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SPECTRAL PROBLEMS OF NONSELFADJOINT 1D SINGULAR HAMILTONIAN SYSTEMS

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Abstract. In this paper, the maximal dissipative one dimensional singular Hamiltonian operators (in limit-circle case at singular end point b) are considered in the Hilbert space $\mathcal{L}_W^2([a,b);\mathbb{C}^2)(-\infty < a < b \leq \infty)$. The maximal dissipative operators with general boundary conditions are investigated. A selfadjoint dilation of the dissipative operator and its incoming and outgoing spectral representations are constructed. These representations allows us to determine the scattering matrix of the dilation. Further a functional model of the dissipative operator is constructed and its characteristic function in terms of the scattering matrix of dilation is considered. Finally, the theorem on completeness of the system of root vectors of the dissipative operators is proved.

1. Introduction

One of the methods of the spectral analysis of dissipative operators is the functional model theory that is an application of dilation theory [1-6, 15, 17-18]. This theory is associated with the characteristic function. The spectral analysis of the dissipative operators can be studied with the help of the characteristic function. Using the theory of Sz.-Nagy-Foiaş, the dissipative operator cen be handled as the model operator [15, 17, 18]. The factorization of the characteristic function may help us to learn that whether the system of root vectors is complete or not in some Hilbert space. The characteristic function is established by the selfadjoint dilation of the dissipative operator and corresponding scattering matrix [13]. The spectral analysis of the dissipative Sturm-Liouville and Dirac-type operators are investigated in [1-6, 17-18].

In this paper, the minimal symmetric one dimensional singular differential Hamiltonian (or Dirac-type) operator $T_{\min}$ with defect index $(2,2)$ (in Weyl's limit-circle case

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at singular end point b) is considered in the Hilbert space $\mathcal{L}_W^2([a,b);\mathbb{C}^2)$ $(-\infty < a < b \leq \infty)$. All maximal dissipative and selfadjoint extensions of such a symmetric operator are described with the help of the boundary conditions at the end points a and b. The maximal dissipative operators with general (coupled, and separated) boundary conditions are investiged. If we consider separated boundary conditions, the nonselfadjoint (dissipative) boundary conditions at the end points a and b are prescribed similarly. At first a selfadjoint dilation of the maximal dissipative operator is constructed and then its incoming and outgoing spectral representations are prescribed. With these representations determining the scattering matrix of the dilation is possible [13]. Then the model of the maximal dissipative operator is constructed and we define its characteristic function in terms of the scattering matrix of dilation. Finally, a theorem about completeness of the system of eigenvectors and associated vectors (or root vectors) of the maximal dissipative Hamiltonian operators is proved.

2. Preliminaries

One dimensional differential Hamiltonian (or Dirac-type) system is considered as with the singular end point b

$$\tau_1(y) := J\frac{dy(x)}{dx} + Q(x)y(x) = \lambda W(x)y(x),\ x \in \mathbb{I} := [a,b)\,. \tag{2.1}$$

Here λ is a complex spectral parameter,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$
$$W(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix},\ Q(x) = \begin{pmatrix} q(x) & r(x) \\ r(x) & p(x) \end{pmatrix},$$

$W(x) > 0$ (for almost all $x \in \mathbb{I}$); entries of the matrices $W(x)$ and $Q(x)$ are real-valued, Lebesgue measurable and locally integrable functions on $\mathbb{I}$.

Let $\mathfrak{H} := \mathcal{L}_W^2(\mathbb{I};E)$ $(E{:=}\,\mathbb{C}^2)$ be the Hilbert space consisting of all vector-valued functions y with values in E such that

$$\int_a^b (W(x)y(x),y(x))_E dx < +\infty,$$

and with the inner product $(y,z) := \int_a^b (W(x)y(x),z(x))_E dx$. This space allows us to pass to the operators from the differential expression $\tau(y) := W^{-1}\tau_1(y)$.

Consdier the set

$$\mathcal{D}_{\max} = \left\{ y \in \mathfrak{H} : y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ \begin{array}{l} y_1, y_2 \in AC_{loc}(\mathbb{I}), \\ \tau(y) \in \mathfrak{H} \end{array} \right\},$$

where $AC_{loc}(\mathbb{I})$ denotes all locally absolutely continuous functions on $\mathbb{I}$. The operator $T_{\max}$ is defined on $\mathcal{D}_{\max}$ as $T_{\max}y = \tau(y)$.

For all vectors $y, z \in \mathcal{D}_{\max}$, Green's formula is

$$(T_{\max}y, z) - (y, T_{\max}z) = [y, z](b) - [y, z](a), \tag{2.2}$$

where $[y, z](x) := \mathcal{W}(y, \bar{z})(x) := y_1(x)\bar{z}_2(x) - y_2(x)\bar{z}_1(x)$, $x \in \mathbb{I}$, $[y, z](b) := \lim_{x\to b^-} [y, z](x)$.

In $\mathfrak{H}$ let us consider the dense linear set $\mathcal{D}'_{\min}$ that consists of smooth, compactly supported vector-valued functions on $\mathbb{I}$. Let $T'_{\min}$ denote the restriction of the operator $T_{\max}$ to $\mathcal{D}'_{\min}$. From (2.2) one gets that $T'_{\min}$ is symmetric. This admits the closure which we denote it by $T_{\min}$. The domain of $T_{\min}$ consists of precisely those vectors $y \in \mathcal{D}_{\max}$ satisfying the conditions $y_1(a) = y_2(a) = 0, [y, z](b) = 0,$ for arbitrary $z \in \mathcal{D}_{\max}$. $T_{\min}$ is a symmetric operator with defect index $(1, 1)$ or $(2, 2)$ (see [7, 9-12, 14, 20-24]), and $T_{\max} = T^*_{\min}$. The operators $T_{\min}$ and $T_{\max}$ are called the *minimal* and *maximal operators,* respectively.

We remind that the linear operator S with dense domain $\mathcal{D}(S)$ acting in some Hilbert space $\mathcal{H}$ is called *dissipative (accretive)* if $\mathrm{Im}(Sf, f) \geq 0(\mathrm{Im}(Sf, f) \leq 0)$ for all $f \in \mathcal{D}(S)$ and *maximal dissipative* (*maximal accretive*) if it does not have a proper dissipative (accretive) extension.

In the case of the defect index of $T_{\min}$ is $(1, 1)$ which is Weyl's limit-point case, all maximal dissipative (maximal accretive) extensions T_α of the symmetric operator $T_{\min}$ are described with the boundary conditions ($y \in \mathcal{D}_{\max}$): $y_2(a) - \alpha y_1(a) = 0$, where $\mathrm{Im}\alpha \geq 0$ or $\alpha = \infty$ ($\mathrm{Im}\alpha \leq 0$ or $\alpha = \infty$). In the case of $\mathrm{Im}\alpha = 0$ or $\alpha = \infty$, the selfadjoint extensions of $T_{\min}$ are obtained. For $\alpha = \infty$, the corresponding boundary condition has the form $y_1(a) = 0$. The maximal dissipative operators T_α with $\mathrm{Im}\alpha > 0$ are investigated in [2].

