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HÖLDER CONTINUITY OF THE SOLUTION MAP TO AN ELLIPTIC OPTIMAL CONTROL PROBLEM WITH MIXED CONSTRAINTS

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Abstract. The goal of the paper is to investigate the Hölder continuity of the solution map to a parametric optimal control problem which is governed by elliptic equations with mixed control-state constraints and convex cost functions. By reducing the problem to a programming problem and parametric variational inequality, we get sufficient conditions under which the solution map is Hölder continuous in parameters.

1. Introduction

Let Ω be a bounded domain in R^N with the Lipschitz boundary $\partial\Omega$ and $N \in \{2, 3\}$. We consider the following parametric optimal control problem for the elliptic equations with mixed control-state constraints:

Find a control function $u \in L^p(\Omega)$, $p \geq 2$ and a state $y \in H_0^1(\Omega) \cap C(\overline{\Omega})$ which minimize the cost

$$F(y, u, \mu) = \int_\Omega f(x, y(x), u(x), \mu(x))dx \tag{1}$$

with the state equation

$$\begin{cases} Ay = u + \lambda_1 & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

and pointwise constraints

$$\begin{cases} u \geq \lambda_2 & \text{in } \Omega \\ \epsilon u \geq \delta y + \lambda_3 & \text{in } \Omega, \end{cases} \tag{3}$$

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where $f : \Omega \times R \times R \times R^k \to R \cup \{+\infty\}$ with $k \geq 1$, is given function,

$$\mu = (\mu_1, \ldots, \mu_k) \in L^\infty(\Omega)^k, \lambda = (\lambda_1, \lambda_2, \lambda_3) \in L^p(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$$

are parameters, A denotes a second-order elliptic operator of the form

$$Ay(x) = -\sum_{i,j=1}^{N} D_j(a_{ij}(x) D_i y(x)) + a_0(x) y(x),$$

where coefficients $a_{ij} \in L^\infty(\Omega)$ satisfy the strongly elliptic condition

$$\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq \lambda_A |\xi|^2 \quad \forall \xi \in R^N, \text{a. e. } x \in \Omega$$

for some $\lambda_A > 0$ and $a_0 \in L^\infty(\Omega), a_0(x) \geq 0$ almost everywhere $x \in \Omega$, $\delta \in L^\infty(\Omega)$ and $\epsilon \in L^\infty(\Omega)$.

Let us put

$$Y = H_0^1(\Omega) \cap C(\overline{\Omega}), U = L^p(\Omega), Z = Y \times U$$

and

$$M = L^\infty(\Omega)^k, \Lambda = L^p(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega).$$

The norms of $y \in Y, \mu \in M$ and $\lambda \in \Lambda$ are defined by

$$\|y\|_Y = \|y\|_{H_0^1(\Omega)} + \|y\|_{C(\overline{\Omega})}, \qquad \|\mu\|_M = \max\{\|\mu_i\|_{L^\infty(\Omega)} : 1 \leq i \leq k\}$$
$$\text{and} \quad \|\lambda\|_\Lambda = \|\lambda_1\|_{L^p(\Omega)} + \|\lambda_2\|_{L^\infty(\Omega)} + \|\lambda_3\|_{L^\infty(\Omega)},$$

respectively.

In the sequel, we denote by B_X and $\overline{B}_X$ the open unit ball and the closed unit ball in a norm space X, respectively. Also, given $x \in X$ and $\delta > 0$, $B_X(x, \delta)$ and $\overline{B}_X(x, \delta)$ stand for an open ball and a closed ball, respectively with center x and radius δ.

Let us define a set-valued map $K : \Lambda \rightrightarrows Z$ by setting

$$K(\lambda) = \{z = (y, u) \in Y \times U | (2) \text{ and } (3) \text{ are satisfied}\}. \tag{4}$$

Then problem (1)-(3) can be formulated in the form

$$P(\mu, \lambda) \qquad \begin{cases} F(z, \mu) \to \inf \\ z \in K(\lambda). \end{cases}$$

We denote by $\mathcal{S}(\mu, \lambda)$ the solution set of $P(\mu, \lambda)$. In this paper, we always assume that $\mathcal{S}(\overline{\mu}, \overline{\lambda}) = \{\overline{z}\}$, that is, problem $P(\overline{\mu}, \overline{\lambda})$ has a unique solution $\overline{z} = \overline{z}(\overline{\mu}, \overline{\lambda}) = \big(\overline{y}(\overline{\mu}, \overline{\lambda}), \overline{u}(\overline{\mu}, \overline{\lambda})\big)$.

Our main concern is to investigate the behavior of $\mathcal{S}(\mu,\lambda)$ when (μ,λ) varies around $(\overline{\mu},\overline{\lambda})$. This problem interested some authors in the last decade. For papers which have a closed connection to the present work, we refer the readers to [2, 10, 15, 16] and the references given therein. When f is a quadratic function, that is

$$f(x,y,u,\mu)=\frac{1}{2}|y-y_d(x)|^2+\frac{\gamma}{2}|u-u_d(x)|^2-\mu_1 y-\mu_2 u, \tag{5}$$

where y_d and u_d are given in $L^2(\Omega)$, and $\gamma>0$ is a constant, [2, 10] and [15] showed that the solution map is singleton and Lipschitz continuous in parameters.

It is noted that the obtained result of [10] is for problem with pure state constraints, the obtained result of [15] is for problem with pure control constraints while the obtained result of [2] is for problem with mixed control-state constraints (3) with $\epsilon=\epsilon_0, \delta=-1$ and under additional condition that

$$\exists\sigma>0,\quad S_1^\sigma\cap S_2^\sigma=\emptyset, \tag{6}$$

where

$$S_1^\sigma:=\{x\in\Omega: 0\le\overline{u}_0(x)\le\sigma\},$$
$$S_2^\sigma:=\{x\in\Omega: 0\le\epsilon_0\overline{u}_0(x)+\overline{y}_0(x)-y_c(x)\le\sigma\}$$

with $y_c\in L^\infty(\Omega), (\overline{y}_0,\overline{u}_0)$ is a solution of $P(\overline{\mu},\overline{\lambda})$ corresponding to $\overline{\mu}=0$ and $\overline{\lambda}=(0,0,y_c)$.

In this paper we continue to develop results of [2] by considering problem (1)-(3) under weaker conditions and for a larger class of cost functions F, where the integrand function f is not necessary to be quadratic. Namely, by reducing (1)-(3) to a parametric variational inequality and using technique in [8] and [18], we will show that, under certain conditions but without condition (6), the solution map $\mathcal{S}$ of problem (1)-(3) is singleton and Hölder continuous in (μ,λ).

Let us recall some concepts which are related to our problems. Given a function $\phi\in L^2(\Omega)$, a function $y\in H_0^1(\Omega)$ is called a weak solution of the elliptic partial differential equation

$$\begin{cases} Ay=\phi & \text{in } \Omega,\\ y=0 & \text{on } \partial\Omega\end{cases} \tag{7}$$

if

$$\int_\Omega\Big(\sum_{i,j=1}^N a_{ij}(x)D_iy(x)D_jv(x)+a_0(x)y(x)v(x)\Big)dx=\int_\Omega\phi(x)v(x)dx\quad\forall v\in H_0^1(\Omega).$$

Given a Banach space E and a nonempty closed convex set K in E, the *normal cone* to K at a point $z_0\in Z$ is define by

$$N(z_0;K)=\{z^*\in E^*:\langle z^*,z-z_0\rangle\le 0,\forall z\in K\}.$$

For definition of normal cones and their properties, we refer the readers to [13, Chapter 4].

Let us impose the following conditions for problem (1)-(3).

$(A1)$ Ω is a bounded domain in $R^N, N \in \{2,3\}$, with the Lipschitz boundary $\partial\Omega$ and $\epsilon, \delta \in L^\infty(\Omega), \epsilon(x) \geq \epsilon_0 > 0,$ a.e. x in Ω.

$(A2)$ $f(\cdot, y, u, \mu)$ is measurable for all $(y, u, \mu) \in R \times R \times R^k$ and $f(x, \cdot, \cdot, \cdot)$ is continuous a.e. x in Ω. Besides, there exist a positive number ϵ_1 and a continuous nonnegative function $g : \overline{\Omega} \times R^3 \to R$ such that for all $(x, \mu) \in \Omega \times R^k$ with $|\mu - \overline{\mu}(x)| \leq \epsilon_1$, one has

$$\Big|f\big(x, \overline{y}(x), \overline{u}(x), \mu\big) - f\big(x, \overline{y}(x), \overline{u}(x), \overline{\mu}(x)\big)\Big| \leq g\big(x, |\overline{y}(x)|, |\mu|, |\overline{\mu}(x)|\big) H_1\big(|\overline{u}(x)|\big),$$

where $H_1(\cdot)$ is the following form

$$H_1(t) = \sum_{i=1}^{m_1} t^{s_i} \quad \text{with} \quad m_1 \geq 1, 0 \leq s_i \leq p, \forall i = \overline{1, m_1}.$$

$(A3)$ There exist constant numbers $\epsilon_2, \rho > 0$ such that for a. e. $x \in \Omega$ the function $(y, u) \mapsto f(x, y, u, \mu)$ is continuously differentiable and convex on subset $D(x)$ and the following condition holds

$$\big(f_z(x, z_1, \mu) - f_z(x, z_2, \mu)\big)(z_1 - z_2) \geq \rho |u_1 - u_2|^p$$

for all $z_i = (y_i, u_i) \in D(x)$ and for all $\mu \in R^k$ with $|\mu - \overline{\mu}(x)| \leq \epsilon_1$, where $D(x) = (\overline{y}(x) - \epsilon_2, \overline{y}(x) + \epsilon_2) \times R$.

