus D$, then $b \in P'$ for some $P' \in Spec(N)$ and so we have $b_{P'} \in \bigcap_{Q' \in Spec(N) \setminus \{P'\}} Q' \setminus \mathcal{I}$. Then $b_{P'} \in D$ and $bNb_{P'} \subseteq \mathcal{I}$. Consequently, since every $a_P \in D$, $D(a_P) = \{P\}$, $P \in Spec(N)$ and by Proposition 2.3(ii), D induces a complete subgraph of $\Gamma_{\mathcal{I}}(N)$. Hence D is a connected dominating set. Therefore $\gamma_c(\Gamma_{\mathcal{I}}(N)) \leq |D| = Spec(N)$. ■

Remark 3.8. The bound in Proposition 3.7, is sharp. Consider the near-ring N with $Spec(N) = \{P_1, P_2\}$ and $|P_i \setminus \mathcal{I}| > 1$ for $i = 1, 2$. Let $a \in V(\Gamma_{\mathcal{I}}(N))$. Without loss of generality $a \in P_1 \setminus \mathcal{I}$, then $aNb \subseteq \mathcal{I}$ for every $b \in P_2 \setminus \mathcal{I}$ and $aNa' \nsubseteq \mathcal{I}$ for every $a' \in P_1 \setminus \mathcal{I}$, so for all $a \in P_1 \setminus \mathcal{I}$, $b \in P_2 \setminus \mathcal{I}$, $\{a, b\}$ is a minimum connected dominating set.

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REFERENCES

1. G. Alan Cannon, Kent M. Neuerburg and Shane P. Redmond, Zero-divisor graphs of Near-rings and Semigroups, *Near-rings and Near-fields:Proceedings of the Conference on Near-rings and Near-fields II*, 2005, pp. 189-200, doi: 10.1007/1-4020-3391-5_8.
2. D. F. Anderson and P. S. Livingston, The zero divisor graph of a commutative ring, *J. Algebra*, **217** (1999), 434-447.
3. I. Beck, Coloring of Commutative rings, *J. Algebra*, **116** (1988), 208-226.
4. N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, 1973.
5. G. F. Birkenmeier, H. E. Heatherly and E. K Lee, Completely prime ideals and radicals in near-rings, *Proc. Fredericton Conference on Near-rings and Near-fields*, Kluwer Acad. Publ., Dordrecht, 1995, pp. 63-67.
6. G. Chartrand and P. Zhang, *Introduction to Graph Theory*, Wadsworth and Brooks/Cole, Monterey, CA, 1986.
7. P. Dheena and B. Elavarasan, An ideal-based zero-divisor graph of 2-primal near-rings, *Bull. Korean Math. Soc.*, **46** (2009), 1051-1060.
8. R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.
9. J. R. Munkres, *Topology*, Prentice-Hall of India, New Delhi, 2005.
10. G. Pilz, *Near-rings*, North Holland, Amsterdam, 1983.
11. S. P. Redmond, An ideal-based zero-divisor graph of a commutative ring, *Comm. Algebra*, **31** (2003), 4425-4443.
12. K. Samei, The zero-divisor graph of a reduced ring, *J. Pure Appl. Algebra*, **209** (2007), 813-821.
13. T. Tamizh Chelvam and S. Nithya, Zero-divisor graph of an ideal of a near-ring, *Discrete Math. Algorithms Appl.*, to appear.
14. S. Veldsman, On equiprime near-rings, *Comm. Algebra* **20(9)** (1992), 2569-2587.

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