Throughtout the paper it is assumed that $T_{\min}$ has defect index $(2, 2)$, that is, the Weyl's limit-circle case holds for the Hamiltonian system (2.1) (see [7, 9-12, 14, 20-24]).

Let $\varphi(x)$ and $\psi(x)$ be the solutions of the system

$$\tau(y) = 0,\ x \in \mathbb{I} \tag{2.3}$$

satisfying the conditions

$$\varphi_1(a) = 1, \varphi_2(a) = 0, \psi_1(a) = 0, \psi_2(a) = 1. \tag{2.4}$$

It is known that the Wronskian of the two solutions of (2.3) does not depend on x, and the two solutions of this system are linearly independent if and only if their Wronskian is nonzero. Hence from the conditions (2.4) and the constancy of the Wronskian one gets for $a \leq x \leq b$ that

$$\mathcal{W}(\varphi, \psi)(x) = \mathcal{W}(\varphi, \psi)(a) = 1. \tag{2.5}$$

Hence, φ and ψ form a fundamental system of solutions of (2.3). Since $T_{\min}$ has defect index (2, 2), $\varphi, \psi \in \mathfrak{H}$. Further φ and ψ belong to $\mathcal{D}_{\max}$.

For all $y, z \in \mathcal{D}_{\max}$, the following equality is obtained ([5])

$$[y, z](x) = [y, \varphi](x) [\bar{z}, \psi](x) - [y, \psi](x) [\bar{z}, \varphi](x), \ a \le x \le b. \tag{2.6}$$

The domain $\mathcal{D}_{\min}$ of the operator $T_{\min}$ consists of the vectors $y \in \mathcal{D}_{\max}$ satisfying the conditions ([5])

$$y_1(a) = y_2(a) = 0, [y, \varphi](b) = [y, \psi](b) = 0. \tag{2.7}$$

Let Υ_1 and Υ_2 be the linear mappings from $\mathcal{D}_{\max}$ into E as

$$\Upsilon_1 y = \begin{pmatrix} -y_1(a) \\ [y, \varphi](b) \end{pmatrix}, \ \Upsilon_2 y = \begin{pmatrix} y_2(a) \\ [y, \psi](b) \end{pmatrix}. \tag{2.8}$$

Then we have (see [5]);

Theorem 2.1. *For arbitrary contraction L in E the restriction of the operator $T_{\max}$ to the set of vectors $y \in \mathcal{D}_{\max}$ satisfying the boundary condition*

$$(L - I)\Upsilon_1 y + i(L + I)\Upsilon_2 y = 0 \tag{2.9}$$

or

$$(L - I)\Upsilon_1 y - i(L + I)\Upsilon_2 y = 0 \tag{2.10}$$

is, respectively, a maximal dissipative or a maximal accretive extension of the operator $T_{\min}$. Conversely, every maximal dissipative (maximal accretive) extension of $T_{\min}$ is the restriction of $T_{\max}$ to the set of vectors $y \in \mathcal{D}_{\max}$ satisfying (2.9) ((2.10)), and the contraction L is uniquely determined by the extensions. These conditions give a selfadjoint extension if and only if L is unitary. In the latter case, (2.9) and (2.10) are equivalent to the condition $(\cos S)\Upsilon_1 y - (\sin S)\Upsilon_2 y = 0$, where S is a selfadjoint operator (Hermitian matrix) in E.

In particular, the boundary conditions

$$y_2(a) - \beta_1 y_1(a) = 0 \tag{2.11}$$

$$[y, \psi](b) + \beta_2 [y, \varphi](b) = 0 \tag{2.12}$$

with $\mathrm{Im}\beta_1 \ge 0$ or $\beta_1 = \infty$, and $\mathrm{Im}\beta_2 \ge 0$ or $\beta_2 = \infty$ ($\mathrm{Im}\beta_1 \le 0$ or $\beta_1 = \infty$, and $\mathrm{Im}\beta_2 \le 0$ or $\beta_2 = \infty$) describe all maximal dissipative (maximal accretive) extensions of $T_{\min}$ with separated boundary conditions. The selfadjoint extensions of $T_{\min}$ are obtained precisely when $\mathrm{Im}\beta_1 = 0$ or $\beta_1 = \infty$ and $\mathrm{Im}\beta_2 = 0$ or $\beta_2 = \infty$. Here, *for* $\beta_1 = \infty$ $(\beta_2 = \infty)$, *condition* (*2.11*) ((*2.12*)) *should be replaced by* $y_1(a) = 0$ $([y, \varphi](b) = 0)$.

In the following we consider the maximal dissipative operator T_L, where L satisfies the inequality $\|L\|_E < 1$, generated by the expression τ and boundary condition (2.9).

The condition $\|L\|_E < 1$ implies that the operator $L + I$ must be invertible. Moreover (2.9) is equivalent to the condition

$$\Upsilon_2 y + B\Upsilon_1 y = 0, \tag{2.13}$$

where $B = -i(L+I)^{-1}(L-I)$, $\mathrm{Im}B \geq 0$, and $-L$ is the Cayley transform of the dissipative operator B. We denote by $\widetilde{T}_B (= T_L)$ the maximal dissipative operator generated by the expression τ and the boundary condition (2.13).

Let

$$B = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix},$$

where $\mathrm{Im}\beta_1 > 0$, $\mathrm{Im}\beta_2 > 0$. Then the boundary condition (2.13) coincides with the separated boundary conditions (2.11) and (2.12).

3. Selfadjoint Dilation of the Maximal Dissipative Operator

The useful method to investigate the spectral analysis of the maximal dissipative operators is about the construction a selfadjoint dilation of the maximal dissipative operators belongs to Sz.-Nagy-Foiaş ([15]). In the literature there are a lot of works that contains this method. For this aim, let us add the space $\mathfrak{H}$ to the 'incoming' and 'outgoing' subspaces $\mathcal{L}^2(\mathbb{R}_-; E)$ $(\mathbb{R}_- := (-\infty, 0])$ and $\mathcal{L}^2(\mathbb{R}_+; E)$ $(\mathbb{R}_+ := [0, \infty))$ of the Hilbert space $\mathbf{H} = \mathcal{L}^2(\mathbb{R}_-; E) \oplus \mathfrak{H} \oplus \mathcal{L}^2(\mathbb{R}_+; E)$ called the *main Hilbert space of the dilation*. Clearly the elements of $\mathbf{H}$ are three-component vector-valued functions $h = \langle \theta_-, y, \theta_+ \rangle$.