$(A4)$ There exist continuous functions $a_i : \overline{\Omega} \times R^2 \to R, b_i : \overline{\Omega} \times R^3 \to R$ and positive numbers $\alpha_i, i = 1, 2$ such that

$$\big|f_y(x, y, u, \overline{\mu}(x))\big| \leq a_1\big(x, |y|, |\overline{\mu}(x)|\big) H_1\big(|u|\big),$$
$$\big|f_u(x, y, u, \overline{\mu}(x))\big| \leq a_2\big(x, |y|, |\overline{\mu}(x)|\big) H_2\big(|u|\big)$$

for all $x \in \Omega, y, u \in R$ satisfying $|y - \overline{y}(x)| \leq \epsilon_2$ and

$$\big|f_y(x, y, u, \mu^1) - f_y(x, y, u, \mu^2)\big| \leq b_1\big(x, |y|, |\mu^1|, |\mu^2|\big) H_1\big(|u|\big) \big|\mu^1 - \mu^2\big|^{\alpha_1},$$
$$\big|f_u(x, y, u, \mu^1) - f_u(x, y, u, \mu^2)\big| \leq b_2\big(x, |y|, |\mu^1|, |\mu^2|\big) H_2\big(|u|\big) \big|\mu^1 - \mu^2\big|^{\alpha_2}$$

for all $x \in \Omega, y, u \in R, \mu^i \in R^k$ satisfying $|\mu^i - \overline{\mu}(x)| \leq \epsilon_1, i = 1, 2, |y - \overline{y}(x)| \leq \epsilon_2$, where

$$H_2(t) = \sum_{j=1}^{m_2} t^{s_j} \quad \text{with } m_2 \geq 1, 0 \leq s_j \leq p - 1 \ \forall j = \overline{1, m_2}.$$

Under conditions $(A1)$, $(A2)$ and by Lemma 2.1, for each $\phi \in L^p(\Omega)$, equation (7) has a unique solution $y_\phi \in H_0^1(\Omega) \cap C(\overline{\Omega})$ which satisfies the estimation

$$\|y_\phi\|_{H_0^1(\Omega)} + \|y_\phi\|_{C(\overline{\Omega})} \le C\|\phi\|_{L^p(\Omega)}. \tag{8}$$

In the paper, we also need the following assumption.
$(A5)$ For a.e. $x \in \Omega$,

$$\delta(x) \le \delta_0 := \frac{\epsilon_0}{4C\max\{1; |\Omega|^{1/p}\}}, \tag{9}$$

where $|\Omega|$ is the volume of Ω and C is positive constant which is given in (8).

We now state our main result

Theorem 1.1. *Suppose that assumptions $(A1) - (A5)$ are satisfied. Then there exist a neighborhood $M_1 \times \Lambda_1$ of $(\overline{\mu}, \overline{\lambda})$ and a neighborhood $Z_1 = Y_1 \times U_1$ of $(\overline{y}, \overline{u})$ such that for each $(\mu, \lambda) \in M_1 \times \Lambda_1$, $P(\mu, \lambda)$ has a unique solution $z(\mu, \lambda) = \big(y(\mu, \lambda), u(\mu, \lambda)\big) \in Z_1$ and the map $z(\cdot, \cdot)$ is Hölder continuous, that is, there exist positive constants l_1 and l_2 such that*

$$\begin{aligned} &\|y(\mu^1, \lambda^1) - y(\mu^2, \lambda^2)\|_Y + \|u(\mu^1, \lambda^1) - u(\mu^2, \lambda^2)\|_{L^p(\Omega)} \\ &\le l_1\|\mu^1 - \mu^2\|_M^{\alpha/p} + l_2\|\lambda^1 - \lambda^2\|_\Lambda^{1/p} \end{aligned}$$

for all $(\mu^i, \lambda^i) \in M_1 \times \Lambda_1$ with $i = 1, 2$. Here $\alpha = \min\{\alpha_1, \alpha_2\}$.

In order to prove Theorem 1.1 we will establish some auxiliary results which are provided in section 2. Section 3 contains the proof of Theorem 1.1. Section 4 is destined for some examples illustrating Theorem 1.1.

2. Auxiliary Results

In this section we will give some properties of the set-valued map $K : \Lambda \rightrightarrows Z$, where $K(\lambda)$ is defined by (4). We begin with the following important result on the continuity of solutions of PDEs, which is due to E. Casas who did the associated pionieering work (see [5, Theorem 2.1] and [6, Theorem 2.1]) . For complement, we provide here a brief proof.

Lemma 2.1. [5, Theorem 2.1] *Assume that conditions $(A1)$ and $(A2)$ are satisfied. Then for each $\phi \in L^p(\Omega)$ equation (7) has a unique weak solution $y_\phi \in H_0^1(\Omega) \cap C(\overline{\Omega})$ which has the a priori estimate*

$$\|y_\phi\|_{H_0^1(\Omega)} + \|y_\phi\|_{C(\overline{\Omega})} \le C\|\phi\|_{L^p(\Omega)}, \tag{10}$$

where C is a constant independent of ϕ, and if $\phi_n \rightharpoonup \phi$ weakly in $L^p(\Omega)$ then $y_{\phi_n} \to y_\phi$ strongly in $H_0^1(\Omega) \cap C(\overline{\Omega})$. Moreover, the maximum principle holds, that is,

$$\phi \ge 0 \quad \textit{implies} \quad y_\phi \ge 0.$$

Proof. Since $\phi \in L^p(\Omega) \hookrightarrow L^2(\Omega)$, the Lax-Milgram theorem and G. Stampacchia (see [7, Theorem 12.4]) imply that (7) has a unique solution $y_\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and there exists a constant $C_1 > 0$ independent of ϕ such that

$$\|y_\phi\|_{H_0^1(\Omega)} + \|y_\phi\|_{L^\infty(\Omega)} \leq C_1 \|\phi\|_{W^{-1,r}(\Omega)}, \tag{11}$$

where $2 \leq r < \frac{2N}{N-2}$. The continuity of $y_\phi(\cdot)$ is followed from [9, Theorem 8.30]. Since the imbedding $L^2(\Omega) \hookrightarrow W^{-1,r}(\Omega)$ is compact, there exists a constant C independent of ϕ such that (10) is satisfied. If $\phi_n \rightharpoonup \phi$ weakly in $L^p(\Omega)$ then $\phi_n \to \phi$ strongly in $W^{-1,r}(\Omega)$. Combining this with (11), we see that $y_{\phi_n} \to y_\phi$ strongly in $H_0^1(\Omega) \cap C(\overline{\Omega})$. Finally, by [2, Lemma 2.2], the maximum principle is proved. ■

From Lemma 2.1, we can define a linear continuous solution mapping

$$\begin{aligned} S : L^p(\Omega) &\to Y \\ \phi &\mapsto y, \end{aligned}$$

where y is a unique solution of (7) corresponding to ϕ.

Lemma 2.2. *Under assumptions* $(A1), (A2)$ *and* $(A5)$*, for each* $\lambda \in \Lambda$*,* $K(\lambda)$ *is a nonempty and closed convex set in* Z*.*

Proof. For each $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda$. Obviously, $K(\lambda)$ is convex. Now we show that $K(\lambda)$ is nonempty subset. In fact, we choose $u(x) = \max\{u_0; -\lambda_1(x)\}$, where u_0 is given by

$$u_0 := \max\Big\{\frac{\epsilon_0|\Omega|^{-1/p}\|\lambda_1\|_{L^p(\Omega)} + 4\|\lambda_3\|_{L^\infty(\Omega)}}{3\epsilon_0}; \|\lambda_2\|_{L^\infty(\Omega)}\Big\}.$$

This implies $u + \lambda_1 = \frac{1}{2}\big(u_0 + \lambda_1 + |u_0 + \lambda_1|\big) \geq 0$ and $u \geq \lambda_2$ in Ω.
Moreover, we set $y = S(u + \lambda_1)$ then (y, u) satisfies (2) and $y \geq 0$.
From Lemma 2.1, we get

$$\begin{aligned} \|y\|_{C(\overline{\Omega})} &\leq C\|u + \lambda_1\|_{L^p(\Omega)} \\ &\leq C\big(\|u_0\|_{L^p(\Omega)} + \|\lambda_1\|_{L^p(\Omega)}\big) \\ &\leq C\big(u_0|\Omega|^{1/p} + \|\lambda_1\|_{L^p(\Omega)}\big). \end{aligned}$$

