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ON THE RELAXED HYBRID-EXTRAGRADIENT METHOD FOR SOLVING CONSTRAINED CONVEX MINIMIZATION PROBLEMS IN HILBERT SPACES

L. C. Ceng and C. Y. Chou*

Abstract. In 2006, Nadezhkina and Takahashi [N. Nadezhkina, W. Takahashi, Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings, *SIAM J. Optim.*, 16(4) (2006), 1230-1241.] introduced an iterative algorithm for finding a common element of the fixed point set of a nonexpansive mapping and the solution set of a variational inequality in a real Hilbert space via combining two well-known methods: hybrid and extragradient. In this paper, motivated by Nadezhkina and Takahashi's hybrid-extragradient method we propose and analyze a relaxed hybrid-extragradient method for finding a solution of a constrained convex minimization problem, which is also a common element of the solution set of a variational inclusion and the fixed point set of a strictly pseudocontractive mapping in a real Hilbert space. We obtain a strong convergence theorem for three sequences generated by this algorithm. Based on this result, we also construct an iterative algorithm for finding a solution of the constrained convex minimization problem, which is also a common fixed point of two mappings taken from the more general class of strictly pseudocontractive mappings.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a

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*Corresponding author.

nonempty closed convex subset of H and let P_C be the metric projection from H onto C. A mapping A of C into H is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in C.$$

A mapping A of C into H is called k-Lipschitz continuous if there exists a constant $k > 0$ such that

$$\|Au - Av\| \leq k\|u - v\|, \quad \forall u, v \in C.$$

Let the mapping A from C to H be monotone and Lipschitz continuous. The variational inequality is to find a $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

The solution set of the variational inequality (1.1) is denoted by $\mathrm{VI}(C, A)$. The variational inequality was first discussed by Lions [16] and now is well known; there are various approaches to solving this problem in finite-dimensional and infinite-dimensional spaces, and the research is intensively continued. This problem has many applications in partial differential equations, optimal control, mathematical economics, optimization, mathematical programming, mechanics, and other fields; see, e.g., [10, 20, 31]. In the meantime, to construct a mathematical model which is as close as possible to a real complex problem, we often have to use more than one constraint. Solving such problems, we have to obtain some solution which is simultaneously the solution of two or more subproblems or the solution of one subproblem on the solution set of another subproblem. Actually, these subproblems can be given by problems of different types. For example, Antipin considered a finite-dimensional variant of the variational inequality, where the solution should satisfy some related constraint in inequality form [1] or some system of constraints in inequality and equality form [2]. Yamada [30] considered an infinite-dimensional variant of the solution of the variational inequality on the fixed point set of some mapping.

A mapping A of C into H is called α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Au - Av, u - v \rangle \geq \alpha\|Au - Av\|^2, \quad \forall u, v \in C;$$

see [6]. It is obvious that an α-inverse strongly monotone mapping A is monotone and Lipschitz continuous. A mapping S of C into itself is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|, \quad \forall u, v \in C;$$

see [28,33]. We denote by $F(S)$ the fixed point set of S; i.e., $F(S) = \{x \in C : Sx = x\}$.

A set-valued mapping M with domain $D(M)$ and range $R(M)$ in H is called monotone if its graph $G(M) = \{(x, f) \in H \times H : x \in D(M), f \in Mx\}$ is a monotone set in $H \times H$; i.e., M is monotone if and only if

$$(x, f), (y, g) \in G(M) \ \Rightarrow \ \langle x - y, f - g \rangle \geq 0.$$

A monotone set-valued mapping M is called maximal if its graph $G(M)$ is not properly contained in the graph of any other monotone mapping in H.

Let Φ be a single-valued mapping of C into H and M be a multivalued mapping with $D(M) = C$. Consider the following variational inclusion: find $u \in C$, such that

$$0 \in \Phi(u) + Mu. \tag{1.2}$$

We denote by $\mathrm{VI}(C, \Phi, M)$ the solution set of the variational inclusion (1.2). In particular, if $\Phi = M = 0$, then $\mathrm{VI}(C, \Phi, M) = C$.

In 1998, Huang [7] studied problem (1.2) in the case where M is maximal monotone and Φ is strongly monotone and Lipschitz continuous with $D(M) = C = H$. Subsequently, Zeng, Guu and Yao [13] further studied problem (1.2) in the case which is more general than Huang's one [7]. Moreover, the authors [13] obtained the same strong convergence conclusion as in Huang's result [7]. In addition, the authors also gave the geometric convergence rate estimate for approximate solutions.

In 2003, for finding an element of $F(S)\cap\mathrm{VI}(C, A)$ under the assumption that a set $C \subset H$ is nonempty, closed and convex, a mapping S of C into itself is nonexpansive and a mapping A of C into H is α-inverse strongly monotone, Takahashi and Toyoda [29] introduced the following iterative algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \tag{1.3}$$

for every $n = 0, 1, 2, ...$, where $x_0 = x \in C$ chosen arbitrarily, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $F(S) \cap \mathrm{VI}(C, A)$ is nonempty, the sequence $\{x_n\}$ generated by (1.3) converges weakly to some $z \in F(S) \cap \mathrm{VI}(C, A)$.

