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A NOTE ON EXTREMAL VALUES OF THE SCATTERING NUMBER

Peter Dankelmann, Wayne Goddard*, Charles A. McPillan and Henda C. Swart

Abstract. Let $c(H)$ denote the number of components of graph H. The scattering number of a graph G is the maximum of $c(G-S)-|S|$ taken over all cut-sets S of G. In this note we explore the minimum and maximum scattering number for several families. For example, we show that the minimum scattering number of a triangle-free graph on n vertices is approximately $-n/3$. We also consider the scattering number of some graph products.

1. Introduction

The scattering number of a graph was defined by Jung [5]. He introduced it as a measure related to the hamiltonicity of the graph, but the scattering number is now also regarded as a measure of the vulnerability of a graph, in the same vein as connectivity, integrity and toughness. It is most closely related to toughness; indeed Jung called it the additive dual of toughness.

We will use the notation $c(H)$ to denote the number of components of graph H. Then the ***scattering number*** $\mathrm{sc}(G)$ is:

$$\mathrm{sc}(G) = \max\{\, c(G-S) - |S| : S \subseteq V \text{ and } S \text{ a cut-set}\,\}.$$

A ***scatter set*** is an S which achieves this maximum. We take the view that a set of all but one vertex is by definition a cut-set, and so the scattering number of the complete graph K_n is $2-n$. It is unusual to have a graph parameter which can take on both positive and negative values.

In this note we explore the minimum and maximum scattering number for several families. For example, we show that the minimum scattering number of a triangle-free graph on n vertices is approximately $-n/3$. We also present a couple of results on graph products. Some of our results correct mistakes in the literature, in particular from [6, 7].

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*Corresponding author.

2. Preliminaries

Here are some well-known values:

Proposition 1.
(a) [5] *For the cycle,* $\mathrm{sc}(C_n) = 0$ *for* $n \geq 4$.
(b) [5] *For the path,* $\mathrm{sc}(P_n) = 1$ *for* $n \geq 3$.
(c) [11] *For the complete bipartite graph,* $\mathrm{sc}(K_{a,b}) = b - a$ *if* $a \leq b$ *and* $b \geq 2$.

The following bounds have been observed:

Proposition 2. *For graph* G *of order* n*, independence number* α *and connectivity* κ*:*
(a) [11] $\mathrm{sc}(G) \geq 2\alpha - n$,
(b) [11] *if* G *noncomplete then* $\mathrm{sc}(G) \geq 2 - \kappa$,
(c) [11, 6] $\mathrm{sc}(G) \leq \alpha - \kappa$.

There is a useful upper bound:

Proposition 3. [11]. *For a graph with order* n *and minimum degree* δ*,* $\mathrm{sc}(G) \leq n - 2\delta$.

We will also need the simple formula for the disjoint union:

Proposition 4. [5, 4]. *For any graphs* G *and* H*,* $\mathrm{sc}(G \cup H) = \max(1, \mathrm{sc}(G)) + \max(1, \mathrm{sc}(H))$.

3. Extremal Values for Classes

3.1. Claw-free Graphs and Regular Graphs

Recall that the toughness $t(G)$ of a graph was defined by Chvátal [1] as

$$t(G) = \min\left\{ \frac{|S|}{c(G-S)} : S \subseteq V \text{ and } S \text{ a cut-set} \right\}.$$

The obvious necessary condition for hamiltonicity is that the graph has toughness at least 1. The relationship with toughness is the turning point between positivity and negativity: $\mathrm{sc}(G) \leq 0$ if and only if $t(G) \geq 1$.

The following result corrects typos in [6]:

Theorem 1. *For a graph* G *with connectivity* κ*, vertex cover number* β *and toughness* t*,*

$$\mathrm{sc}(G) \leq \begin{cases} \kappa(\frac{1}{t} - 1) & \text{if } t \geq 1, \\ \beta(\frac{1}{t} - 1) & \text{if } t \leq 1. \end{cases}$$

Proof. By the definition of toughness, $c(G-S) \leq |S|/t$ for any cut-set S. Hence, $c(G-S) - |S| \leq |S|(1/t - 1)$. If $t > 1$, then $(1/t - 1)$ is negative, and so the bound is maximized at S as small as possible, viz. a minimum cut-set. If $t < 1$, the bound is maximized at S as large as possible, viz. a minimum vertex cover. ■

We obtain the following as a consequence of Theorem 1 and Proposition 2b:

Corollary 1. *If $\kappa \geq 2$ and $t = \kappa/2$, then* $\mathrm{sc} = 2 - \kappa$.