Combining this with $(A5)$ yields

$$\begin{aligned} \delta y + \lambda_3 &\leq \delta_0\|y\|_{C(\overline{\Omega})} + \|\lambda_3\|_{L^\infty(\Omega)} \\ &\leq \frac{\epsilon_0}{4C|\Omega|^{1/p}} C\big(u_0|\Omega|^{1/p} + \|\lambda_1\|_{L^p(\Omega)}\big) + \|\lambda_3\|_{L^\infty(\Omega)} \\ &\leq \frac{\epsilon_0 u_0}{4} + \frac{\epsilon_0\|\lambda_1\|_{L^p(\Omega)}}{4|\Omega|^{1/p}} + \|\lambda_3\|_{L^\infty(\Omega)}. \end{aligned}$$

On the other hand, since $u \geq u_0 \geq 0$ and $\epsilon \geq \epsilon_0 > 0$,

$$\begin{aligned} \epsilon u &\geq \epsilon_0 u_0 \\ &\geq \frac{\epsilon_0 u_0}{4} + \frac{1}{4}\big(\epsilon_0 |\Omega|^{-1/p} \|\lambda_1\|_{L^p(\Omega)} + 4\|\lambda_3\|_{L^\infty(\Omega)}\big). \end{aligned}$$

Hence $\epsilon u \geq \delta y + \lambda_3$, and so (y, u) satisfies (3). Consequently, $K(\lambda) \neq \emptyset$.
Finally we show that $K(\lambda)$ is closed.

Indeed. Assume that $z_n = (y_n, u_n) \in K(\lambda)$ and $z_n \to z = (y, u)$ in Z. Then $z_n \to z$ in $L^2(\Omega) \times L^p(\Omega)$. By passing a subsequence where $z_n \to z$ as $n \to \infty$ a. e. in Ω (see, [4, Theorem 4.9, pp. 94]). In other words, there exists a subset B which has measure zero such that

$$z_n(x) \to z(x) = \big(y(x), u(x)\big) \text{ for all } x \in \Omega \backslash B \quad \text{as } n \to \infty.$$

Since

$$\begin{cases} u_n \geq \lambda_2 & \text{in } \Omega, \\ \epsilon u_n \geq \delta y_n + \lambda_3 & \text{in } \Omega, \end{cases}$$

there exists subset P_n which has measure zero such that

$$\begin{cases} u_n(x) \geq \lambda_2(x) & \text{for all } \ x \in \Omega \backslash P_n, \\ \epsilon(x) u_n(x) \geq \delta(x) y_n(x) + \lambda_3(x) & \text{for all } \ x \in \Omega \backslash P_n. \end{cases}$$

Setting $T = \bigcup_{n \geq 1} P_n \cup B$, we see that T has measure zero. Letting $n \to \infty$, we obtain from the above that

$$\begin{cases} u(x) \geq \lambda_2(x) & \text{for all } x \in \Omega \backslash T, \\ \epsilon(x) u(x) \geq \delta(x) y(x) + \lambda_3(x) & \text{for all } x \in \Omega \backslash T. \end{cases}$$

Hence $z = (y, u)$ satisfies (3). It remains to prove that (y, u) satisfies (2). In fact, we set $\overline{y} = S(u + \lambda_1)$. We then have

$$\begin{aligned} \|y - \overline{y}\|_Y &\leq \|y - y_n\|_Y + \|y_n - \overline{y}\|_Y \\ &\leq \|y - y_n\|_Y + \|S(u_n + \lambda_1) - S(u + \lambda_1)|\|_Y \\ &\leq \|y - y_n\|_Y + C\|u_n - u\|_{L^p(\Omega)}. \end{aligned}$$

Letting $n \to \infty$, we obtain $y = \overline{y}$. Consequently, (y, u) satisfies (2) and $(y, u) \in K(\lambda)$. Hence $K(\lambda)$ is closed. The proof of the lemma is complete. ■

Lemma 2.3. *Under assumptions of Lemma 2.2, the set-valued map* $K : \Lambda \rightrightarrows Z$ *is Lipschitz continuous, that is, there exists a positive constant* k *such that*

$$K(\lambda) \subset K(\beta) + k\|\lambda - \beta\|_\Lambda \overline{B}_Z, \quad \forall \lambda, \beta \in \Lambda. \tag{12}$$

Proof. Take any $\lambda = (\lambda_1, \lambda_2, \lambda_3), \beta = (\beta_1, \beta_2, \beta_3) \in \Lambda$. For convenience we put

$$\gamma = \|\lambda - \beta\|_\Lambda = \|\lambda_1 - \beta_1\|_{L^p(\Omega)} + \|\lambda_2 - \beta_2\|_{L^\infty(\Omega)} + \|\lambda_3 - \beta_3\|_{L^\infty(\Omega)},$$
$$\tau(x) = \gamma + |\lambda_1(x) - \beta_1(x)|, \quad \theta = \max\Big\{1; \frac{4+\epsilon_0}{2\epsilon_0}\Big\}.$$

Taking any $z_\lambda = (y_\lambda, u_\lambda) \in K(\lambda)$, we choose $u_\beta = u_\lambda + \theta\tau$ and set $y_\beta = S(u_\beta + \beta_1)$ is a unique solution to the following elliptic equation

$$\begin{cases} Ay = u_\beta + \beta_1 & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $u_\lambda \geq \lambda_2$, we have

$$u_\beta \geq \beta_2. \tag{13}$$

Moreover

$$\begin{aligned} y_\beta = S(u_\beta + \beta_1) &= S(u_\lambda + \lambda_1) + S(\theta\tau + \beta_1 - \lambda_1) \\ &= y_\lambda + \sigma, \end{aligned}$$

where $\sigma = S(\theta\tau + \beta_1 - \lambda_1)$. Since $\theta\tau + \beta_1 - \lambda_1 \geq 0$, Lemma 2.1 implies that $\sigma \geq 0$. Hence

$$\begin{aligned} &\epsilon u_\beta - \delta y_\beta - \beta_3 \\ &= \epsilon u_\lambda - \delta y_\lambda - \lambda_3 + \theta\epsilon\tau - \delta\sigma + \lambda_3 - \beta_3 \\ &\geq \theta\epsilon_0\tau - \delta_0\|\sigma\|_{C(\overline{\Omega})} - \|\lambda_3 - \beta\|_{L^\infty(\Omega)} \quad (\text{because of } \epsilon u_\lambda - \delta y_\lambda - \lambda_3 \geq 0) \\ &\geq \theta\epsilon_0\tau - \delta_0 C\|\theta\tau + \beta_1 - \lambda_1\|_{L^p(\Omega)} - \|\lambda_3 - \beta_3\|_{L^\infty(\Omega)} \\ &\geq \theta\epsilon_0\tau - \delta_0 C\big[\theta\|\tau\|_{L^p(\Omega)} + \|\lambda_1 - \beta_1\|_{L^p(\Omega)}\big] - \|\lambda_3 - \beta_3\|_{L^\infty(\Omega)}. \end{aligned}$$

This implies

$$\begin{aligned} &\epsilon u_\beta - \delta y_\beta - \beta_3 \\ &\geq \theta\epsilon_0\tau - \delta_0 C\big[\theta|\Omega|^{1/p}\gamma + (\theta+1)\|\lambda_1 - \beta_1\|_{L^p(\Omega)}\big] - \|\lambda_3 - \beta_3\|_{L^\infty(\Omega)} \\ &\geq \theta\epsilon_0\tau - \frac{1}{4}\theta\epsilon_0\gamma - \frac{1}{4}(\theta+1)\epsilon_0\|\lambda_1 - \beta_1\|_{L^p(\Omega)} - \|\lambda_3 - \beta_3\|_{L^\infty(\Omega)} \\ &\geq \theta\epsilon_0\tau - \frac{1}{4}\epsilon_0(2\theta+1)\gamma - \|\lambda_3 - \beta_3\|_{L^\infty(\Omega)} \\ &\geq (\theta\epsilon_0 - 1)\gamma - \frac{1}{4}\epsilon_0(2\theta+1)\gamma \quad (\text{because of } \tau \geq \gamma \text{ and } \|\lambda_3 - \beta_3\|_{L^\infty(\Omega)} \leq \gamma) \\ &\geq \frac{1}{4}(2\theta\epsilon_0 - 4 - \epsilon_0)\gamma \geq 0. \end{aligned}$$

Combining this with (13) yields $z_\beta = (y_\beta, u_\beta)$ satisfying (3). This implies $z_\beta \in K(\beta)$. On the other hand, we have

$$\begin{aligned} \|u_\beta - u_\lambda\|_{L^p(\Omega)} = \|\theta\tau\|_{L^p(\Omega)} &\le \theta|\Omega|^{1/p}\gamma + \theta\|\beta_1 - \lambda_1\|_{L^p(\Omega)} \\ &\le \theta\big(|\Omega|^{1/p} + 1\big)\gamma. \end{aligned} \tag{14}$$