In 2006, to solve this problem (i.e., to find an element of $F(S)\cap\mathrm{VI}(C, A)$), Iiduka and Takahashi [12] introduced the following iterative scheme by a hybrid method:

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x, \end{cases} \tag{1.4}$$

for every $n = 0, 1, 2, ...$, where $x_0 = x \in C$ chosen arbitrarily, $0 \le \alpha_n \le c < 1$ and $0 < a \le \lambda_n \le b < 2\alpha$. They proved that if $F(S) \cap \mathrm{VI}(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(S)\cap\mathrm{VI}(C,A)}x$. Generally speaking, the algorithm suggested by Iiduka and Takahashi is based on two well-known types of methods, i.e., on the projection-type method for solving variational inequality and so-called hybrid or outer-approximation method for solving fixed point problem. The idea of "hybrid" or "outer-approximation " types of methods was originally introduced by Haugazeau in 1968 and was successfully generalized and extended in many papers; see, e.g., [3-5, 18].

It is easy to see that the class of α-inverse strongly monotone mappings in the above mentioned problem of Takahashi and Toyoda [29] is a quite important class of mappings in various classes of well-known mappings. It is also easy to see that while α-inverse strongly monotone mappings are tightly connected with the important class of nonexpansive mappings, α-inverse strongly monotone mappings are also tightly connected with a more general and also quite important class of strictly pseudocontractive mappings. (A mapping $T : C \to C$ is called κ-strictly pseudocontractive if there exists a constant $0 \le \kappa < 1$ such that $\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2$ for all $x, y \in C$.) That is, if a mapping $T : C \to C$ is nonexpansive, then the mapping $I - T$ is $\frac{1}{2}$-inverse strongly monotone; moreover, $F(T) = \mathrm{VI}(C, I - T)$ (see, e.g., [29]). At the same time, if a mapping $T : C \to C$ is κ-strictly pseudocontractive, then the mapping $I - T$ is $\frac{1-\kappa}{2}$-inverse-strongly monotone and $\frac{2}{1-\kappa}$-Lipschitz continuous.

In 1976, for finding a solution of the nonconstrained variational inequality in the finite-dimensional Euclidean space $\mathbf{R}^n$ under the assumption that a set $C \subset \mathbf{R}^n$ is nonempty, closed and convex and a mapping $A : C \to \mathbf{R}^n$ is monotone and k-Lipschitz-continuous, Korpelevich [15] introduced the following so-called extragradient method:

$$(1.5) \qquad \begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases}$$

for every $n = 0, 1, 2, ...$, where $x_0 = x \in C$ chosen arbitrarily and $\lambda \in (0, \frac{1}{k})$. She showed that if $\mathrm{VI}(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.5) converge to the same point $z \in \mathrm{VI}(C, A)$. The idea of the extragradient iterative algorithm introduced by Korpelevich [15] was successfully generalized and extended not only in Euclidean but also in Hilbert and Banach spaces; see, e.g., [9,11,19,24].

In 2006, by combining hybrid and extragradient methods, Nadezhkina and Takahashi [22] introduced an iterative algorithm for finding a common element of the fixed point set of a nonexpansive mapping and the solution set of the variational inequality for a monotone, Lipschitz-continuous mapping in a real Hilbert space. They gave a strong convergence theorem for three sequences generated by this algorithm.

Theorem 1.1. (see [22, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone and k-Lipschitz-continuous mapping and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

for every $n = 0, 1, 2, ...,$ *where* $x_0 = x \in C$ *chosen arbitrarily,* $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, \frac{1}{k})$ *and* $\{\alpha_n\} \subset [0, c]$ *for some* $c \in [0, 1)$. *Then the sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *converge strongly to* $P_{F(S)\cap \mathrm{VI}(C,A)}x$.

On the other hand, Xu [17] very recently considered the following constrained convex minimization problem

$$\text{minimize } \{f(x) : x \in C\}, \tag{1.6}$$

where $f : C \to \mathbf{R}$ is a real-valued convex function. If f is (Frechet) differentiable, then the gradient-projection method (for short, GPM) generates a sequence $\{x_n\}$ via the recursive formula

$$x_{n+1} = P_C(x_n - \lambda \nabla f(x_n)), \quad \forall n \geq 0, \tag{1.7}$$

or more generally,

$$x_{n+1} = P_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0, \tag{1.8}$$

where in both (1.7) and (1.8) the initial guess x_0 is taken from C arbitrarily, and the parameters, λ or λ_n, are positive real numbers. The convergence of the algorithms (1.7) and (1.8) depends on the behavior of the gradient ∇f. As a matter of fact, it is known that if ∇f is strongly monotone and Lipschitz continuous; that is, there are constants $\eta, L > 0$ satisfying the properties

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq \eta \|x - y\|^2 \tag{1.9}$$

and

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \tag{1.10}$$

for all $x, y \in C$, then, for $0 < \lambda < 2\eta/L^2$, the operator

$$T := P_C(I - \lambda \nabla f) \tag{1.11}$$

is a contraction; hence, the sequence $\{x_n\}$ defined by algorithm (1.7) converges in norm to the unique solution of the minimization (1.6). More generally, if the sequence $\{\lambda_n\}$ is chosen to satisfy the property

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\eta/L^2 \tag{1.12}$$

then the sequence $\{x_n\}$ defined by algorithm (1.8) converges in norm to the unique minimizer of (1.6).