Now let us consider the following mappings;

$$\mathcal{P}: \quad \mathbf{H} \to \mathfrak{H}, \quad \langle \theta_-, y, \theta_+ \rangle \to y, \quad \mathcal{P}_1: \quad \mathfrak{H} \to \mathbf{H}, \quad y \to \langle 0, y, 0 \rangle$$

and

$$\mathcal{P}^+: \quad \mathbf{H} \to \mathcal{L}^2(\mathbb{R}_+; E), \quad \langle \theta_-, y, \theta_+ \rangle \to \theta_+,$$

$$\mathcal{P}_1^+: \quad \mathcal{L}^2(\mathbb{R}_+; E) \to \mathbf{H}, \quad \theta \to \langle 0, 0, \theta \rangle.$$

Let us consider the operator $\mathbf{T}_B$ in the main Hilbert space $\mathbf{H}$ generated by differential expression

$$\mathbf{T}\langle \theta_-, y, \theta_+ \rangle = \langle i\frac{d\theta_-}{d\xi}, \tau(y), i\frac{d\theta_+}{d\zeta} \rangle \tag{3.1}$$

on the set of elements $\mathcal{D}(\mathbf{T}_B)$ satisfying the boundary conditions

$$\Upsilon_2 y + B\Upsilon_1 y = A\theta_-(0)\,,\ \Upsilon_2 y + B^*\Upsilon_1 y = A\theta_+(0)\,, \tag{3.2}$$

where $\theta_- \in W_2^1(\mathbb{R}_-;E)$, $\theta_+ \in W_2^1(\mathbb{R}_+;E)$, $y \in \mathcal{D}_{\max}$, $A^2 := 2\mathrm{Im}B$, $A > 0$, and $W_2^1(\mathbb{R}_\mp)$ is the Sobolev space.

One gets the following;

Theorem 3.1. *The operator $\mathbf{T}_B$ is selfadjoint in $\mathbf{H}$. Further $\mathbf{T}_B$ is a selfadjoint dilation of the maximal dissipative operator $\widetilde{T}_B\,(=T_L)$.*

Proof. For $h = \langle\theta_-, y, \theta_+\rangle, g = \langle\vartheta_-, z, \vartheta_+\rangle \in \mathcal{D}(\mathbf{T}_B)$ we get that

$$\begin{aligned}(\mathbf{T}_B h, g)_{\mathbf{H}} - (h, \mathbf{T}_B g)_{\mathbf{H}} &= i(\theta_-(0)\,, \vartheta_-(0))_E - i(\theta_+(0)\,, \vartheta_+(0))_E \\ &\quad + [y,z](b) - [y,z](a).\end{aligned} \tag{3.3}$$

Further using (3.2) and (2.6), one gets that

$$i(\theta_-(0)\,, \vartheta_-(0))_E - i(\theta_+(0)\,, \vartheta_+(0))_E + [y,z](b) - [y,z](a) = 0.$$

This implies that the operator $\mathbf{T}_B$ is symmetric, and $\mathcal{D}(\mathbf{T}_B) \subseteq \mathcal{D}(\mathbf{T}_B^*)$.

$\mathbf{T}_B$ and $\mathbf{T}_B^*$ are generated by the same expression (3.1). This can be seen with a direct calculation. Now we shall form the domain of $\mathbf{T}_B^*$. We shall compute the terms outside the integral sign, which are obtained by integration by parts in bilinear form $(\mathbf{T}_B h, g)_{\mathbf{H}}, h \in \mathcal{D}(\mathbf{T}_B), g \in \mathcal{D}(\mathbf{T}_B^*)$. Their sum is equal to zero:

$$[y,z](b) - [y,z](a) + i(\theta_-(0)\,, \vartheta_-(0))_E - i(\theta_+(0)\,, \vartheta_+(0))_E = 0. \tag{3.4}$$

Now solving the boundary conditions (3.2) for $\Upsilon_1 y$ and $\Upsilon_2 y$, we obtain that

$$\Upsilon_1 y = -iA^{-1}(\theta_-(0) - \theta_+(0)), \Upsilon_2 y = A\theta_-(0) + iBA^{-1}(\theta_-(0) - \theta_+(0)).$$

From (2.6) and (2.8), one gets that (3.4) is equivalent to the equality

$$\begin{aligned}&i(\theta_+(0)\,, \vartheta_+(0))_E - i(\theta_-(0)\,, \vartheta_-(0))_E = [y,z](b) - [y,z](a) \\ &= [y,\varphi](b)\,[\bar{z},\psi](b) - [y,\psi](b)\,[\bar{z},\varphi](b) - [y,\varphi](a)\,[\bar{z},\psi](a) \\ &\quad + [y,\psi](a)\,[\bar{z},\varphi](a) = (\Upsilon_1 y, \Upsilon_2 z)_E - (\Upsilon_2 y, \Upsilon_1 z)_E \\ &= -i(A^{-1}(\theta_-(0) - \theta_+(0)), \Upsilon_2 z)_E - (A\theta_-(0)\,, \Upsilon_1 z)_E \\ &\quad -i(BA^{-1}(\theta_-(0) - \theta_+(0)), \Upsilon_1 z)_E.\end{aligned}$$

Since the values $\theta_\pm(0)$ are arbitrary, we get from comparing the coefficients of $(\theta_\pm)_i(0)$ $(i = 1,2)$ that

$$\Upsilon_2 z + B\Upsilon_1 z = A\vartheta_-(0)\,, \Upsilon_2 z + B^*\Upsilon_1 z = A\vartheta_+(0)\,,$$

and this proves that the vector $g = \langle \vartheta_-, z, \vartheta_+ \rangle$ satisfies the boundary conditions (3.2). Hence $\mathcal{D}(\mathbf{T}_B^*) \subseteq \mathcal{D}(\mathbf{T}_B)$, and $\mathbf{T}_B = \mathbf{T}_B^*$.

Now let us consider the operator $\mathcal{Z}_t = \mathcal{P}\mathcal{U}_t\mathcal{P}_1, t \geq 0$, where $\mathcal{U}_t := \exp[i\mathbf{T}_B t]$ $(t \in \mathbb{R} := (-\infty, \infty))$ is the unitary group on $\mathbf{H}$. It is known that the operator family $\{\mathcal{Z}_t\}_{t\geq 0}$ is the strictly continuous semigroup of completely nonunitary contractions on $\mathfrak{H}$ ([15-18]). Let C_B be the generator of this semigroup: $C_B y = \lim_{t\to+0}(it)^{-1}(\mathcal{Z}_t y - y)$. It is clear that the domain of C_B consists of all vectors for which the limit exists and C_B is maximal dissipative. The operator $\mathbf{T}_B$ is called the *selfadjoint dilation* of C_B ([15-18]). We shall show that $\widetilde{T}_B = C_B$.