By Lemma 2.1,

$$\begin{aligned} \|y_\beta - y_\lambda\|_Y = \|\sigma\|_Y &\le C\|\theta\tau + (\beta_1 - \lambda_1)\|_{L^p(\Omega)} \\ &\le C\big(\theta\|\tau\|_{L^p(\Omega)} + \|\beta_1 - \lambda_1\|_{L^p(\Omega)}\big) \\ &\le C\big(\theta|\Omega|^{1/p}\gamma + (\theta+1)\|\beta_1 - \lambda_1\|_{L^p(\Omega)}\big) \\ &\le C\big(\theta|\Omega|^{1/p} + (\theta+1)\big)\gamma. \end{aligned} \tag{15}$$

Combining (14) with (15) we have the following inequality

$$\|y_\beta - y_\lambda\|_Y + \|u_\beta - u_\lambda\|_{L^p(\Omega)} \le k\gamma,$$

where $k = \theta\big(|\Omega|^{1/p} + 1\big) + C\big(\theta|\Omega|^{1/p} + (\theta+1)\big)$. The proof is complete. ∎

3. Proof of the Main Result

From Lemma 2.3 , we get

$$K(\overline{\lambda}) \subset K(\lambda) + k\|\overline{\lambda} - \lambda\|_\Lambda \overline{B}_Z, \quad \forall \lambda \in \Lambda.$$

Fix $r_0 > 0$ such that $kr_0 \le \epsilon_2$, where ϵ_2 is given in the assumption $(A3)$. Then we have

$$K(\lambda) \cap (\overline{z} + \epsilon_2 \overline{B}_Z) \ne \emptyset, \quad \forall \lambda \in \overline{B}_\Lambda(\overline{\lambda}, r_0). \tag{16}$$

Let us put

$$Y_0 = \overline{B}_Y(\overline{y}, \epsilon_2), U_0 = \overline{B}_U(\overline{u}, \epsilon_2), Z_0 = Y_0 \times U_0,$$

$$M_0 = \overline{B}_M(\overline{\mu}, \epsilon_1) \quad \text{and} \quad \Lambda_0 = \overline{B}_\Lambda(\overline{\lambda}, r_0).$$

Easily, we see that

$$\overline{B}_Z(\overline{z}, \epsilon_2) \subset Z_0. \tag{17}$$

Combining this with (16) yields

$$K(\lambda) \cap Z_0 \ne \emptyset \quad \forall \lambda \in \Lambda_0. \tag{18}$$

Lemma 3.1. *Suppose that assumptions $(A1)-(A5)$ are fulfilled. Then the following assertions hold:*

(i) *For each $\mu \in M_0$, the function $F(\cdot,\mu)$ is Gâteaux differentiable and its derivative is given by*

$$\begin{aligned}\langle F_z(z,\mu),h\rangle &= \langle F_y(y,u,\mu),h_1\rangle + \langle F_u(y,u,\mu),h_2\rangle\\ &= \int_\Omega f_y(x,y(x),u(x),\mu(x))h_1(x)dx + \int_\Omega f_u(x,y(x),u(x),\mu(x))h_2(x)dx,\end{aligned}$$

for all $h=(h_1,h_2)\in Z$. Moreover, $F_z(\cdot,\cdot)$ is uniformly bounded on $Z_0\times M_0$.

(ii) *There exists a positive constant l_0 such that*

$$\|F_z(z,\mu^1)-F_z(z,\mu^2)\|_{Z^*} \le l_0\|\mu^1-\mu^2\|_M^\alpha, \quad \forall z\in Z_0, \mu^1,\mu^2\in M_0. \tag{19}$$

(iii) *$F_z(\cdot,\mu)$ is strongly monotone, that is*

$$\langle F_z(z_1,\mu)-F_z(z_2,\mu), z_1-z_2\rangle \ge \rho\|u_1-u_2\|^p_{L^p(\Omega)} \quad \forall z_1,z_2\in Z_0, \tag{20}$$

where $z_1=(y_1,u_1)$ and $z_2=(y_2,z_2)$.

Proof. By $(A4)$, for each $\mu\in M_0$, the first variation $F_z(z,\mu)(h)$ of $F(\cdot,\mu)$ at a point $z=(y,u)\in Z$ does exist and defined by

$$\begin{aligned}F_z(z,\mu)(h) &= \langle F_z(z,\mu),h\rangle\\ &= \langle F_y(y,u,\mu),h_1\rangle + \langle F_u(y,u,\mu),h_2\rangle\\ &= \int_\Omega f_y(x,y(x),u(x),\mu(x))h_1(x)dx + \int_\Omega f_u(x,y(x),u(x),\mu(x))h_2(x)dx,\end{aligned}$$

for all $h=(h_1,h_2)\in Z$. Obviously, $F_z(z,\mu)(\cdot)$ is a linear mapping. We now show that $F_z(\cdot,\cdot)$ is uniformly bounded on $Z_0\times M_0$.
Indeed. For any $z=(y,u)\in Z_0$, we have

$$\|F_z(z,\overline{\mu})\|_{Z^*} \le \|F_y(y,u,\overline{\mu})\|_{Y^*} + \|F_u(y,u,\overline{\mu})\|_{L^q(\Omega)}, \tag{21}$$

where q is the conjugate number of p.
By the Hölder inequality, there exist constants $c_j>0, j=1,2$ such that

$$\int_\Omega |u|^s|h_1|dx \le c_1\|u\|^s_{L^p(\Omega)}\|h_1\|_{C(\overline{\Omega})} \le c_1\|u\|^s_{L^p(\Omega)}\|h_1\|_Y \quad \forall\, h_1\in Y,$$

(22) and $$\int_\Omega |u|^d|h_2|dx \le c_2\|u\|^d_{L^p(\Omega)}\|h_2\|_{L^p(\Omega)} \quad \forall\, h_2\in L^p(\Omega),$$

where $0\le s\le p$ and $0\le d\le p-1$.

By definitions of $H_i (i = 1, 2)$ and (22), there exist positive constants C_{H_i} such that

$$\int_\Omega H_1(|u|)|h_1|dx \le C_{H_1} H_1\big(\|u\|_{L^p(\Omega)}\big)\|h_1\|_Y \quad \forall\ h_1 \in Y, \tag{23}$$

and

$$\int_\Omega H_2(|u|)|h_2|dx \le C_{H_2} H_2\big(\|u\|_{L^p(\Omega)}\big)\|h_2\|_{L^p(\Omega)} \quad \forall\ h_2 \in L^p(\Omega). \tag{24}$$

We put

$$A_i = \max\Big\{a_i\big(x, |t_1|, |t_2|\big) : (x, t_1, t_2) \in \overline{\Omega} \times [0, \delta_1] \times [0, \delta_2]\Big\}, i = 1, 2,$$

where $\delta_1 := \|\overline{y}\|_{C(\overline{\Omega})} + \epsilon_2, \delta_2 := \|\overline{\mu}\|_M + \epsilon_1$.
By $(A4)$,

$$\begin{aligned}
\|F_y(y, u, \overline{\mu})\|_{Y^*} &= \sup\Big\{\langle F_y(y, u, \overline{\mu}), h_1\rangle : h_1 \in Y, \|h_1\|_Y \le 1\Big\}\\
&= \sup\Big\{\int_\Omega f_y(x, y(x), u(x), \overline{\mu}(x))h_1(x)dx : \|h_1\|_Y \le 1\Big\}\\
&\le \sup\Big\{\int_\Omega a_1\big(\cdot, |y|, |\overline{\mu}|\big) H_1\big(|u|\big)|h_1|dx : \|h_1\|_Y \le 1\Big\}\\
&\le A_1 \sup\Big\{\int_\Omega H_1\big(|u|\big)|h_1|dx : \|h_1\|_Y \le 1\Big\}.
\end{aligned} \tag{25}$$

Combining (23) with (25) yields

$$\begin{aligned}
\|F_y(y, u, \overline{\mu})\|_{Y^*} &\le A_1 C_{H_1} \sup\Big\{H_1\big(\|u\|_{L^p(\Omega)}\big)\|h_1\|_Y : \|h_1\|_Y \le 1\Big\}\\
&\le A_1 C_{H_1} H_1\big(\|u\|_{L^p(\Omega)}\big)\\
&\le A_1 C_{H_1} H_1\big(\|\overline{u}\|_{L^p(\Omega)} + \epsilon_2\big).
\end{aligned} \tag{26}$$

Using similar arguments, we obtain

$$\|F_u(y, u, \overline{\mu})\|_{L^q(\Omega)} \le A_2 C_{H_2} H_2\big(\|\overline{u}\|_{L^p(\Omega)} + \epsilon_2\big). \tag{27}$$

Combining (21) with (26) and (27) we conclude that

$$\|F_z(z, \overline{\mu})\|_{Z^*} \le A_1 C_{H_1} H_1\big(\|\overline{u}\|_{L^p(\Omega)} + \epsilon_2\big) + A_2 C_{H_2} H_2\big(\|\overline{u}\|_{L^p(\Omega)} + \epsilon_2\big). \tag{28}$$