However, if the gradient ∇f fails to be strongly monotone, the operator T defined in (1.11) could fail to be contractive; consequently, the sequence $\{x_n\}$ generated by algorithm (1.7) may fail to converge strongly (see [17, Section 4]). The following states that if the Lipschitz condition (1.10) holds, then algorithms (1.7) and (1.8) can still converge in the weak topology.

Theorem 1.2. *Assume the minimization* (*1.6*) *is consistent and let* Ω *denote its solution set. Assume the gradient* ∇f *satisfies the Lipschitz condition* (*1.10*). *Let the sequence* $\{\lambda_n\}$ *of parameters satisfy the condition*

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{L}. \tag{1.13}$$

Then the sequence $\{x_n\}$ *generated by the gradient-projection algorithm (1.8) converges weakly to a minimizer of* (*1.1*).

The proof of Theorem 1.2 given in the current existing literature heavily depends on the function f; see Levitin and Polyak [14]. However, Xu [17] gave an alternative operator-oriented approach to algorithm (1.8); namely, an averaged mapping approach. In [17], he gave his averaged mapping approach to the gradient-projection algorithm (1.8) and the relaxed gradient-projection algorithm. Moreover, he constructed a counterexample which shows that algorithm (1.7) does not converge in norm in an infinite-dimensional space, and also presented two modifications of gradient projection algorithms which are shown to have strong convergence. The following is one of two modifications.

Theorem 1.3. (see [17, Theorem 4.4]). *Assume the minimization* (*1.1*) *is consistent and let* Ω *be its solution set. Assume the gradient* ∇f *satisfies the Lipschitz condition* (*1.10*). *Let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} y_n = P_C(x_n - \lambda_n \nabla f(x_n)), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

for every $n = 0, 1, 2, ...,$ *where* $x_0 = x \in C$ *chosen arbitrarily, and the sequence* $\{\lambda_n\}$ *satisfies the condition* (*1.13*). *Then* $\{x_n\}$ *converges strongly to* $P_\Omega x$.

Furthermore, related iterative methods for solving fixed point problems, variational inequalities, equilibrium problems and optimization problems can be found in [8,24-26,32,36-38].

In this paper, let C be a nonempty closed convex subset of a real Hilbert space H. Assume the minimization (1.6) is consistent and let Ω denote its solution set. Let $\Phi : C \to H$ be an α-inverse strongly monotone mapping, M be a maximal monotone mapping with $D(M)=C$ and $S:C\to C$ be a κ-strictly pseudocontractive mapping such that $F(S)\cap\Omega\cap\mathrm{VI}(C,\Phi,M)\neq\emptyset$. Assume the gradient ∇f satisfies the Lipschitz condition (1.10). Motivated by Nadezhkina and Takahashi's hybrid-extragradient method [22] we introduce the following relaxed hybrid-extragradient algorithm

$$\left\{\begin{array}{l} y_n = P_C(x_n-\lambda_n\nabla f(x_n)),\\ t_n = P_C(x_n-\lambda_n\nabla f(y_n)),\\ z_n = (1-\alpha_n-\hat{\alpha}_n)x_n+\alpha_n J_{M,\mu_n}(t_n-\mu_n\Phi(t_n))\\ \qquad +\hat{\alpha}_n S J_{M,\mu_n}(t_n-\mu_n\Phi(t_n)),\\ C_n=\{z\in C:\|z_n-z\|\le\|x_n-z\|\},\\ Q_n=\{z\in C:\langle x_n-z,x-x_n\rangle\ge 0\},\\ x_{n+1}=P_{C_n\cap Q_n}x \end{array}\right. \tag{1.14}$$

for every $n=0,1,2,...$, where $J_{M,\mu_n}=(I+\mu_n M)^{-1}$, $x_0=x\in C$ chosen arbitrarily, $\{\lambda_n\}\subset(0,\frac{1}{L})$, $\{\mu_n\}\subset(0,2\alpha]$ and $\{\alpha_n\},\{\hat{\alpha}_n\}\subset(0,1]$ such that $\alpha_n+\hat{\alpha}_n\le 1$. It is proven that under very mild conditions three sequences $\{x_n\},\{y_n\},\{z_n\}$ generated by (1.14) converge strongly to the same point $P_{F(S)\cap\Omega\cap\mathrm{VI}(C,\Phi,M)}x$. It is worth pointing out that whenever $\Phi=M=0$ and $S=I$, we have $F(S)=\mathrm{VI}(C,\Phi,M)=C$. In this case, the problem of finding an element of $F(S)\cap\Omega\cap\mathrm{VI}(C,\Phi,M)$ reduces to the one of finding an element of Ω. Thus, our result improves and extends Xu's corresponding one [17], i.e., the above Theorem 1.3. Based on our main result, we also construct an iterative algorithm for finding a solution of the minimization (1.6), which is also a common fixed point of two mappings taken from the more general class of strictly pseudocontractive mappings.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and C be a nonempty closed convex subset of H. We write $\to$ to indicate that the sequence $\{x_n\}$ converges strongly to x and $\rightharpoonup$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n):=\{x:x_{n_i}\rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