Now let us consider the equality

$$\mathcal{P}(\mathbf{T}_B - \lambda I)^{-1}\mathcal{P}_1 y = (\widetilde{T}_B - \lambda I)^{-1} y, y \in \mathfrak{H}, \ \mathrm{Im}\lambda < 0. \tag{3.5}$$

Let $(\mathbf{T}_B - \lambda I)^{-1}\mathcal{P}_1 y = f = \langle \vartheta_-, z, \vartheta_+ \rangle$. Then the equialities $(\mathbf{T}_B - \lambda I) f = \mathcal{P}_1 y$, and $\tau(z) - \lambda z = y$, $\vartheta_-(\xi) = \vartheta_-(0)\, e^{-i\lambda\xi}$, $\vartheta_+(\zeta) = \vartheta_+(0)\, e^{-i\lambda\zeta}$ hold. Since $f \in \mathcal{D}(\mathbf{T}_B)$, hence $\vartheta_- \in W_2^1(\mathbb{R}_-; E)$ and so $\vartheta_-(0) = 0$. This implies that z satisfies the boundary condition $\Upsilon_2 z + B\Upsilon_1 z = 0$. Therefore, $z \in \mathcal{D}(\widetilde{T}_B)$, and since a point λ with $\mathrm{Im}\lambda < 0$ cannot be an eigenvalue of a dissipative operator, then $z = \mathcal{R}_\lambda(\widetilde{T}_B) y := (\widetilde{T}_B - \lambda I)^{-1} y$. Hence for $y \in \mathfrak{H}$ and $\mathrm{Im}\lambda < 0$ we have

$$(\mathbf{T}_B - \lambda I)^{-1}\mathcal{P}_1 y = \langle 0, \mathcal{R}_\lambda(\widetilde{T}_B) y, \ A^{-1}(\Upsilon_2 y + B^*\Upsilon_1 y) e^{-i\lambda\xi} \rangle.$$

Applying the mapping $\mathcal{P}$ to this equality, we obtain (3.5) and

$$\begin{aligned} \mathcal{R}_\lambda(\widetilde{T}_B) &= \mathcal{P}(\mathbf{T}_B - \lambda I)^{-1}\mathcal{P}_1 = -i\mathcal{P}\int_0^\infty \mathcal{U}_t e^{-i\lambda t} dt \mathcal{P}_1 \\ &= -i\int_0^\infty \mathcal{Z}_t e^{-i\lambda t} dt = (C_B - \lambda I)^{-1}, \ \mathrm{Im}\lambda < 0. \end{aligned}$$

This implies that $\widetilde{T}_B = C_B$ and the theorem is proved.

4. Scattering Theory of the Dilation, Functional Model and Completeness Theorem for the System of Root Vectors of the Dissipative Operator

Lax and Phillips constructed their scattering theory [13] in the decomposition of the main Hilbert space $\mathbf{H} = \mathfrak{D}_- \oplus \mathfrak{H} \oplus \mathfrak{D}_+$ in which the unitary group $\{\mathcal{U}_t\}$ $(t \in \mathbb{R})$ has the following properties

(1) $\mathcal{U}_t\mathfrak{D}_- \subset \mathfrak{D}_-, t \leq 0$; $\mathcal{U}_t\mathfrak{D}_+ \subset \mathfrak{D}_+, t \geq 0$;

(2) $\bigcap_{t\leq 0}\mathcal{U}_t\mathfrak{D}_- = \bigcap_{t\geq 0}\mathcal{U}_t\mathfrak{D}_+ = \{0\}$;

(3) $\overline{\bigcup_{t\geq 0}\mathcal{U}_t\mathfrak{D}_-} = \overline{\bigcup_{t\leq 0}\mathcal{U}_t\mathfrak{D}_+} = \mathbf{H}$;

(4) $\mathfrak{D}_- \perp \mathfrak{D}_+$.

Let $\mathfrak{D}_- = \langle \mathcal{L}^2(\mathbb{R}_-;E), 0, 0\rangle$ and $\mathfrak{D}_+ = \langle 0, 0, \mathcal{L}^2(\mathbb{R}_+;E)\rangle$. We shall show that the properties (**1**)-(**4**) are satisfied. Inner product in the main Hilbert space $\mathbf{H}$ shows that the property (**4**) is satisfied.

To show that the property (**1**) is satisfied for $\mathfrak{D}_+$ (for $\mathfrak{D}_-$, the proof is analogous), let $\mathcal{R}_\lambda = (\mathbf{T}_B - \lambda I)^{-1}$. Then for all λ with Im$\lambda < 0$ and for all $h = \langle 0, 0, \theta_+\rangle \in \mathfrak{D}_+$ we get that

$$\mathcal{R}_\lambda h = \langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}\theta_+(s)\,ds\rangle.$$

This implies that $\mathcal{R}_\lambda h \in \mathfrak{D}_+$. Hence for $g \perp \mathfrak{D}_+$ one obtains

$$0 = (\mathcal{R}_\lambda h, g)_{\mathbf{H}} = -i\int_0^\infty e^{-i\lambda t}(\mathcal{U}_t h, g)_{\mathbf{H}} dt, \mathrm{Im}\lambda < 0.$$

We get from the last equality that $(\mathcal{U}_t h, g)_{\mathbf{H}} = 0$ for all $t \geq 0$. Hence for $t \geq 0$, $\mathcal{U}_t\mathfrak{D}_+ \subset \mathfrak{D}_+$. This proves the property (**1**) is satisfied.

For proving the property (**2**) consider $\mathcal{U}_t^+ = \mathcal{P}^+\mathcal{U}_t\mathcal{P}_1^+, t \geq 0$. It is known that the semigroup of isometries $\mathcal{U}_t^+ = \mathcal{P}^+\mathcal{U}_t\mathcal{P}_1^+, t \geq 0$ is the one-sided shift in $\mathcal{L}^2(\mathbb{R}_+;E)$. In fact, the generator of the semigroup of the shift $\mathcal{V}_t$ in $\mathcal{L}^2(\mathbb{R}_+;E)$ is the differential operator $i(d/d\sigma)$ with the boundary condition $v(0) = 0$. On the other hand, the generator $\mathcal{A}$ of semigroup of isometries $\mathcal{U}_t^+, t \geq 0$ is given by $\mathcal{A}\varphi = \mathcal{P}^+\mathcal{L}_B\mathcal{P}_1^+ v = \mathcal{P}^+\mathcal{L}_B\langle 0, 0, v\rangle = \mathcal{P}^+\langle 0, 0, i(dv/d\sigma)\rangle = i(dv/d\sigma)$, where $v \in W_2^1(\mathbb{R}_+;E)$ and $v(0) = 0$. Since a semigroup is uniquely determined by its generator, $\mathcal{U}_t^+ = \mathcal{V}_t$, and

$$\bigcap_{t\geq 0}\mathcal{U}_t\mathfrak{D}_+ = \langle 0, 0, \bigcap_{t\geq 0}\mathcal{V}_t\mathcal{L}^2(\mathbb{R}_+;E) = \{0\}.$$

This proves that the property (**2**) is satisfied.

According to the Lax-Phillips scattering theory, the scattering matrix is defined in terms of the theory of spectral representations. Using these representations we will have also proved property (**3**) of the incoming and outgoing subspaces.

The linear operator S (with domain $\mathcal{D}(S)$) acting in the Hilbert space H is called *completely nonselfadjoint* (or *simple*) if invariant subspace $K \subseteq \mathcal{D}(S)$ ($K \neq \{0\}$) of the operator S on which restriction S on K is selfadjoint does not exist.