For example, the claw-free graphs are known to have toughness half their connectivity [8]. A special case of those is the Cartesian products of two complete graphs. (The ***Cartesian product*** of graphs G and H is $G \square H$ with vertex set $V(G) \times V(H)$ and (u_1, v_1) adjacent to (u_2, v_2) iff $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$.)

Corollary 2. [10]. *For $m, n \geq 2$,* $\mathrm{sc}(K_m \square K_n) = 4 - m - n$.

It also follows that $2-r$ is the minimum value of the scattering number of r-regular graphs.

3.2. Triangle-free Graphs

We will need the following special case of Theorem 1 of [2] (where by maximal we mean that the addition of any edge creates a triangle):

Theorem 2. [2]. *If G is a maximal triangle-free graph on n vertices with minimum degree at least $(n+2)/3$, then there exist two nonadjacent vertices u and v such that $N(u) = N(v)$.*

Theorem 3. *For a triangle-free graph G of order $n \geq 5$,* $\mathrm{sc}(G) \geq (5-n)/3$.

Proof. Let δ denote the minimum degree. We know that $\mathrm{sc}(G) \geq 2 - \delta$ by Proposition 2b. Also, since the neighborhood of a vertex is independent in a triangle-free graph, we have that $\mathrm{sc}(G) \geq 2\delta - n$ by Proposition 2a. Double the first bound added to the second gives that $3\,\mathrm{sc}(G) \geq 4 - n$.

Equality requires that $\delta = (n+2)/3$. Indeed, it follows that for triangle-free G to have $\mathrm{sc}(G) = (4-n)/3$, it must be that $\delta = \alpha = \kappa = (n+2)/3$. For $n = 4$, such a graph is C_4. Since adding edges can only decrease the scattering number, we may assume that G is maximal triangle-free. Then Theorem 2 says that there are two nonadjacent vertices u and v such that $N(u) = N(v)$. If we let $S = N(u)$, it follows that $c(G-S) \geq 3$ provided $n > 4$, and thus $\mathrm{sc}(G) \geq 3 - \delta$. Hence for $n \neq 4$, $3\,\mathrm{sc}(G) > 4 - n$ and thus $\mathrm{sc}(G) \geq (5-n)/3$. ■

The above theorem is sharp in that there are triangle-free graphs with this scattering number. Let $M_m = \overline{C_{3m+2}^m}$, the complement of the m^{th} power of the $(3m+2)$-cycle. For example, $M_1 = C_5$ and M_2 is the Möbius ladder shown in Figure 1. This graph is $(m+1)$-regular.

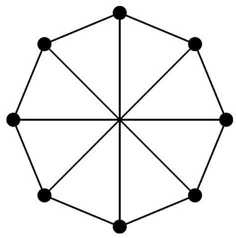

Fig. 1. A triangle-free graph M_2 with smallest scattering number.

We sketch the argument that M_m has the claimed scattering number. We note that every induced copy F of $2K_2$ in M_m is a dominating set (that is, every other vertex is adjacent to a vertex of F). It follows that if S is a cut-set whose removal produces more than 2 components, it must be that case that at most one of these components is nontrivial. If we let T denote the set of vertices of the trivial components of $M_m - S$, it is not hard to show that $|N(T)| \geq |T| + \delta - 1$ (implicit in [9]). Clearly $S \supseteq N(T)$. Hence

$$c(G-S) - |S| \leq (|T|+1) - (|T| + \delta - 1) = 2 - \delta.$$

It follows that $\mathrm{sc}(M_m) = 1 - m$.

3.3. Planar Graphs

The question of whether there is an infinite family of $5/2$-tough planar graphs remains unresolved [3]. In contrast, the question of the minimum scattering number for planar graphs is readily resolved, since the scattering number is at least $2 - \kappa$ (Proposition 2b) and hence at least -3, since the maximum connectivity of a planar graph is 5.

A planar graph with scattering number -3 is given by the following. Construct H_m as follows. Start with vertices x and y, and two cycles A and B of length m. Say the cycles are $a_1, \ldots, a_m, a_1$ and $b_1, \ldots, b_m, b_1$. Then add edges a_ib_i and b_ia_{i+1} (indices modulo m). Further join x to all of A and y to all of B. The graph H_7 is shown in Figure 2.

Theorem 4. *For $m \geq 5$, $\mathrm{sc}(H_m) = -3$.*

Proof. The graph H_5 is the icosahedron—this is 5-connected claw-free, and so by Corollary 1 has $\mathrm{sc}(H_5) = -3$. So, let $m \geq 6$.