For every point $x\in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x-P_C x\|\le\|x-y\|,\quad\forall y\in C.$$

P_C is called the metric projection of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C; that is, there holds the following relation

$$\langle P_C x-P_C y,x-y\rangle\ge\|P_C x-P_C y\|^2,\quad\forall x,y\in H.$$

Consequently, P_C is nonexpansive and monotone. It is also known that P_C is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \tag{2.1}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.2}$$

for all $x \in H, y \in C$; see [28,34] for more details. Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality, this implies that

$$x \in \mathrm{VI}(C, A) \quad \Leftrightarrow \quad x = P_C(x - \lambda Ax) \ \forall \lambda > 0. \tag{2.3}$$

It is also known that H satisfies the Opial condition [21]. That is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.4}$$

holds for every $y \in H$ with $y \neq x$.

A set-valued mapping $M : D(M) \subset H \to 2^H$ is called monotone if for all $x, y \in D(M)$, $f \in Mx$ and $g \in My$ imply

$$\langle f - g, x - y \rangle \geq 0.$$

A set-valued mapping M is called maximal monotone if M is monotone and $(I + \lambda M)D(M) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(M)$ the graph of M. It is known that a monotone mapping M is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y \rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$.

Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \ \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$; see [23].

Assume that $M : D(M) \subset H \to 2^H$ is a maximal monotone mapping. Then, for $\lambda > 0$, associated with M, the resolvent operator $J_{M,\lambda}$ can be defined as

$$J_{M,\lambda} x = (I + \lambda M)^{-1} x, \quad \forall x \in H.$$

In terms of Huang [7] (see also [13]), there holds the following property for the resolvent operator $J_{M,\lambda} : H \to H$.

Lemma 2.1. *$J_{M,\lambda}$ is single-valued and firmly nonexpansive, i.e.,*

$$\langle J_{M,\lambda}x - J_{M,\lambda}y, x - y\rangle \geq \|J_{M,\lambda}x - J_{M,\lambda}y\|^2, \quad \forall x, y \in H.$$

Consequently, $J_{M,\lambda}$ is is nonexpansive and monotone.

Lemma 2.2. (see [35]). *There holds the relation:*

$$\|\lambda x + \mu y + \nu z\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 + \nu\|z\|^2 - \lambda\mu\|x - y\|^2 - \mu\nu\|y - z\|^2 - \lambda\nu\|x - z\|^2$$

for all $x, y, z \in H$ and $\lambda, \mu, \nu \in [0, 1]$ with $\lambda + \mu + \nu = 1$.

Lemma 2.3. *Let M be a maximal monotone mapping with $D(M) = C$. Then for any given $\lambda > 0$, $u \in C$ is a solution of problem (1.2) if and only if $u \in C$ satisfies*

$$u = J_{M,\lambda}(u - \lambda\Phi(u)).$$

Proof.

$$\begin{aligned} 0 \in \Phi(u) + Mu \quad &\Leftrightarrow \; u - \lambda\Phi(u) \in u + \lambda Mu \\ &\Leftrightarrow \; u = (I + \lambda M)^{-1}(u - \lambda\Phi(u)) \\ &\Leftrightarrow \; u = J_{M,\lambda}(u - \lambda\Phi(u)). \end{aligned}$$

∎

Given a nonempty closed convex subset C of a real Hilbert space H and a self-mapping $S : C \to C$. Recall that S is a strict pseudocontraction if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.5}$$

Recall also that $S : C \to C$ is called a quasi-strict pseudocontraction if the fixed point set of S, $F(S)$, is nonempty and if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Sx - p\|^2 \leq \|x - p\|^2 + \kappa\|x - Sx\|^2 \quad \text{for all } x \in C \text{ and } p \in F(S). \tag{2.6}$$

Note that we also say that S is a κ-strict pseudocontraction if condition (2.5) holds and respectively, S is a κ-quasi-strict pseudocontraction if condition (2.6) holds. Here we state some properties of these mappings as follows.

Lemma 2.4. (see [27, Proposition 2.1]). *Assume C is a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a self-mapping of C.*

(i) *If S is a κ-strict pseudocontraction, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1 + \kappa}{1 - \kappa}\|x - y\|, \quad \forall x, y \in C. \tag{2.7}$$

(ii) *If S is a κ-strict pseudocontraction, then the mapping $I - S$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - S)x_n \to 0$, then $(I - S)\tilde{x} = 0$.*

(*iii*) *If S is a κ-quasi-strict pseudocontraction, then the fixed point set $F(S)$ of S is closed and convex so that the projection $P_{F(S)}$ is well defined.*

Remark 2.1. In Lemma 2.4, the statement (ii), i.e., the demiclosedness principle for strict pseudocontractions in Hilbert spaces, can be proven by the Opial condition for Hilbert space; see the proof in Marino and Xu [27, Proposition 2.1].