Lemma 4.1. *The operator $\widetilde{T}_B$ is completely nonselfadjoint (simple).*

Proof. Let $\widetilde{T}_B'$ is the selfadjoint part of $\widetilde{T}_B$ in $\mathfrak{H}_0 \subset \mathfrak{H}$. The subspace $\mathfrak{H}_0$ is invariant with respect to semigroup of isometries $Z_t = \exp(i\widetilde{T}_B' t)$ ($Z_t^* = \exp(-i\widetilde{T}_B' t)$, $Z_t^{-1} = Z_t^*, t > 0$). Let $f \in \mathfrak{H}_0 \cap \mathcal{D}(\widetilde{T}_B)$. Then $f \in \mathcal{D}(\widetilde{T}_B^*)$, and $\Upsilon_2 f + B\Upsilon_1 f = 0, \Upsilon_2 f + B^*\Upsilon_1 f = 0$. Hence one gets that $\Upsilon_1 f = \Upsilon_2 f = 0$. For eigenvectors $z_\lambda \in \mathfrak{H}_0$ of the operator $\widetilde{T}_B$, we have $z_{1\lambda}(a) = 0$, $z_{2\lambda}(a) = 0$. The uniqueness theorem of the Cauchy problem for the system $\tau(z) = \lambda z$ $(x \in \mathbb{I})$ implies that $z_\lambda \equiv 0$. Since all solutions of $\tau(z) = \lambda z$ $(x \in \mathbb{I})$ belong to $\mathcal{L}_W^2(\mathbb{I};E)$, it can be concluded that the resolvent $\mathcal{R}_\lambda(\widetilde{T}_B)$ of the operator $\widetilde{T}_B$ is a compact operator, and the spectrum of $\widetilde{T}_B$ is purely discrete. Hence, by the theorem on expansion in eigenvectors of the selfadjoint operator $\widetilde{T}'_B$, we have $\mathfrak{H}_0 = \{0\}$. Hence the operator $\widetilde{T}_B$ is simple. The lemma is proved.

Now to prove the property **(3)** consider the equalities

$$\mathbf{H}_- = \overline{\underset{t\geq 0}{\cup}\mathcal{U}_t\mathfrak{D}_-},\ \mathbf{H}_+ = \overline{\underset{t\leq 0}{\cup}\mathcal{U}_t\mathfrak{D}_+}.$$

Lemma 4.2. *The equality* $\mathbf{H}_-\dot{+}\mathbf{H}_+ = \mathbf{H}$ *holds.*

Proof. Using the property **(1)** of the subspaces $\mathfrak{D}_\pm$, we shall show that the subspace $\mathbf{H}' = \mathbf{H}\ominus(\mathbf{H}_-\dot{+}\mathbf{H}_+)$ is invariant with respect to the group $\{\mathcal{U}_t\}$. $\mathbf{H}'$ has the form $\mathbf{H}' = \langle 0, \mathfrak{H}', 0\rangle$, where $\mathfrak{H}'$ is a subspace of $\mathfrak{H}$. Let the subspace $\mathbf{H}'$ (and hence also $\mathfrak{H}'$) be nontrivial and let the unitary group $\{\mathcal{U}'_t\}$ restricted to this subspace be a unitary part of the group $\{\mathcal{U}_t\}$. This implies that the restriction $\widetilde{T}'_B$ of the operator $\widetilde{T}_B$ to $\mathfrak{H}'$ is the selfadjoint operator in $\mathfrak{H}'$. From the simplicity of the operator $\widetilde{T}'_B$ this shows that $\mathfrak{H}' = \{0\}$, i.e. $\mathbf{H}' = \{0\}$. So the lemma is proved.

Let u and v be the solutions of the system $\tau(y) = \lambda y$ $(x \in \mathbb{I})$ satisfying the conditions

$$u_1(a,\lambda) = 0,\ u_2(a,\lambda) = -1,\ v_1(a,\lambda) = 1,\ v_2(a,\lambda) = 0. \tag{4.1}$$

Consider the matrix-valued function $\mathcal{M}(\lambda)$ satisfying the conditions

$$\mathcal{M}(\lambda)\Upsilon_1 u = \Upsilon_2 u,\ \mathcal{M}(\lambda)\Upsilon_1 v = \Upsilon_2 v. \tag{4.2}$$

With a direct calculation one can obtain that $\mathcal{M}(\lambda)$ has the form

$$\mathcal{M}(\lambda) = \begin{pmatrix} m_\infty(\lambda) & -\dfrac{1}{[u,\varphi](b)} \\ -\dfrac{1}{[u,\varphi](b)} & \dfrac{[u,\psi](b)}{[u,\varphi](b)} \end{pmatrix}, \tag{4.3}$$

where $m_\infty(\lambda)$ is the Weyl-Titchmarsh function of the selfadjoint operator T_∞ generated by the expression τ with the boundary conditions $y_1(a) = 0$ and $[y,\varphi](b) = 0$. Then we have

$$m_\infty(\lambda) = -\frac{[v,\varphi](b)}{[u,\varphi](b)}.$$

It is easy to show that the matrix-valued function $\mathcal{M}(\lambda)$ is meromorphic in $\mathbb{C}$ with all its poles on real axis $\mathbb{R}$, and that it has the following properties:

(i) $\mathrm{Im}\mathcal{M}(\lambda) \leq 0$ for $\mathrm{Im}\lambda > 0$, and $\mathrm{Im}\mathcal{M}(\lambda) \geq 0$ for $\mathrm{Im}\lambda < 0$;

(ii) $\mathcal{M}^*(\lambda) = \mathcal{M}(\bar{\lambda})$ for all $\lambda \in \mathbb{C}$, except the real poles of $\mathcal{M}(\lambda)$.

Let $\chi_j(x)$ and $\theta_j(x)$ $(j = 1, 2)$ be the solutions of the system $\tau(y) = \lambda y$ $(x \in \mathbb{I})$ satisfying the conditions

(4.4) $$\Upsilon_1 \chi_j = (\mathcal{M}(\lambda) + B)^{-1} A e_j, \Upsilon_1 \theta_j = (\mathcal{M}(\lambda) + B^*)^{-1} A e_j \ (j = 1, 2),$$

where e_1 and e_2 are the orthonormal basis for E.

Now consider the vectors $\mathcal{V}_{\lambda j}^-$ $(j = 1, 2)$ as

$$\mathcal{V}_{\lambda j}^-(x, \sigma, \zeta) = \langle e^{-i\lambda\sigma} e_j, \chi_j(x), A^{-1}(\mathcal{M} + B^*)(\mathcal{M} + B)^{-1} A e^{-i\lambda\zeta} e_j \rangle.$$

For all $\lambda \in \mathbb{R}$, the vectors $\mathcal{V}_{\lambda j}^-$ $(j = 1, 2)$ do not belong to $\mathbf{H}$. However, $\mathcal{V}_{\lambda j}^-$ $(j = 1, 2)$ satisfies the equation $\mathcal{L}\mathcal{V} = \lambda \mathcal{V}$ and the boundary conditions (3.2).