Let S be a cut-set of H_m. Let $S' = S - \{x, y\}$ and $H' = H_m - \{x, y\}$. There are three cases. If both x and y are in S, then $|S| = |S'| + 2$. Since H' is 4-connected and claw-free, it follows that $c(H' - S') - |S'| \leq -2$ so that $c(H_m - S) - |S| \leq -4$. On the other hand, if neither x nor y is in S, then $c(H_m - S) \leq 2$ and so $c(H_m - S) - |S| \leq 2 - |S| \leq -3$. So suppose exactly one of x or y is in S, say x. If $N(y) \subseteq S$, then $c(H_m - S) - |S| \leq (\lfloor m/2 \rfloor + 1) - (m+1) = -\lfloor m/2 \rfloor \leq -3$. Otherwise, y is not isolated in $H_m - S$, and so $c(H_m - S) \leq c(H' - S')$. Since $|S| = |S'| + 1$, it follows that $c(H_m - S) - |S| \leq c(H' - S') - |S'| - 1 \leq \mathrm{sc}(H') - 1 \leq -3$. Therefore, $\mathrm{sc}(H_m) = -3$ for $m \geq 5$. ■

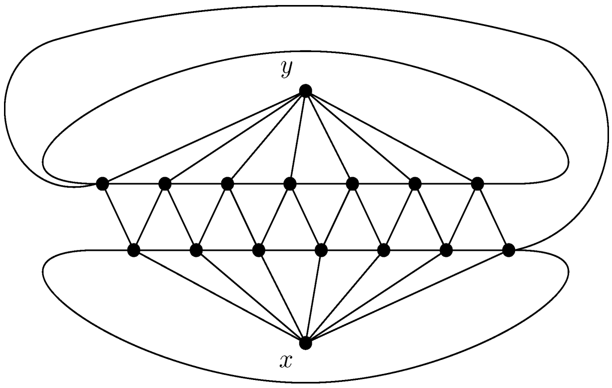

Fig. 2. A planar graph H_7 with minimum scattering number.

4. Products and Thorn Graphs

In this section we consider thorn graphs and related product graphs, first considered in [6, 7].

4.1. Coronas and Thorn Graphs

Recall that the ***corona*** $\mathrm{cor}(G)$ of a graph G is obtained by adding for each vertex in G a new end-vertex adjacent only to it. More generally, given a graph G with vertex set $\{v_1, \ldots, v_n\}$ and nonnegative integers $\{p_1, \ldots, p_n\}$, the ***thorn graph*** $G^*(p_1, \ldots, p_n)$ is obtained from G by adding p_i new vertices of degree 1 adjacent to v_i for each i. The corona is the case where all $p_i = 1$.

Kirlangiç and Aytaç [7] established results for some choices of the p_i (though their proofs are incomplete). We give here a succinct formula for the scattering number of a thorn graph that generalizes results of [7, 6]:

Theorem 5. *Consider the thorn graph $G^*(p_1, \ldots, p_n)$ with all $p_i \geq 1$. Then*

$$\mathrm{sc}(G^*) = \sum_{i=1}^{n} p_i \; - n + \alpha(G - L),$$

where $L = \{\, v_i : p_i \geq 2 \,\}$ and α is the independence number.

Proof. Out of all scatter sets of G^*, let S be a largest scatter set. It is easy to see that for any graph (except K_2) a scatter set cannot contain a vertex of degree 1 (since removing such a vertex from S cannot decrease the number of components of $G - S$). Thus, $S \subseteq V(G)$.

We claim that S contains all of L. For, if not, adding $v_i \in L$ to S increases the number of components of $G - S$ and so $c(G - S) - |S|$ does not decrease. Further, we claim that if v_1 and v_2 are adjacent vertices of $G - L$, then S contains at least one of them. For, if not, adding v_1 to S increases the number of components, as before. That is, S consists of L and a vertex cover of the graph $G - L$.

It follows that every end-vertex of G^* is in a separate component of $G^* - S$, and so $c(G^* - S) = \sum_{i=1}^{n} p_i$. At the same time, S must be the smallest set with the established properties; that is, S consists of L and a minimum vertex cover of $G - L$. This implies that $|S| = n - \alpha(G - L)$, and the result follows. ∎

Corollary 3. [7]. *For graph G of independence number α,* $\mathrm{sc}(\mathrm{cor}(G)) = \alpha$.

Corollary 4. [7]. *Consider the thorn graph $G^*(p_1, \ldots, p_n)$ with all $p_i \geq 2$. Then* $\mathrm{sc}(G^*) = \sum_{i=1}^{n} p_i - n$.

As a consequence we obtain the formula for the scattering number of binomial trees. The ***binomial tree*** B_i is defined recursively as $B_0 = K_1$ and $B_{i+1} = \mathrm{cor}(B_i)$ for $i \geq 0$. The binomial tree B_i has 2^i vertices.

Corollary 5. [6]. *For $m \geq 2$, the binomial tree has* $\mathrm{sc}(B_m) = 2^{m-2}$.