Proposition 2.1. (see [13]). *Let M be a maximal monotone mapping with $D(M) = C$ and let $V : C \to H$ be a strongly monotone, continuous and single-valued mapping. Then for each $z \in H$, the equation $z \in Vx+\lambda Mx$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 2.5. *Let M be a maximal monotone mapping with $D(M) = C$ and $A : C \to H$ be a monotone, continuous and single-valued mapping. Then $(I + \lambda(M + A))C = H$ for each $\lambda > 0$. In this case, $M + A$ is maximal monotone.*

Proof. For each fixed $\lambda > 0$, put $V = I + \lambda A$. Then $V : C \to H$ is a strongly monotone, continuous and single-valued mapping. In terms of Proposition 2.1, we obtain $(V + \lambda M)C = H$. That is, $(I + \lambda(M + A))C = H$. It is clear that $M + A$ is monotone. Therefore, $M + A$ is maximal monotone.

3. Strong Convergence Theorem

In this section we prove a strong convergence theorem by the relaxed hybrid-extragradient method for finding a solution of the constrained convex minimization problem (1.6), which is also a common solution of the variational inclusion (1.2) and the fixed-point problem of a κ-strict pseudocontraction in a real Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume the minimization (1.6) is consistent and let Ω be its solution set. Let $\Phi : C \to H$ be an α-inverse strongly monotone mapping, M be a maximal monotone mapping with $D(M) = C$ and $S : C \to C$ be a κ-strictly pseudocontractive mapping such that $F(S)\cap\Omega\cap\mathrm{VI}(C, \Phi, M) \neq \emptyset$. Assume the gradient ∇f satisfies the Lipschitz condition (1.10). For $x_0 = x \in C$ chosen arbitrarily, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences generated by*

$$(1.14)\qquad \begin{cases} y_n = P_C(x_n - \lambda_n \nabla f(x_n)), \\ t_n = P_C(x_n - \lambda_n \nabla f(y_n)), \\ z_n = (1 - \alpha_n - \hat{\alpha}_n)x_n + \alpha_n J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) \\ \qquad +\hat{\alpha}_n S J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\ C_n = \{z \in C : \|z_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0, 1, 2, \ldots$, *where the following conditions hold:*

(*i*) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, \frac{1}{L})$;

(*ii*) $\{\mu_n\} \subset [\epsilon, 2\alpha]$ *for some* $\epsilon \in (0, 2\alpha]$;

(*iii*) $\alpha_n + \hat{\alpha}_n \leq 1$ *for every* $n = 0, 1, 2, \ldots$;

(*iv*) $\{\alpha_n\} \subset [c, 1]$ *for some* $c \in (\kappa, 1]$ *and* $\{\hat{\alpha}_n\} \subset [\hat{c}, 1]$ *for some* $\hat{c} \in (0, 1]$.

Then the sequences $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *converge strongly to* $P_{F(S)\cap\Omega\cap\mathrm{VI}(C,\Phi,M)}x$.

Proof. Putting $\hat{t}_n = J_{M,\mu_n}(t_n - \mu_n \Phi(t_n))$, we have $z_n = (1 - \alpha_n - \hat{\alpha}_n)x_n + \alpha_n\hat{t}_n + \hat{\alpha}_n S\hat{t}_n$ for every $n = 0, 1, 2, \ldots$. Note that the Lipschitz condition (1.10) implies that the gradient ∇f is $\frac{1}{L}$-inverse strongly monotone [1,39] (see also [17]). It is obvious that C_n is closed and Q_n is closed and convex for every $n = 0, 1, 2, \ldots$. As $C_n = \{z \in C : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z\rangle \leq 0\}$, we also have that C_n is convex for every $n = 0, 1, 2, \ldots$. As $Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}$, we have $\langle x_n - z, x - x_n\rangle \geq 0$ for all $z \in Q_n$ and hence $x_n = P_{Q_n}x$ by (2.1).

For the rest of the proof, we divide it into several steps.

Step 1. We claim that $F(S)\cap\Omega\cap\mathrm{VI}(C, \Phi, M) \subset C_n \cap Q_n$ for every $n = 0, 1, 2, \ldots$. Indeed, observe first that $q \in C$ solves the minimization (1.6) if and only if q solves the fixed point equation

$$q = P_C(I - \lambda\nabla f)q,$$

where $\lambda > 0$ is any fixed positive number. Thus, it follows immediately that $q \in C$ solves the minimization (1.6) if and only if q solves the variational inequality of finding $q \in C$ such that

$$\langle\nabla f(q), x - q\rangle \geq 0, \quad \forall x \in C.$$

Now, take a fixed $u \in F(S)\cap\Omega\cap\mathrm{VI}(C, \Phi, M)$ arbitrarily. From (2.2), monotonicity of ∇f, and $u \in \mathrm{VI}(C, \nabla f)$, we have

$$\begin{aligned}
\|t_n - u\|^2 &\leq \|x_n - \lambda_n\nabla f(y_n) - u\|^2 - \|x_n - \lambda_n\nabla f(y_n) - t_n\|^2 \\
&= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle\nabla f(y_n), u - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n(\langle\nabla f(y_n) \\
&\quad -\nabla f(u), u - y_n\rangle + \langle\nabla f(u), u - y_n\rangle + \langle\nabla f(y_n), y_n - t_n\rangle) \\
&\leq \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle\nabla f(y_n), y_n - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle \\
&\quad -\|y_n - t_n\|^2 + 2\lambda_n\langle\nabla f(y_n), y_n - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n\nabla f(y_n) - y_n, t_n - y_n\rangle.
\end{aligned}$$