With the help of the vectors $\mathcal{V}_{\lambda j}^-$ $(j = 1, 2)$, let us consider the transformation $\Phi_- : h \to \tilde{h}_-(\lambda)$, where $h = \langle \theta_-, y, \theta_+ \rangle$, $(\Phi_- h)(\lambda) := \tilde{h}_-(\lambda) := \sum_{j=1}^{2} \tilde{h}_j^-(\lambda) e_j$, θ_-, θ_+, and y are smooth, compactly supported functions, and $\tilde{h}_j^-(\lambda) = \frac{1}{\sqrt{2\pi}}(h, \mathcal{V}_{\lambda j}^-)_{\mathbf{H}}$ $(j = 1, 2)$.

Lemma 4.3. $\mathbf{H}_-$ *is isometrically mapped by the transformation* Φ_- *onto* $\mathcal{L}^2(\mathbb{R}; E)$. *For all vectors* $h, g \in \mathbf{H}_-$, *the Parseval equality*

$$(h, g)_{\mathbf{H}} = (\tilde{h}_-, \tilde{g}_-)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{2} \tilde{h}_j^-(\lambda) \overline{\tilde{g}_j^-(\lambda)} d\lambda,$$

and the inversion formula

$$h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=1}^{2} \mathcal{V}_{\lambda j}^- \tilde{h}_j^-(\lambda) d\lambda,$$

hold, where $\tilde{h}_-(\lambda) = (\Phi_- f)(\lambda)$, $\tilde{g}_-(\lambda) = (\Phi_- g)(\lambda)$.

Proof. Let $H_\pm^2(E)$ denote the Hardy classes in $\mathcal{L}^2(\mathbb{R}; E)$ consisting of the vector-valued functions analytically extendable to the upper and lower half-planes, respectively. For $h, g \in \mathfrak{D}_-$, $h = \langle h_-, 0, 0 \rangle$, $g = \langle g_-, 0, 0 \rangle$, $h_-, g_- \in \mathcal{L}^2(\mathbb{R}_-; E)$, we have

$$\tilde{h}_j^-(\lambda) = \frac{1}{\sqrt{2\pi}}(h, \mathcal{V}_{\lambda j}^-)_{\mathbf{H}} = \frac{1}{2\pi} \int_{-\infty}^{0} \left(h_-(\sigma), e^{-i\lambda\sigma} e_j \right)_E d\sigma \in H_-^2,$$

$$\tilde{h}_-(\lambda) = \sum_{j=1}^{2} \tilde{h}_j^-(\lambda) e_j \in H_-^2(E),$$

and the Parseval equality:

$$(h, g)_{\mathbf{H}} = (\tilde{h}_-, \tilde{g}_-)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{2} \tilde{h}_j^-(\lambda) \overline{\tilde{g}_j^-(\lambda)} d\lambda.$$

We shall extend this equality to the all of the space $\mathbf{H}_-$. For this aim, let us consider in $\mathbf{H}_-'$ the dense set $\mathbf{H}_-$ of vectors, obtained on smooth, compactly supported functions belonging to $\mathfrak{D}_-$ by the following way: $h \in \mathbf{H}'_-, h = \mathcal{U}_{t_f} h_0, h_0 = \langle \theta_-, 0, 0 \rangle, \theta_- \in C_0^\infty(\mathbb{R}_-; E)$. For these vectors, noting $\mathbf{T}_B = \mathbf{T}_B^*$, and using the fact that $\mathrm{U}_{-t} h \in \langle C_0^\infty(\mathbb{R}_-; E), 0, 0 \rangle$, and $(\mathcal{U}_{-t} h, \mathcal{V}_{\lambda j}^-)_{\mathbf{H}} = e^{-i\lambda t} (h, \mathcal{V}_{\lambda j}^-)_{\mathbf{H}}\, (j = 1, 2)$ for $t > t_f, t_g$, we have

$$
\begin{aligned}
(h, g)_{\mathbf{H}} &= (\mathcal{U}_{-t} h, \mathcal{U}_{-t} g)_{\mathbf{H}} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^{2} (\mathcal{U}_{-t} h, \mathcal{V}_{\lambda j}^-)_{\mathbf{H}} \overline{(\mathcal{U}_{-t} g, \mathcal{V}_{\lambda j}^-)_{\mathbf{H}}} d\lambda = \int_{-\infty}^{\infty} \sum_{j=1}^{2} \tilde{h}_j^-(\lambda) \overline{\tilde{g}_j^-(\lambda)} d\lambda.
\end{aligned}
$$

Taking the closure, one obtains that the Parseval equality holds for all of the space $\mathbf{H}_-$. The inversion formula follows from the Parseval equality if all integrals in it are understood as limits in the mean of integrals over finite intervals. Finally, we have

$$
\Phi_- \mathbf{H}_- = \overline{\underset{t \geq 0}{\cup} \Phi_- \mathcal{U}_t \mathfrak{D}_-} = \overline{\underset{t \geq 0}{\cup} e^{i\lambda t} H_-^2(E)} = \mathcal{L}^2(\mathbb{R}; E).
$$

This implies that Φ_- maps $\mathbf{H}_-$ onto whole $\mathcal{L}^2(\mathbb{R}; E)$. So, the lemma is proved.

Now let us consider the vectors

$$
\mathcal{V}_{\lambda j}^+(x, \sigma, \zeta) = \langle S_B(\lambda) e^{-i\lambda\sigma} e_j, \theta_j(x), e^{-i\lambda\zeta} e_j \rangle \; (j = 1, 2),
$$

where

$$
S_B(\lambda) = A^{-1} (\mathcal{M}(\lambda) + B) (\mathcal{M}(\lambda) + B^*)^{-1} A. \tag{4.5}
$$

It should be noted that vectors $\mathcal{V}_{\lambda j}^+$ $(j = 1, 2)$ for all $\lambda \in \mathbb{R}$ do not belong to $\mathbf{H}$. However, $\mathcal{V}_{\lambda j}^+ (j = 1, 2)$ satisfies the equation $\mathbf{T}\mathcal{V} = \lambda \mathcal{V}$ and the boundary conditions (3.2).

Let $\Phi_+ : h \to \tilde{h}_+(\lambda)$ be the transformation as $(\Phi_+ h)(\lambda) := \tilde{h}_+(\lambda) := \sum_{j=1}^2 \tilde{f}_j^+(\lambda) e_j$, where $h = \langle \theta_-, y, \theta_+ \rangle$, θ_-, θ_+, and y are smooth, compactly supported functions, and $\tilde{h}_j^+(\lambda) = \frac{1}{\sqrt{2\pi}} (h, \mathcal{V}_{\lambda j}^+)_{\mathbf{H}}\; (j = 1, 2)$. The proof of the next result is analogous to that of Lemma 4.3.