On the other hand, for any $z = (y, u) \in Z_0$ and $\mu^1, \mu^2 \in M_0$, we have

$$\begin{aligned}
&\|F_z(z, \mu^1) - F_z(z, \mu^2)\|_{Z^*}\\
&\quad \le \|F_y(y, u, \mu^1) - F_y(y, u, \mu^2)\|_{Y^*} + \|F_u(y, u, \mu^1) - F_u(y, u, \mu^2)\|_{L^q(\Omega)}.
\end{aligned} \tag{29}$$

In the same manner, using $(A4)$, we get

(30) $$\begin{aligned}\|F_y(y,u,\mu^1)-F_y(y,u,\mu^2)\|_{Y^*} &\le B_1C_{H_1}H_1\big(\|u\|_{L^p(\Omega)}\big)\|\mu^1-\mu^2\|_M^{\alpha_1}\\ &\le B_1C_{H_1}H_1\big(\|\overline{u}\|_{L^p(\Omega)}+\epsilon_2\big)\|\mu^1-\mu^2\|_M^{\alpha_1}\end{aligned}$$

and

(31) $$\begin{aligned}\|F_u(y,u,\mu^1)-F_u(y,u,\mu^2)\|_{L^q} &\le B_2C_{H_2}H_2\big(\|u\|_{L^p(\Omega)}\big)\|\mu^1-\mu^2\|_M^{\alpha_2}\\ &\le B_2C_{H_2}H_2\big(\|\overline{u}\|_{L^p(\Omega)}+\epsilon_2\big)\|\mu^1-\mu^2\|_M^{\alpha_2},\end{aligned}$$

where $B_i := \max\Big\{b_i(x,t_1,t_2,t_3) : (x,t_1,t_2,t_3)\in\overline{\Omega}\times[0,\delta_1]\times[0,\delta_2]^2\Big\}, i=1,2.$ From (29)-(31) we deduce that

(32) $$\|F_z(z,\mu^1)-F_z(z,\mu^2)\|_{Z^*}\le l_0\|\mu^1-\mu^2\|_M^{\alpha},$$

where $l_0 := B_1C_{H_1}H_1\big(\|\overline{u}\|_{L^p(\Omega)}+\epsilon_2\big)(2\epsilon_1)^{\alpha_1-\alpha}+B_2C_{H_2}H_2\big(\|\overline{u}\|_{L^p(\Omega)}+\epsilon_2\big)(2\epsilon_1)^{\alpha_2-\alpha}$. We obtain assertion (ii).

Since (28) and (32), we get

$$\begin{aligned}\|F_z(z,\mu)\|_{Z^*} &\le \|F_z(z,\mu)-F_z(z,\overline{\mu})\|_{Z^*}+\|F_z(z,\overline{\mu})\|_{Z^*}\\ &\le l_0\|\mu-\overline{\mu}\|_M^{\alpha}+A_1C_{H_1}H_1\big(\|\overline{u}\|_{L^p(\Omega)}+\epsilon_2\big)+A_2C_{H_2}H_2\big(\|\overline{u}\|_{L^p(\Omega)}+\epsilon_2\big).\end{aligned}$$

Hence

(33) $$\|F_z(z,\mu)\|_{Z^*}\le l,$$

where $l = l_0\epsilon_1^{\alpha}+A_1C_{H_1}H_1\big(\|\overline{u}\|_{L^p(\Omega)}+\epsilon_2\big)+A_2C_{H_2}H_2\big(\|\overline{u}\|_{L^p(\Omega)}+\epsilon_2\big)$. This implies that $F_z(\cdot,\cdot)$ is uniformly bounded on $Z_0\times M_0$. Consequently, the function $F(\cdot,\mu)$ is Gâteaux differentiable for all $\mu\in M_0$. Hence, assertion (i) is obtained.

Fix any $\mu\in M_0$. Taking any $z_i=(y_i,u_i)\in Z_0, i=1,2$, we have

$$\begin{aligned}&\big\langle F_z(z_1,\mu)-F_z(z_2,\mu),z_1-z_2\big\rangle\\ &=\int_\Omega\big(f_z(x,z_1(x),\mu(x))-f_z(x,z_2(x),\mu(x)\big)\big(z_1(x)-z_2(x)\big)dx.\end{aligned}$$

From this and $(A3)$ we obtain

$$\big\langle F_z(z_1,\mu)-F_z(z_2,\mu),z_1-z_2\big\rangle\ge\rho\|u_1-u_2\|^p_{L^p(\Omega)}.$$

The proof of the lemma is complete. ∎

Lemma 3.2. *Under assumptions of Lemma 3.1, for each* $(\mu, \lambda) \in M_0 \times \Lambda_0$, *the problem*

$$P_0(\mu,\lambda) \qquad \begin{cases} F(z,\mu) \to \inf \\ z \in K(\lambda) \cap Z_0 \end{cases}$$

has a unique solution.

Proof. Put

$$\xi = \inf\left\{F(z,\mu) : z \in K(\lambda) \cap Z_0\right\}.$$

Then there exists a sequence $z_n = (y_n, u_n) \in K(\lambda) \cap Z_0$ such that

$$\xi = \lim_{n\to\infty} F(z_n, \mu).$$

Since $\{u_n\}$ is bounded and $L^p(\Omega)$ is a reflexive Banach space, we can assume that

$$u_n \rightharpoonup \hat{u} \quad \text{in} \quad L^p(\Omega).$$

By Lemma 2.1, we get

$$y_n \to \hat{y} \quad \text{in} \quad Y$$

for some $\hat{z} = (\hat{y}, \hat{u}) \in Y \times L^p(\Omega)$. By Lemma 2.2, $K(\lambda)$ is a weakly closed set. Consequently, $\hat{z} = (\hat{y}, \hat{u}) \in K(\lambda)$. Since Z_0 is a weakly closed subset, $\hat{z} \in Z_0$. Thus we get $\hat{z} \in K(\lambda) \cap Z_0$ and

$$F(\hat{z}, \mu) \geq \xi. \tag{34}$$

On the other hand, by a property of convex functions, we have

$$\begin{aligned} &f\big(x, y_n(x), u_n(x), \mu(x)\big) \geq f\big(x, \hat{y}(x), \hat{u}(x), \mu(x)\big) \\ &+ \big\langle f_y\big(x, \hat{y}(x), \hat{u}(x), \mu(x)\big), y_n(x) - \hat{y}(x)\big\rangle + \big\langle f_u\big(x, \hat{y}(x), \hat{u}(x), \mu(x)\big), u_n(x) - \hat{u}(x)\big\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} F(y_n, u_n, \mu) \geq F(\hat{y}, \hat{u}, \mu) &+ \int_\Omega f_y\big(x, \hat{y}(x), \hat{u}(x), \mu(x)\big)\big(y_n(x) - \hat{y}(x)\big)dx \\ &+ \int_\Omega f_u\big(x, \hat{y}(x), \hat{u}(x), \mu(x)\big)\big(u_n(x) - \hat{u}(x)\big)dx. \end{aligned}$$

By $(A2)$ and $(A4)$ we can show that $f_y(\cdot, \hat{y}, \hat{u}, \mu) \in L^2(\Omega)$ and $f_u(\cdot, \hat{y}, \hat{u}, \mu) \in L^q(\Omega)$. Letting $n \to \infty$, we obtain from the above that $\xi \geq F(\hat{y}, \hat{u}, \mu)$. Combining this with (34) we have $\xi = F(\hat{y}, \hat{u}, \mu)$.

We now prove that ξ is finite. To do this we first show that $F(\overline{z}, \cdot)$ is bounded on M_0. In fact, for any $\mu \in M_0$ from $(A2)$, we get

$$\begin{aligned}\Big|F(\overline{z},\mu)-F(\overline{z},\overline{\mu})\Big| &\le \int_\Omega \Big|f\big(x,\overline{y}(x),\overline{u}(x),\mu(x)\big)-f\big(x,\overline{y}(x),\overline{u}(x),\overline{\mu}(x)\big)\Big|dx\\ &\le \int_\Omega g\big(x,|\overline{y}(x)|,|\mu(x)|,|\overline{\mu}(x)|\big)H_1\big(|\overline{u}(x)|\big)dx\\ &\le \eta\int_\Omega H_1\big(|\overline{u}(x)|\big)dx,\\ &\le \eta C_{H_1}H_1(\|\overline{u}\|_{L^p(\Omega)}),\end{aligned}$$

where

$$\eta=\max\Big\{g(x,t_1,t_2,t_3):(x,t_1,t_2,t_3)\in\overline{\Omega}\times\big[0,\|\overline{y}\|_{C(\overline{\Omega})}\big]\times\big[0,\|\overline{\mu}\|_M+\epsilon_1\big]\times\big[0,\|\overline{\mu}\|_M\big]\Big\}.$$

Consequently,

$$\Big|F(\overline{z},\mu)-F(\overline{z},\overline{\mu})\Big|\le \eta C_{H_1}H_1\big(\|\overline{u}\|_{L^p(\Omega)}\big). \tag{35}$$

We obtain the desired conclusion.