Further, since $y_n = P_C(x_n - \lambda_n\nabla f(x_n))$ and ∇f is L-Lipschitz-continuous, from

(2.1) we have

$$\begin{aligned}&\langle x_n-\lambda_n\nabla f(y_n)-y_n,t_n-y_n\rangle\\&=\langle x_n-\lambda_n\nabla f(x_n)-y_n,t_n-y_n\rangle+\langle\lambda_n\nabla f(x_n)-\lambda_n\nabla f(y_n),t_n-y_n\rangle\\&\le\langle\lambda_n\nabla f(x_n)-\lambda_n\nabla f(y_n),t_n-y_n\rangle\\&\le\lambda_nL\|x_n-y_n\|\|t_n-y_n\|.\end{aligned}$$

So, we obtain

$$\begin{aligned}&\|t_n-u\|^2\\&\le\|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-t_n\|^2+2\lambda_nL\|x_n-y_n\|\|t_n-y_n\|\\&\le\|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-t_n\|^2+\lambda_n^2L^2\|x_n-y_n\|^2+\|y_n-t_n\|^2\\&=\|x_n-u\|^2+(\lambda_n^2L^2-1)\|x_n-y_n\|^2\\&\le\|x_n-u\|^2.\end{aligned}\tag{3.1}$$

Also, since $z_n=(1-\alpha_n-\hat{\alpha}_n)x_n+\alpha_n\hat{t}_n+\hat{\alpha}_nS\hat{t}_n,\ u=Su$ and $u=J_{M,\mu_n}(u-\mu_n\Phi(u))$, utilizing Lemma 2.2 we get from (3.1)

$$\begin{aligned}&\|z_n-u\|^2=\|(1-\alpha_n-\hat{\alpha}_n)(x_n-u)+\alpha_n(\hat{t}_n-u)+\hat{\alpha}_n(S\hat{t}_n-u)\|^2\\&\le(1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+\alpha_n\|\hat{t}_n-u\|^2+\hat{\alpha}_n\|S\hat{t}_n-u\|^2-\alpha_n\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2\\&\le(1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+\alpha_n\|\hat{t}_n-u\|^2\\&\quad+\hat{\alpha}_n(\|\hat{t}_n-u\|^2+\kappa\|\hat{t}_n-S\hat{t}_n\|^2)-\alpha_n\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2\\&=(1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)\|\hat{t}_n-u\|^2+(\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2\\&=(1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)\|J_{M,\mu_n}(t_n-\mu_n\Phi(t_n))\\&\quad-J_{M,\mu_n}(u-\mu_n\Phi(u))\|^2+(\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2\\&\le(1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)\|(t_n-\mu_n\Phi(t_n))-(u-\mu_n\Phi(u))\|^2\\&\quad+(\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2\\&\le(1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)[\|t_n-u\|^2+\mu_n(\mu_n-2\alpha)\|\Phi(t_n)\\&\quad-\Phi(u)\|^2]+(\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2\\&\le(1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)\|t_n-u\|^2+(\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2\\&\le(1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)[\|x_n-u\|^2\\&\quad+(\lambda_n^2L^2-1)\|x_n-y_n\|^2]+(\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2\\&=\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)(\lambda_n^2L^2-1)\|x_n-y_n\|^2+(\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2\\&\le\|x_n-u\|^2\end{aligned}\tag{3.2}$$

for every $n=0,1,2,...$ and hence $u\in C_n$. So, $F(S)\cap\Omega\cap\mathrm{VI}(C,\Phi,M)\subset C_n$ for every $n=0,1,2,....$ Next, let us show by mathematical induction that $\{x_n\}$ is well-defined and $F(S)\cap\Omega\cap\mathrm{VI}(C,\Phi,M)\subset C_n\cap Q_n$ for every $n=0,1,2,....$ For $n=0$

we have $Q_0 = C$. Hence we obtain $F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M) \subset C_0 \cap Q_0$. Suppose that x_k is given and $F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M) \subset C_k \cap Q_k$ for some integer $k \geq 0$. Since $F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M)$ is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of C. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k} x$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0$ for every $z \in C_k \cap Q_k$. Since $F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M) \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0$ for $z \in F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M)$ and hence $F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M) \subset Q_{k+1}$. Therefore, we obtain $F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M) \subset C_{k+1} \cap Q_{k+1}$.

Step 2. We claim that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_n - z_n\| = 0$. Indeed, let $l_0 = P_{F(S)\cap\Omega\cap\mathrm{VI}(C,\Phi,M)} x$. From $x_{n+1} = P_{C_n \cap Q_n} x$ and $l_0 \in F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M) \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \leq \|l_0 - x\| \tag{3.3}$$

for every $n = 0, 1, 2, \ldots$. Therefore, $\{x_n\}$ is bounded. From (3.1) and (3.2) we also obtain that $\{t_n\}$ and $\{z_n\}$ are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n} x$, we have

$$\|x_n - x\| \leq \|x_{n+1} - x\|$$

for every $n = 0, 1, 2, \ldots$. Therefore, there exists $\lim_{n\to\infty} \|x_n - x\|$. Since $x_n = P_{Q_n} x$ and $x_{n+1} \in Q_n$, utilizing (2.2), we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2$$

for every $n = 0, 1, 2, \ldots$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have $\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$ and hence