Lemma 4.4. $\mathbf{H}_+$ *is isometrically mapped by the transformation* Φ_+ *onto* $\mathcal{L}^2(\mathbb{R}; E)$. *For all vectors* $h, g \in \mathbf{H}_+$, *the Parseval equality*

$$
(h, g)_{\mathbf{H}} = (\tilde{h}_+, \tilde{g}_+)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{2} \tilde{h}_j^+(\lambda) \overline{\tilde{g}_j^+(\lambda)} d\lambda,
$$

and the inversion formula

$$
h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=1}^{2} \mathcal{V}_{\lambda j}^+ \tilde{h}_j^+(\lambda) d\lambda,
$$

are valid, where $\tilde{h}_j^+(\lambda) = (\Phi_+ h)(\lambda)$, $\tilde{g}_+(\lambda) = (\Phi_+ g)(\lambda)$.

The matrix-valued function $S_B(\lambda)$ is meromorphic in $\mathbb{C}$ and all poles are in the lower half-plane. From (4.5) one can obtain that $\|S_B(\lambda)\|_E \leq 1$ for Im$\lambda > 0$ and $S_B(\lambda)$ is the unitary matrix for all $\lambda \in \mathbb{R}$. Since $S_B(\lambda)$ is the unitary matrix for $\lambda \in \mathbb{R}$, then, it follows from the definitions of the vectors $\mathcal{V}_{\lambda j}^{+}$ and $\mathcal{V}_{\lambda j}^{-}$ that

$$\mathcal{V}_{\lambda j}^{+} = \sum_{k=1}^{2} S_{jk}(\lambda)\, \mathcal{V}_{\lambda k}^{-}\, (j = 1, 2),$$

where $S_{jk}(\lambda)\,(j, k = 1, 2)$ are entries of the matrix $S_B(\lambda)$. This implies from Lemmas 4.3 and 4.4 that $\mathbf{H}_{-} = \mathbf{H}_{+}$. Together with Lemma 4.2, this shows that $\mathbf{H}_{-}$=$\mathbf{H}_{+}$=$\mathbf{H}$, and property **(3)** above has been established for the incoming and outgoing subspaces.

Thus, $\mathbf{H}$ is isometrically mapped by transformation Φ_{-} onto $\mathcal{L}^2(\mathbb{R}; E)$; the subspace $\mathfrak{D}_{-}$ is mapped onto $H_{-}^2(E)$, while the operators $\mathcal{U}_t$ go over into the operators of multiplication by $e^{i\lambda t}$. According to the Lax-Phillips scattering theory ([13]) Φ_{-} is an incoming spectral representation of the group $\{\mathcal{U}_t\}$. Similarly, Φ_{+} is an outgoing spectral representation of $\{\mathcal{U}_t\}$. From the explicit formulas for $\mathcal{V}_{\lambda j}^{-}$ and $\mathcal{V}_{\lambda j}^{+}$ $(j = 1, 2)$, it follows that the passage from the Φ_{-}-representation of a vector $h \in \mathbf{H}$ to its Φ_{+}-representation is accomplished as follows: $\tilde{h}_{+}(\lambda) = S_B^{-1}(\lambda)\tilde{h}_{-}(\lambda)$. Hence [13], we have now proved the following result.

Theorem 4.5. *The matrix* $S_B^{-1}(\lambda)$ *is the scattering matrix of the group* $\mathcal{U}_t$ *(of the operator* $\mathbf{T}_B$*).*

Remind that the analytic matrix-valued function $\mathcal{S}(\lambda)$ on the upper half-plane $\mathbb{C}_{+}$ is called *inner function* on $\mathbb{C}_{+}$ if $\|\mathcal{S}(\lambda)\| \leq 1$ for $\lambda \in \mathbb{C}_{+}$ and $\mathcal{S}(\lambda)$ is a unitary matrix for almost all $\lambda \in \mathbb{R}$. To sum up the equivalence of the characteristic function of Sz.-Nagy-Foiaş and the scattering function of Lax-Phillips, let $\mathcal{S}(\lambda)$ be an arbitrary nonconstant (matrix-valued) inner matrix-valued function [15] on the upper half-plane. Consider the space $\mathcal{K} = H_{+}^2 \ominus \mathcal{S}H_{+}^2$. It is known that $\mathcal{K} \neq \{0\}$ and is a subspace of the Hilbert space H_{+}^2. We consider the semigroup of the operators $\mathcal{Z}_t$ $(t \geq 0)$ acting in $\mathcal{K}$ according to the formula $\mathcal{Z}_t\varphi = \mathcal{P}\left[e^{i\lambda t}u\right]$, $u := u(\lambda) \in \mathcal{K}$, where $\mathcal{P}$ is the orthogonal projection from H_{+}^2 onto $\mathcal{K}$. The generator of the semigroup $\{\mathbf{Z}_t\}$ is denoted by $\mathcal{A} : \mathcal{A}\varphi = \lim_{t\to+0}(it)^{-1}(\mathcal{Z}_t\varphi - \varphi)$, which is a maximal dissipative operator acting in $\mathcal{K}$ with the domain $\mathcal{D}(\mathcal{A})$ consisting of all vectors $u \in \mathcal{K}$, such that the limit exists. The operator $\mathcal{A}$ is called a *model dissipative operator.* Here, this model dissipative operator is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş ([15-17]). The basic assertion is that $\mathcal{S}(\lambda)$ is the *characteristic function* of the operator $\mathcal{A}$.

Using the unitary transformation Φ_{-} we have:

$$\mathbf{H} \to \mathcal{L}^2(\mathbb{R}; E),\ h \to \tilde{h}_{-}(\lambda) = (\Phi_{-}h)(\lambda),\ \mathfrak{D}_{-} \to H_{-}^2(E),$$

$$\mathfrak{D}_{+} \to S_B H_{+}^2(E),\ \mathbf{H} \ominus (\mathfrak{D}_{-} \oplus \mathfrak{D}_{+}) \to H_{+}^2(E) \ominus S_B H_{+}^2(E),$$

$$\mathcal{U}_t h \to (\Phi_{-}\mathcal{U}_t\Phi_{-}^{-1}\tilde{h}_{-})(\lambda) = e^{i\lambda t}\tilde{h}_{-}(\lambda).$$

These mappings show that the operator $\widetilde{T}_B$ (T_L) is unitary equivalent to the model dissipative operator with the characteristic function $S_B(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operators coincide [15-18], we have proved the next theorem.