Proof. The tree B_m is the corona of the tree B_{m-1}. The tree B_{m-1} has a matching and hence has independence number half its order. ∎

4.2. Tensor Product

In this section we correct results about the tensor product from [6]. The ***tensor product*** of G and H is $G \times H$ with vertex set $V(G) \times V(H)$ and (u_1, v_1) adjacent to (u_2, v_2) iff $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

Theorem 6. *If graphs G and H have orders m and n and both have a perfect matching, then* $\mathrm{sc}(\mathrm{cor}(G) \times \mathrm{cor}(H)) = mn$.

Proof. Let $F = \mathrm{cor}(G) \times \mathrm{cor}(H)$. The graphs $\mathrm{cor}(G)$ and $\mathrm{cor}(H)$ have spanning subgraphs $(m/2)P_4$ and $(n/2)P_4$. Thus F is a supergraph of $F' = (mn/4)P_4$. It is straight-forward to observe that the scattering number of $P_4 \times P_4$ is 4. Thus by Proposition 4, we have $\mathrm{sc}(F') = mn$, and so $\mathrm{sc}(F) \leq mn$.

On the other hand, let S be the set of the mn vertices in $V(G) \times V(H)$, and consider $F - S$. Consider an end-vertex v' in $\mathrm{cor}(G)$ adjacent to v, and an end-vertex u' in $cor(H)$ adjacent to u. Then the vertex (v', u') has only one neighbor (v, u) in F, and that neighbor is in S. Also the vertex (v, u') has only neighbors of the form (w, u) in F, and hence its only neighbor outside S is (v', u). This means that $F - S$ is the union on mn isolates and mn K_2's. That is, $\mathrm{sc}(F - S) = 2mn$. Hence $\mathrm{sc}(F) \geq mn$. ∎

As a consequence we obtain the correct formula for the scattering number of the tensor product of two binomial trees:

Corollary 6. $\mathrm{sc}(B_m \times B_n) = 2^{n+m-2}$

4.3. Cartesian Products

Finally, we consider results about the Cartesian product. Theorem 3.1 of [7] claims a formula for the scattering number of the Cartesian product of thorn graphs. But the formula is incorrect. For example, the theorem claims a negative value for the Cartesian product of any two coronas. But actually, since every thorn graph has an end-vertex, the Cartesian product of two thorn graphs has a vertex of degree 2, and so by Proposition 2b, $\mathrm{sc}(G^*\Box H^*) \geq 0$. (It is unclear what proportion of the time the claimed formula in [7] is correct. Whatever the case, the "proof" is definitely incomplete).

We were unable to determine a formula for $\mathrm{sc}(G^*\Box H^*)$ in general. One special case is when $G^*\Box H^*$ is hamiltonian, since that implies the scattering number is non-positive, so that $\mathrm{sc}(G^*\Box H^*) = 0$. In that regard, here is a partial result.

Theorem 7. *If the Cartesian product $G\Box H$ of graphs G and H is hamiltonian and has even order, then the Cartesian product of their coronas is also hamiltonian, and so* $\mathrm{sc}(\mathrm{cor}(G)\Box\,\mathrm{cor}(H)) = 0$.

Proof. For any vertex x of G or H, let x' denote the vertex adjacent to x introduced in the corona. Partition the vertex set of $\mathrm{cor}(G)\Box\,\mathrm{cor}(H)$ into the quartets $\{(x, y), (x', y), (x, y'), (x', y')\}$ for each $x \in V(G)$ and $y \in V(H)$. Each quartet induces a 4-cycle. We will build a hamiltonian cycle D of $\mathrm{cor}(G)\Box\,\mathrm{cor}(H)$ that visits all vertices of a quartet consecutively.

Let C be the hamiltonian cycle of $G\Box H$. Say the first two vertices of C are (x_1, y) and (x_2, y). Then the hamiltonian cycle D of $\mathrm{cor}(G)\Box\,\mathrm{cor}(H)$ starts the first quartet at (x_1, y) and ends at (x_1, y'). Then it goes to (x_2, y') and ends that quartet at (x_2, y). The process continues—D traverses the entire quartet for each vertex of C. Eventually we re-enter the first quartet; since there is an even number of vertices in C, this re-entry is at (x_1, y), and so D is completed. ∎

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Peter Dankelmann
Department of Mathematics
University of Johannesburg
South Africa

Wayne Goddard
Department of Mathematical Sciences
Clemson University
U.S.A.
E-mail: goddard@clemson.edu

Charles A. McPillan
Department of Mathematical Sciences
Clemson University
U.S.A.

Henda C. Swar
School of Mathematical Sciences
University of KwaZulu-Natal
South Africa