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \leq 2\|x_{n+1} - x_n\|$$

for every $n = 0, 1, 2, \ldots$. From $\|x_{n+1} - x_n\| \to 0$ it follows that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0.$$

Step 3. We claim that

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - t_n\| = \lim_{n\to\infty} \|S\hat{t}_n - \hat{t}_n\| = \lim_{n\to\infty} \|\hat{t}_n - t_n\| = 0.$$

Indeed, for $u \in F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M)$, we obtain from (3.2)

$$\|z_n - u\|^2 \leq \|x_n - u\|^2 + (\alpha_n + \hat{\alpha}_n)(\lambda_n^2 L^2 - 1)\|x_n - y_n\|^2 + (\kappa - \alpha_n)\hat{\alpha}_n\|\hat{t}_n - S\hat{t}_n\|^2.$$

Therefore, we have

$$\begin{aligned}
&\|x_n - y_n\|^2 + \frac{(c-\kappa)\hat{c}}{1-a^2L^2}\|\hat{t}_n - S\hat{t}_n\|^2 \\
&\le \|x_n - y_n\|^2 + \frac{(\alpha_n-\kappa)\hat{\alpha}_n}{(\alpha_n+\hat{\alpha}_n)(1-\lambda_n^2L^2)}\|\hat{t}_n - S\hat{t}_n\|^2 \\
&\le \frac{1}{(\alpha_n+\hat{\alpha}_n)(1-\lambda_n^2L^2)}(\|x_n-u\|^2 - \|z_n-u\|^2) \\
&= \frac{1}{(\alpha_n+\hat{\alpha}_n)(1-\lambda_n^2L^2)}(\|x_n-u\| - \|z_n-u\|) \\
&\quad \times(\|x_n-u\| + \|z_n-u\|) \\
&\le \frac{1}{(\alpha_n+\hat{\alpha}_n)(1-\lambda_n^2L^2)}(\|x_n-u\| + \|z_n-u\|)\|x_n-z_n\| \\
&\le \frac{1}{(c+\hat{c})(1-b^2L^2)}(\|x_n-u\| + \|z_n-u\|)\|x_n-z_n\|.
\end{aligned} \tag{3.4}$$

Since $\|x_n - z_n\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we deduce that

$$\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|\hat{t}_n - S\hat{t}_n\| = 0.$$

By the same process as in (3.1), we also have

$$\begin{aligned}
&\|t_n - u\|^2 \\
&\le \|x_n-u\|^2 - \|x_n-y_n\|^2 - \|y_n-t_n\|^2 + 2\lambda_n L\|x_n-y_n\|\|t_n-y_n\| \\
&\le \|x_n-u\|^2 - \|x_n-y_n\|^2 - \|y_n-t_n\|^2 + \|x_n-y_n\|^2 + \lambda_n^2L^2\|y_n-t_n\|^2 \\
&= \|x_n-u\|^2 + (\lambda_n^2L^2-1)\|y_n-t_n\|^2 \\
&\le \|x_n-u\|^2.
\end{aligned}$$

Then, in contrast with (3.2),

$$\begin{aligned}
&\|z_n-u\|^2 = \|(1-\alpha_n-\hat{\alpha}_n)(x_n-u) + \alpha_n(\hat{t}_n-u) + \hat{\alpha}_n(S\hat{t}_n-u)\|^2 \\
&\le (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2 + \alpha_n\|\hat{t}_n-u\|^2 + \hat{\alpha}_n\|S\hat{t}_n-u\|^2 - \alpha_n\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2 \\
&\le (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2 + (\alpha_n+\hat{\alpha}_n)\|\hat{t}_n-u\|^2 + (\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2 \\
&= (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2 + (\alpha_n+\hat{\alpha}_n)\|J_{M,\mu_n}(t_n-\mu_n\Phi(t_n)) \\
&\quad -J_{M,\mu_n}(u-\mu_n\Phi(u))\|^2 + (\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2 \\
&\le (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2 + (\alpha_n+\hat{\alpha}_n)\|(t_n-\mu_n\Phi(t_n)) - (u-\mu_n\Phi(u))\|^2 \\
&\quad +(\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2 \\
&\le (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2 + (\alpha_n+\hat{\alpha}_n)[\|t_n-u\|^2 + \mu_n(\mu_n-2\alpha)\|\Phi(t_n)-\Phi(u)\|^2] \\
&\quad +(\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2 \\
&\le (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2 + (\alpha_n+\hat{\alpha}_n)\|t_n-u\|^2 + (\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2 \\
&\le (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2 + (\alpha_n+\hat{\alpha}_n)[\|x_n-u\|^2 + (\lambda_n^2L^2-1)\|y_n-t_n\|^2] \\
&\quad +(\kappa-\alpha_n)\hat{\alpha}_n\|\hat{t}_n-S\hat{t}_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - u\|^2 + (\alpha_n + \hat{\alpha}_n)(\lambda_n^2 L^2 - 1)\|y_n - t_n\|^2 + (\kappa - \alpha_n)\hat{\alpha}_n \|\hat{t}_n - S\hat{t}_n\|^2 \\
&\le \|x_n - u\|^2
\end{aligned}
$$