From (35), the uniform boundedness of $F_z(\cdot,\cdot)$ on $Z_0\times M_0$ and the mean value theorem, for all $\mu\in M_0$, we get

$$\begin{aligned}\Big|F(\hat{z},\mu)-F(\overline{z},\overline{\mu})\Big| &\le \Big|F(\hat{z},\mu)-F(\overline{z},\mu)\Big|+\Big|F(\overline{z},\mu)-F(\overline{z},\overline{\mu})\Big|\\ &\le \sup_{0\le t\le 1}\|F_z\big(\overline{z}+t(\hat{z}-\overline{z}),\mu\big)\|_{Z^*}\|\hat{z}-\overline{z}\|_Z+\Big|F(\overline{z},\mu)-F(\overline{z},\overline{\mu})\Big|\\ &\le \sup_{z'\in Z_0}\|F_z(z',\mu)\|_{Z^*}\|\hat{z}-\overline{z}\|_Z+\eta C_{H_1}H_1\big(\|\overline{u}\|_{L^p(\Omega)}\big)\\ &< +\infty.\end{aligned}$$

This implies that $\big|F(\hat{z},\mu)\big|<+\infty$ and so ξ is finite.

It remains to show that problem $P_0(\mu,\lambda)$ has a unique solution. Indeed, we assume that $z_i(\mu,\lambda)=\big(y_i(\mu,\lambda),u_i(\mu,\lambda)\big)$, $i=1,2$ are solutions of $P_0(\mu,\lambda)$. It follows that

$$\big\langle F_z\big(z_i(\mu,\lambda),\mu\big),z-z_i(\mu,\lambda)\big\rangle\ge 0\quad \forall z\in K(\lambda)\cap Z_0, i=1,2.$$

Hence

$$\big\langle F_z\big(z_1(\mu,\lambda),\mu\big)-F_z\big(z_2(\mu,\lambda),\mu\big),z_1(\mu,\lambda)-z_2(\mu,\lambda)\big\rangle\le 0.$$

From this and (iii) of Lemma 3.1, we get

$$\begin{aligned}0 &\ge \big\langle F_z\big(z_1(\mu,\lambda),\mu\big)-F_z\big(z_2(\mu,\lambda),\mu\big),z_1(\mu,\lambda)-z_2(\mu,\lambda)\big\rangle\\ &\ge \rho\|u_1(\mu,\lambda)-u_2(\mu,\lambda)\|^p_{L^p(\Omega)}.\end{aligned}$$

It follows $u_1(\mu,\lambda)=u_2(\mu,\lambda)$. Since $y_i(\mu,\lambda)=S\big(u_i(\mu,\lambda)+\lambda_1\big)$ and Lemma 2.1, we obtain $y_1(\mu,\lambda)=y_2(\mu,\lambda)$. Hence $z_1(\mu,\lambda)=z_2(\mu,\lambda)$. This proves the lemma. ■

Proof of Theorem 1.1. For each $(\mu, \lambda) \in M_0 \times \Lambda_0$, due to Lemma 3.2, problem $P_0(\mu, \lambda)$ has a unique solution $z(\mu, \lambda) = \big(y(\mu, \lambda), u(\mu, \lambda)\big) \in K(\lambda) \cap Z_0$. Since $P_0(\mu, \lambda)$ is a convex problem, it must hold

$$0 \in F_z\big(z(\mu, \lambda), \mu\big) + N\big(z(\mu, \lambda); K(\lambda) \cap Z_0\big). \tag{36}$$

It is equivalent to the variational inequality

$$\big\langle F_z\big(z(\mu, \lambda), \mu\big), z - z(\mu, \lambda)\big\rangle \geq 0 \quad \forall z \in K(\lambda) \cap Z_0, \mu \in M_0, \lambda \in \Lambda_0. \tag{37}$$

We first show that the solution mapping $z(\cdot, \cdot)$ is continuous at $(\overline{\mu}, \overline{\lambda})$.
In fact, fix any $(\mu, \lambda) \in M_0 \times \Lambda_0$. By the Lipschitz continuous property of $K(\cdot)$, there exists an element $z_1 \in K(\overline{\lambda})$ such that

$$\|z(\mu, \lambda) - z_1\|_Z \leq k\|\lambda - \overline{\lambda}\|_\Lambda \leq \epsilon_2.$$

Putting $\lambda = \overline{\lambda}$ and $\beta = \lambda$ in (12), we see that there exists $z_2 \in K(\lambda)$ such that

$$\|\overline{z} - z_2\|_Z \leq k\|\overline{\lambda} - \lambda\|_\Lambda \leq \epsilon_2.$$

Since $\overline{z}$ and $z(\mu, \lambda)$ are solutions of $P(\overline{\mu}, \overline{\lambda})$ and $P(\mu, \lambda)$, respectively, it follows that

$$\langle F_z(\overline{z}, \overline{\mu}), z_1 - \overline{z}\rangle \geq 0 \quad \text{and} \quad \langle F_z(z(\mu, \lambda), \mu), z_2 - z(\mu, \lambda)\rangle \geq 0.$$

By (ii) and (iii) of Lemma 3.1, and using (33), we have

$$\begin{aligned}
&\rho\|u(\mu, \lambda) - \overline{u}\|^p_{L^p(\Omega)} \\
&\leq \big\langle F_z\big(z(\mu, \lambda), \mu\big) - F_z(\overline{z}, \mu), z(\mu, \lambda) - \overline{z}\big\rangle \\
&\leq \big\langle F_z\big(z(\mu, \lambda), \mu\big) - F_z(\overline{z}, \mu), z(\mu, \lambda) - \overline{z}\big\rangle + \big\langle F_z(\overline{z}, \overline{\mu}), z_1 - \overline{z}\big\rangle \\
&\quad + \big\langle F_z\big(z(\mu, \lambda), \mu\big), z_2 - z(\mu, \lambda)\big\rangle \\
&= \big\langle F_z\big(z(\mu, \lambda), \mu\big), z_2 - \overline{z}\big\rangle + \big\langle F_z(\overline{z}, \mu), z_1 - z(\mu, \lambda)\big\rangle \\
&\quad + \big\langle F_z(\overline{z}, \overline{\mu}) - F_z(\overline{z}, \mu), z_1 - \overline{z}\big\rangle \\
&\leq \|F_z(z(\mu, \lambda), \mu)\|_{Z^*}\|z_2 - \overline{z}\|_Z + \|F_z(\overline{z}, \mu)\|_{Z^*}\|z_1 - z(\mu, \lambda)\|_Z \\
&\quad + \|F_z(\overline{z}, \overline{\mu}) - F_z(\overline{z}, \mu)\|_{Z^*}\|z_1 - \overline{z}\|_Z \\
&\leq 2lk\|\lambda - \overline{\lambda}\|_\Lambda + l_0\|z_1 - \overline{z}\|_Z\|\mu - \overline{\mu}\|^\alpha_M.
\end{aligned} \tag{38}$$

On the other hand

$$\|z_1 - \overline{z}\|_Z \leq \|z_1 - z(\mu, \lambda)\|_Z + \|z(\mu, \lambda) - \overline{z}\|_Z \leq \epsilon_2 + 2\epsilon_2 = 3\epsilon_2.$$

Hence (38) implies that

$$\|u(\mu,\lambda)-\overline{u}\|^p_{L^p(\Omega)} \le \frac{2lk}{\rho}\|\lambda-\overline{\lambda}\|_\Lambda + \frac{3\epsilon_2 l_0}{\rho}\|\mu-\overline{\mu}\|^\alpha_M. \tag{39}$$

From Lemma 2.1, it follows that

$$\begin{aligned}\|y(\mu,\lambda)-\overline{y}\|_Y &\le C\|u(\mu,\lambda)+\lambda_1-\overline{u}-\overline{\lambda}_1\|_{L^p(\Omega)}\\ &\le C\Big(\|u(\mu,\lambda)-\overline{u}\|_{L^p(\Omega)}+\|\lambda_1-\overline{\lambda}_1\|_{L^p(\Omega)}\Big)\\ &\le C\Big(\|u(\mu,\lambda)-\overline{u}\|_{L^p(\Omega)}+\|\lambda-\overline{\lambda}\|_\Lambda\Big).\end{aligned}$$

Combining this with (39), we can assert that there exist positive constants C_1, C_2 satisfying

$$\|y(\mu,\lambda)-\overline{y}\|_Y+\|u(\mu,\lambda)-\overline{u}\|_{L^p(\Omega)} \le C_1\|\mu-\overline{\mu}\|^{\alpha/p}_M + C_2\|\lambda-\overline{\lambda}\|^{1/p}_\Lambda. \tag{40}$$

This implies that $\|z(\mu,\lambda)-z(\overline{\mu},\overline{\lambda})\|_Z \to 0$ as $(\mu,\lambda)\to(\overline{\mu},\overline{\lambda})$. We obtain the desired property. It remains to show that the solution mapping $z(\cdot,\cdot)$ is Hölder continuous in a neighborhood of $(\overline{\mu},\overline{\lambda})$. From (40) we can choose neighborhoods $M_1 \subset M_0$ of $\overline{\mu}$ and $\Lambda_1 \subset \Lambda_0$ of $\overline{\lambda}$ such that $z(\mu,\lambda) \in \mathrm{int}B_Z(\overline{z},\epsilon_2)$, for all $\mu \in M_1, \lambda \in \Lambda_1$. Combining this with (17) yields

$$N\big(z(\mu,\lambda);K(\lambda)\big) = N\big(z(\mu,\lambda);K(\lambda)\cap Z_0\big) \quad \forall \mu \in M_1, \lambda \in \Lambda_1.$$