Theorem 4.6. *The characteristic function of the maximal dissipative operator* $\widetilde{T}_B$ (T_L) *coincides with the matrix-valued function* $S_B(\lambda)$ *determined by formula (4.5). The matrix-valued function* $S_B(\lambda)$ *is meromorphic in the complex plane* $\mathbb{C}$ *and is an inner function in the upper half-plane.*

Let $\mathcal{L}$ denote the linear operator acting in the Hilbert space $\mathcal{H}$ with the domain $\mathcal{D}(\mathcal{L})$. The complex number λ_0 is called an *eigenvalue* of the operator $\mathcal{L}$ if there exist a nonzero element $f_0 \in \mathcal{D}(\mathcal{L})$ such that $\mathcal{L}f_0 = \lambda_0 f_0$. Such element f_0 is called the *eigenvector* of the operator $\mathcal{L}$ corresponding to the eigenvalue λ_0. The elements $f_1, f_2, ..., f_k$ are called the *associated vectors* of the eigenvector f_0 if they belong to $\mathcal{D}(\mathcal{L})$ and $\mathcal{L}f_j = \lambda_0 f_j + f_{j-1}$, $j = 1, 2, ..., k$. The element $f \in \mathcal{D}(\mathcal{L})$, $f \neq 0$ is called a *root vector* of the operator $\mathcal{L}$ corresponding to the eigenvalue λ_0, if all powers of $\mathcal{L}$ are defined on this element and $(\mathcal{L} - \lambda_0 I)^n f = 0$ for some integer n. The set of all root vectors of $\mathcal{L}$ corresponding to the same eigenvalue λ_0 with the vector $f = 0$ forms a linear set $\mathcal{N}_{\lambda_0}$ and is called the root lineal. The dimension of the lineal $\mathcal{N}_{\lambda_0}$ is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal $\mathcal{N}_{\lambda_0}$ coincides with the linear span of all eigenvectors and associated vectors of $\mathcal{L}$ corresponding to the eigenvalue λ_0. Consequently, the completeness of the system of all eigenvectors and associated vectors of $\mathcal{L}$ is equivalent to the completeness of the system of all root vectors of this operator.

Sz.-Nagy-Foiaş proved a theorem on completeness of the system of all eigenvectors and associated (or root) vectors of dissipative operator. Showing the absence of the singular factor $s(\lambda)$ in the factorization $\det \tilde{S}_L(\lambda) = s(\lambda) B(\lambda)$ ($B(\lambda)$ is the Blaschke product) ensures the completeness of the system of eigenvectors and associated (or root) vectors of the operator $\widetilde{T}_B$ (T_L) in the space $\mathfrak{H}$ (see [8, 15-18]).

We first use the following result ([1]).

Lemma 4.7. *The characteristic function* $\tilde{S}_L(\lambda)$ *of the operator* T_L $(\widetilde{T}_B)$ *has the form*

$$\tilde{S}_L(\lambda) := S_B(\lambda) = Y_1 (I - L_1 L_1^*)^{-\frac{1}{2}} (\Psi(\sigma) - L_1)(I - L_1^* \Psi(\sigma))^{-1} (I - L_1 L_1^*)^{\frac{1}{2}} Y_2,$$

where $L_1 = -L$ *is the Cayley transformation of the dissipative operator* B, *and* $\Psi(\sigma)$ *is the Cayley transformation of the matrix-valued function* $\mathcal{M}(\lambda)$, $\sigma = (\lambda - i)(\lambda + i)^{-1}$ *and* $Y_1 := (\mathrm{Im} B)^{-\frac{1}{2}} (I - L_1)^{-1} (I - L_1 L_1^*)^{\frac{1}{2}}$, $Y_2 := (I - L_1^* L_1)^{-\frac{1}{2}} (I - L_1^*) (\mathrm{Im} T)^{\frac{1}{2}}$, $|\det Y_1| = |\det Y_2| = 1$.

In order that being the inner matrix-valued function $\tilde{S}_L(\lambda)$ is a Blaschke-Potopov product it is necessary and sufficient that $\det \tilde{S}_L(\lambda)$ is a Blaschke product ([8, 15-18]).

Hence from Lemma 4.7. that the characteristic function $\tilde{S}_L(\lambda)$ is a Blaschke-Potopov product if and only if the matrix-valued function

$$Y_{L_1}(\sigma) = (I - L_1 L_1^*)^{-\frac{1}{2}} (\Psi(\sigma) - L_1)(I - L_1^* \Psi(\sigma))^{-1} (I - L_1^* L_1)^{\frac{1}{2}}$$

is a Blaschke-Potopov product in a unit disk.

To prove a theorem on completeness, we shall formulate the definition of Γ-capacity in a form convenient for what follows (see [8, 19]).

Let $\mathbf{E}$ be an N-dimensional $(N < \infty)$ Euclidean space. In $\mathbf{E}$ we fix an orthonormal basis $e_1, e_2, ..., e_N$ and denote by $\mathbf{E}_n$ $(n = 1, 2, ..., N)$ the linear span vectors $e_1, e_2, ..., e_n$. If $\mathbf{K} \subset \mathbf{E}_n$, then the set of $v \in \mathbf{E}_{n-1}$ with the property $Cap\{\xi : \xi \in \mathbb{C}, (v + \xi e_n) \in \mathbf{K}\} > 0$ will be denoted by $\Gamma_{n-1}\mathbf{K}$. ($CapG$ is the inner logarithmic capacity of the set $G \subset \mathbb{C}$). The Γ-capacity of the set $\mathbf{K} \subset \mathbf{E}$ is a number Γ-$Cap\mathbf{K}$:$= \sup Cap\{\xi : \xi \in \mathbb{C},\ \xi e_1 \subset \Gamma_1\Gamma_2...\Gamma_{m-1}\mathbf{K}\}$, where the sup is taken with respect to all orthonormal bases in $\mathbf{E}$. It is known that every set $\mathbf{K} \subset \mathbf{E}$ of zero Γ-capacity has zero $2N$-dimensional Lebesgue measure (in the decomplexified space $\mathbf{E}$) ([8, 19]), however, the converse is false.

Denote by $[E]$ the set of all linear operators in $E\ (= \mathbb{C}^2)$. To convert $[E]$ into the 4-dimensional Euclidean space, we introduce the inner product $\langle A, C\rangle = trC^*A$, for $A, C \in [E]$ (trC^*A is the trace of the operator C^*A). Hence, we may introduce the Γ-capacity of a set of $[E]$.

We will utilize the following important result of [8].

Lemma 4.8. *Let $Y(\xi)\,(|\xi| \le 1)$ be a holomorphic function with the values to be contractive operators in $[E]$ (i.e. $\|Y(\xi)\| < 1$). Then for Γ-quasi every strictly contractive operators L in E (i.e., for all strictly contractive $L \in [E]$ with the possible exception of a set of Γ-capacity zero) the inner part of the contractive function*

$$Y_L(\xi) := (I - LL^*)^{-\frac{1}{2}} (Y(\xi) - L)(I - L^*Y(\xi))^{-1} (I - L^*L)^{\frac{1}{2}}$$

is a Blaschke-Potopov product.

Then by summing all obtained results for the maximal dissipative operators T_L $(\widetilde{T}_B)$, we have proved the following theorem.

Theorem 4.9. *For Γ-quasi-every strictly contractive $L \in [E]$, the characteristic function $\tilde{S}_L(\lambda)$ of the maximal dissipative operator T_L is a Blaschke-Potopov product, and the spectrum of T_L is purely discrete and belongs to the open upper half-plane. For Γ-quasi-every strictly contractive $L \in [E]$, the operator T_L has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit point at infinity, and the system of all eigenvectors and associated vectors (or root vectors) of this operator is complete in the space $\mathcal{L}^2_W(\mathbb{I}; E)$.*

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