From this and (36) we obtain

$$0 \in F_z\big(z(\mu,\lambda),\mu\big) + N\big(z(\mu,\lambda);K(\lambda)\big) \quad \forall \mu \in M_1, \lambda \in \Lambda_1.$$

This is equivalent to

$$\big\langle F_z\big(z(\mu,\lambda),\mu\big), z - z(\mu,\lambda)\big\rangle \ge 0 \ \forall z \in K(\lambda), \mu \in M_1, \lambda \in \Lambda_1.$$

Consequently, for each $(\mu,\lambda) \in M_1 \times \Lambda_1$, $z(\mu,\lambda)$ is the unique solution of $P(\mu,\lambda)$. Let $(\mu^1,\lambda^1), (\mu^2,\lambda^2) \in M_1 \times \Lambda_1$. Putting $\lambda = \lambda^1$ and $\beta = \lambda^2$ in (12), we can find an element $\zeta_2 \in K(\lambda^2)$ such that

$$\|z(\mu^1,\lambda^1)-\zeta_2\|_Z \le k\|\lambda^1-\lambda^2\|_\Lambda.$$

Also, replacing $\lambda = \lambda^2$ and $\beta = \lambda^1$ in (12), we can find an element $\zeta_1 \in K(\lambda^1)$ such that

$$\|z(\mu^2,\lambda^2)-\zeta_1\|_Z \le k\|\lambda^1-\lambda^2\|_\Lambda.$$

Besides, we have

$$\langle F_z(z(\mu^1,\lambda^1),\mu^1), \zeta_1 - z(\mu^1,\lambda^1)\rangle \ge 0 \quad \text{and} \quad \langle F_z(z(\mu^2,\lambda^2),\mu^2), \zeta_2 - z(\mu^2,\lambda^2)\rangle \ge 0.$$

Combining these with the strong monotonicity of $F_z(\cdot, \mu^1)$ (see Lemma 3.1), we have

$$
\begin{aligned}
&\rho\|u(\mu^1,\lambda^1) - u(\mu^2,\lambda^2)\|_{L^p}^p \\
&\le \langle F_z(z(\mu^1,\lambda^1),\mu^1) - F_z(z(\mu^2,\lambda^2),\mu^1), z(\mu^1,\lambda^1) - z(\mu^2,\lambda^2)\rangle \\
&\le \langle F_z(z(\mu^1,\lambda^1),\mu^1) - F_z(z(\mu^2,\lambda^2),\mu^1), z(\mu^1,\lambda^1) - z(\mu^2,\lambda^2)\rangle \\
&\quad + \langle F_z(z(\mu^1,\lambda^1),\mu^1), \zeta_1 - z(\mu^1,\lambda^1)\rangle + \langle F_z(z(\mu^2,\lambda^2),\mu^2), \zeta_2 - z(\mu^2,\lambda^2)\rangle \\
&= \langle F_z(z(\mu^1,\lambda^1),\mu^1), \zeta_1 - z(\mu^2,\lambda^2)\rangle + \langle F_z(z(\mu^2,\lambda^2),\mu^2), \zeta_2 - z(\mu^1,\lambda^1)\rangle \\
&\quad + \langle F_z(z(\mu^2,\lambda^2),\mu^2) - F_z(z(\mu^2,\lambda^2),\mu^1), z(\mu^1,\lambda^1) - z(\mu^2,\lambda^2)\rangle \\
&\le 2lk\|\lambda^1 - \lambda^2\|_\Lambda + 2\epsilon_2\|F_z(z(\mu^2,\lambda^2),\mu^2) - F_z(z(\mu^2,\lambda^2),\mu^1)\|_{Z^*} \\
&\le 2lk\|\lambda^1 - \lambda^2\|_\Lambda + 2\epsilon_2 l_0\|\mu^1 - \mu^2\|_M^\alpha.
\end{aligned} \tag{41}
$$

Here we used the fact that

$$
\|z(\mu^1,\lambda^1) - z(\mu^2,\lambda^2)\|_Z \le \|z(\mu^1,\lambda^1) - \overline{z}\|_Z + \|\overline{z} - z(\mu^2,\lambda^2)\|_Z \le 2\epsilon_2.
$$

Using the inequality $(a+b)^s \le (a^s + b^s)$, where $a, b \ge 0$ and $0 < s \le 1$ (see [12, Inequality 2.12.2, p.32]), it follows from (41) that

$$
\begin{aligned}
&\|u(\mu^1,\lambda^1) - u(\mu^2,\lambda^2)\|_{L^p(\Omega)} \\
&\le (2\epsilon_2 l_0 \rho^{-1})^{1/p}\|\mu^1 - \mu^2\|_M^{\alpha/p} + (2lk\rho^{-1})^{1/p}\|\lambda^1 - \lambda^2\|_\Lambda^{1/p}.
\end{aligned} \tag{42}
$$

By Lemma 2.1, we obtain

$$
\begin{aligned}
&\|y(\mu^1,\lambda^1) - y(\mu^2,\lambda^2)\|_Y \\
&\le C\|u(\mu^1,\lambda^1) - u(\mu^2,\lambda^2) + (\lambda_1^1 - \lambda_1^2)\|_{L^p(\Omega)} \\
&\le C\big(\|u(\mu^1,\lambda^1) - u(\mu^2,\lambda^2)\|_{L^p(\Omega)} + \|\lambda^1 - \lambda^2\|_\Lambda\big) \\
&\le C(2\epsilon_2 l_0\rho^{-1})^{1/p}\|\mu^1 - \mu^2\|_M^{\alpha/p} + C\big((2lk\rho^{-1})^{1/p} + \|\lambda^1 - \lambda^2\|_\Lambda^{1-1/p}\big)\|\lambda^1 - \lambda^2\|_\Lambda^{1/p} \\
&\le C(2\epsilon_2 l_0\rho^{-1})^{1/p}\|\mu^1 - \mu^2\|_M^{\alpha/p} + C\big((2lk\rho^{-1})^{1/p} + (2r_0)^{1-1/p}\big)\|\lambda^1 - \lambda^2\|_\Lambda^{1/p}.
\end{aligned}
$$

Combining this with (42) yields

$$
\begin{aligned}
&\|y(\mu^1,\lambda^1) - y(\mu^2,\lambda^2)\|_Y + \|u(\mu^1,\lambda^1) - u(\mu^2,\lambda^2)\|_{L^p(\Omega)} \\
&\le l_1\|\mu^1 - \mu^2\|_M^{\alpha/p} + l_2\|\lambda^1 - \lambda^2\|_\Lambda^{1/p},
\end{aligned}
$$

where $l_1 := (1+C)(2\epsilon_2 c\rho^{-1})^{1/p}$ and $l_2 := (1+C)(2lk\rho^{-1})^{1/p} + C(2r_0)^{1-1/p}$. The proof of Theorem 1.1 is complete. ■

4. Some Examples

In this section we will give some examples which illustrate Theorem 1.1.

Example 4.1. Suppose that $k = 2, p = 2, N \in \{2, 3\}$, $\epsilon = \epsilon_0 > 0$ and $\delta = -1$ a.e. in Ω. We consider the problem $P_1(\mu, \lambda)$ of finding $u \in L^2(\Omega)$ and $y \in Y$ which minimize the cost function

$$F(y, u, \mu) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Omega)} - \int_\Omega y\mu_1 dx - \int_\Omega u\mu_2 dx \tag{43}$$

with the state equation

$$\begin{cases} Ay = u + \lambda_1 & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases} \tag{44}$$

and pointwise constraints

$$\begin{cases} u \geq \lambda_2 & \text{in } \Omega \\ \epsilon_0 u + y \geq \lambda_3 & \text{in } \Omega, \end{cases} \tag{45}$$

where Ω is a bounded domain in R^N with the Lipschitz boundary $\partial\Omega$, $y_d, u_d \in L^2(\Omega), \mu = (\mu_1, \mu_2) \in M, \lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda$ with $M = L^\infty(\Omega)^2$, $\Lambda = L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$ and A is a strongly elliptic operator.

For $(\overline{\mu}, \overline{\lambda}) = (0, 0)$, by [2, Lemma 2.4], $P_1(0, 0)$ has a unique solution.

Then all conditions of Theorem 1.1 are satisfied. Moreover, there exist positive constants r_j, k_j with $j = 1, 2$ such that for all $(\mu^i, \lambda^i) \in B_M(0, r_1) \times B_\Lambda(0, r_2)$ with $i = 1, 2$, one has

$$\begin{aligned} &\|y(\mu^1, \lambda^1) - y(\mu^2, \lambda^2)\|_Y + \|u(\mu^1, \lambda^1) - u(\mu^2, \lambda^2)\|_{L^2(\Omega)} \\ &\leq k_1\|\mu^1 - \mu^2\|_M^{1/2} + k_2\|\lambda^1 - \lambda^2\|_\Lambda^{1/2}, \end{aligned}$$

where $(y(\mu, \lambda), u(\mu, \lambda))$ is the unique solution of $P_1(\mu, \lambda)$.

In fact, in this case we have

$$F(y, u, \mu) = \int_\Omega f(x, y(x), u(x), \mu(x))dx,$$

where $f(x, y, u, \mu) = \frac{1}{2}|y - y_d(x)|^2 + \frac{\gamma}{2}|u - u_d(x)|^2 - y\mu_1 - u\mu_2$ or

$$f_y(x, y, u, \mu) = (y - y_d(x)) - \mu_1,$$

$$f_u(x, y, u, \mu) = \gamma(u - u_d(x)) - \mu_2.$$

Hence it is easy to see that assumptions $(A1)$ and $(A5)$ are satisfied. Obviously, $f(x, y, u, \mu)$ is convex in (y, u) and

$$\big|f(x, y, u, \mu) - f(x, y, u, \mu')\big| = \big|y(\mu_1 - \mu_1') + u(\mu_2 - \mu_2')\big| \leq (1 + |y|)(1 + |u|)|\mu - \mu'|.$$

Hence $(A2)$ is valid. Since

$$\big(f_z(x, z_1, \mu) - f_z(x, z_2, \mu)\big)(z_1 - z_2) = (y_1 - y_2)^2 + \gamma(u_1 - u_2)^2 \tag{46}$$

for all $x \in \Omega, z_i = (y_i, u_i) \in R^2$ with $i = 1, 2$ and $\mu \in R^2$, it follows that assumption $(A3)$ is fulfilled with $\rho = \gamma$. Also, $(A4)$ is satisfied with

$$a_1 = 1, a_2(|\mu|) = \gamma + |\mu_1| + |\mu_2|, b_1 = b_2 = 1, H_1(|u|) = 1 + |u|$$

and

$$H_2(|u|) = 1 + |u - u_d(x)|, \alpha_1 = \alpha_2 = 1.$$

Thus all conditions of Theorem 1.1 are fulfilled. The conclusion is followed.

It is noted that when $y_d = 0, u_d = 0, y_c = 0$ a.e. in Ω then condition (6) is not satisfied. Therefore in this case, Theorem 4.2 in [2] is not applicable for Example 4.1.

The next example illustrates Theorem 1.1 for the case where the integrand function f is not a quadratic function.

Example 4.2. Let $k = 4, p = 2, N \in \{2, 3\}$ and $\epsilon(x) = \epsilon_0 > 0, \delta(x) = \overline{\delta}\phi(x)$ a.e. in Ω. Here function $\phi \in L^\infty(\Omega)$ and $\overline{\delta} \in R$ are given. We consider problem $P_2(\lambda, \mu)$ of finding $u \in L^2(\Omega)$ and $y \in Y$ which minimize the cost function

$$F(u, \mu) = \int_\Omega f(y(x), u(x), \mu(x))dx \tag{47}$$

with the state equation

$$\begin{cases} -\Delta y + y = u + \lambda_1 & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases} \tag{48}$$

and constraints

$$\begin{cases} u(x) \geq \lambda_2(x) & \text{a. e. in } \Omega \\ \epsilon_0 u(x) \geq \overline{\delta}\phi(x)y(x) + \lambda_3(x) & \text{a. e. in } \Omega, \end{cases} \tag{49}$$

where $\mu = (\mu_1, \mu_2, \mu_3, \mu_4) \in M = L^\infty(\Omega)^4, (\lambda_1, \lambda_2, \lambda_3) \in \Lambda = L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$ and function

$$f(y, u, \mu) = \frac{1}{2}(y - \mu_1)^2 + \frac{\gamma}{2}(u - \mu_2)^2 + \frac{1}{2}(y - \mu_3 u)^2 + \mu_4 y^3.$$

Here γ is a positive constant.

Easily, we see that $P_2(0,0)$ has a unique optimal solution $(\overline{y},\overline{u}) = (0,0)$ corresponding to $(\overline{\mu},\overline{\lambda}) = (0,0)$. We shall show that for $\overline{\delta}$ small enough, there exist positive numbers r_1, r_2 such that for each $(\mu,\lambda) \in B_M(0,r_1)\times B_\Lambda(0,r_2)$, $P_2(\mu,\lambda)$ satisfies all conditions of Theorem 1.1. Moreover, there exist positive constants k_j with $j=1,2$ such that for all $(\mu^i,\lambda^i) \in B_M(0,r_1)\times B_\Lambda(0,r_2)$ with $i=1,2$, one has

$$\begin{aligned}&\|y(\mu^1,\lambda^1)-y(\mu^2,\lambda^2)\|_Y + \|u(\mu^1,\lambda^1)-u(\mu^2,\lambda^2)\|_{L^2(\Omega)}\\ &\le k_1\|\mu^1-\mu^2\|_M^{1/2} + k_2\|\lambda^1-\lambda^2\|_\Lambda^{1/2},\end{aligned}$$

where $(y(\mu,\lambda),u(\mu,\lambda))$ is the unique solution of $P_2(\mu,\lambda)$.

In fact, since $\overline{\delta}$ is small enough, $(A5)$ is valid. Obviously, $(A1)-(A2)$ are satisfied. It remains to show that $(A3)$ and $(A4)$ are satisfied. We have

$$\begin{aligned}f_y(y,u,\mu) &= (y-\mu_1)+(y-\mu_3 u)+3\mu_4 y^2,\\ f_u(y,u,\mu) &= \gamma(u-\mu_2)-\mu_3(y-\mu_3 u).\end{aligned}$$

The Hessian matrix of f in (y,u) is given by

$$H_f(y,u) = \begin{bmatrix} 2+6\mu_4 y & -\mu_3\\ -\mu_3 & \gamma+\mu_3^2\end{bmatrix}.$$

By a detailed computation, we get

$$f_{yy}f_{uu} - f_{yu}^2 = 2\gamma + \mu_3^2 + 6\mu_4 y(\gamma+\mu_3^2) \ge \gamma,$$

and

$$f_{yy}(y,u,\mu) \ge 2 - \frac{\gamma}{\gamma+1} = \frac{\gamma+2}{\gamma+1}$$

for all $y\in R$, $|y| \le \frac{\gamma}{6(\gamma+1)}$, $u\in R$, $|\mu_3|\le 1$ and $|\mu_4|\le 1$. This implies that for each $\mu = (\mu_1,\mu_2,\mu_3,\mu_4) \in R^4$ with $|\mu_3|\le 1$ and $|\mu_4|\le 1$, the function $f(\cdot,\cdot,\mu)$ is convex on $\big(-\frac{\gamma}{6(\gamma+1)},\frac{\gamma}{6(\gamma+1)}\big)\times R$. Moreover, for $z_i = (y_i,u_i) \in \big(-\frac{\gamma}{6(\gamma+1)},\frac{\gamma}{6(\gamma+1)}\big)\times R$, $\mu = (\mu_1,\mu_2,\mu_3,\mu_4)\in R^4$ with $|\mu_4|\le 1$, we obtain

$$\begin{aligned}&\big(f_z(z_1,\mu)-f_z(z_2,\mu)\big)(z_1-z_2)\\ &= 2(y_1-y_2)^2 + (\gamma+\mu_3^2)(u_1-u_2)^2 + 3\mu_4(y_1^2-y_2^2)(y_1-y_2) - 2\mu_3(y_1-y_2)(u_1-u_2)\\ &= (y_1-y_2)^2\big(1+3\mu_4(y_1+y_2)\big) + \gamma(u_1-u_2)^2 + \big(y_1-y_2-\mu_3(u_1-u_2)\big)^2\\ &\ge \gamma(u_1-u_2)^2.\end{aligned}$$

Here we used the fact that

$$1+3\mu_4(y_1+y_2) \ge 1 - \frac{6\gamma}{6(\gamma+1)} = \frac{1}{\gamma+1} > 0.$$

Hence $(A3)$ is satisfied. On the other hand we get

$$f_y(y,u,0) = 2y, \quad f_u(y,u,0) = \gamma u$$

and for any $y, u \in R, \mu = (\mu_1, \ldots, \mu_4), \mu' = (\mu_1', \ldots, \mu_4') \in R^4$

$$f_y(y,u,\mu) - f_y(y,u,\mu') = -(\mu_1 - \mu_1') - u(\mu_3 - \mu_3') + 3y^2(\mu_4 - \mu_4'),$$
$$f_u(y,u,\mu) - f_u(y,u,\mu') = -\gamma(\mu_2 - \mu_2') + u(\mu_3 - \mu_3')(\mu_3 + \mu_3') - y(\mu_3 - \mu_3').$$

Hence $(A4)$ is valid. Thus all assumptions of Theorem 1.1 are fulfilled for $P_2(\mu, \lambda)$.

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