and, rearranging as in (3.4),

$$
\begin{aligned}
&\|t_n - y_n\|^2 + \frac{(c-\kappa)\hat{c}}{1 - a^2 L^2}\|\hat{t}_n - S\hat{t}_n\|^2 \\
&\le \|t_n - y_n\|^2 + \frac{(\alpha_n - \kappa)\hat{\alpha}_n}{(\alpha_n + \hat{\alpha}_n)(1 - \lambda_n^2 L^2)}\|\hat{t}_n - S\hat{t}_n\|^2 \\
&\le \frac{1}{(\alpha_n + \hat{\alpha}_n)(1 - \lambda_n^2 L^2)}(\|x_n - u\|^2 - \|z_n - u\|^2) \\
&= \frac{1}{(\alpha_n + \hat{\alpha}_n)(1 - \lambda_n^2 L^2)}(\|x_n - u\| - \|z_n - u\|) \\
&\quad \times (\|x_n - u\| + \|z_n - u\|) \\
&\le \frac{1}{(\alpha_n + \hat{\alpha}_n)(1 - \lambda_n^2 L^2)}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\| \\
&\le \frac{1}{(c + \hat{c})(1 - b^2 L^2)}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.
\end{aligned}
$$

Since $\|x_n - z_n\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we deduce that

$$
\lim_{n\to\infty} \|t_n - y_n\| = \lim_{n\to\infty} \|\hat{t}_n - S\hat{t}_n\| = 0.
$$

As ∇f is L-Lipschitz-continuous, we have $\|\nabla f(y_n) - \nabla f(t_n)\| \to 0$. From $\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\|$ we also have $\|x_n - t_n\| \to 0$. Since $z_n = (1 - \alpha_n - \hat{\alpha}_n)x_n + \alpha_n \hat{t}_n + \hat{\alpha}_n S\hat{t}_n$, we have

$$
\begin{aligned}
z_n - x_n &= \alpha_n(\hat{t}_n - x_n) + \hat{\alpha}_n(S\hat{t}_n - x_n) \\
&= \alpha_n(\hat{t}_n - x_n) + \hat{\alpha}_n(S\hat{t}_n - \hat{t}_n + \hat{t}_n - x_n) \\
&= (\alpha_n + \hat{\alpha}_n)(\hat{t}_n - x_n) + \hat{\alpha}_n(S\hat{t}_n - \hat{t}_n).
\end{aligned}
$$

Then

$$
\begin{aligned}
(c + \hat{c})\|\hat{t}_n - x_n\| &\le (\alpha_n + \hat{\alpha}_n)\|\hat{t}_n - x_n\| \\
&= \|z_n - x_n - \hat{\alpha}_n(S\hat{t}_n - \hat{t}_n)\| \\
&\le \|z_n - x_n\| + \hat{\alpha}_n\|S\hat{t}_n - \hat{t}_n\| \\
&\le \|z_n - x_n\| + \|S\hat{t}_n - \hat{t}_n\|
\end{aligned}
$$

and hence $\|\hat{t}_n - x_n\| \to 0$. This together with $\|x_n - t_n\| \to 0$, implies that $\|\hat{t}_n - t_n\| \to 0$.

Step 4. We claim that $\omega_w(x_n) \subset F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M)$. Indeed, as $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to some $u \in \omega_w(x_n)$. We can obtain that $u \in F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M)$. First, we show

$u \in \mathrm{VI}(C, \nabla f)(= \Omega)$. Since $x_n - t_n \to 0$ and $x_n - y_n \to 0$, we conclude that $t_{n_i} \rightharpoonup u$ and $y_{n_i} \rightharpoonup u$. Let

$$Tv = \begin{cases} \nabla f(v) + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

where $N_C v$ is the normal cone to C at $v \in C$. We have already mentioned that in this case the mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, \nabla f)$; see [23]. Let $G(T)$ be the graph of T and let $(v, w) \in G(T)$. Then, we have $w \in Tv = \nabla f(v) + N_C v$ and hence $w - \nabla f(v) \in N_C v$. So, we have $\langle v - t, w - \nabla f(v) \rangle \geq 0$ for all $t \in C$. On the other hand, from $t_n = P_C(x_n - \lambda_n \nabla f(y_n))$ and $v \in C$ we have

$$\langle x_n - \lambda_n \nabla f(y_n) - t_n, t_n - v \rangle \geq 0$$

and hence

$$\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + \nabla f(y_n) \rangle \geq 0.$$

From $\langle v - t, w - \nabla f(v) \rangle \geq 0$ for all $t \in C$ and $t_{n_i} \in C$, we have

$$\begin{aligned}
&\langle v - t_{n_i}, w \rangle \\
&\geq \langle v - t_{n_i}, \nabla f(v) \rangle \\
&\geq \langle v - t_{n_i}, \nabla f(v) \rangle - \langle v - t_{n_i}, \tfrac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + \nabla f(y_{n_i}) \rangle \\
&= \langle v - t_{n_i}, \nabla f(v) - \nabla f(t_{n_i}) \rangle + \langle v - t_{n_i}, \nabla f(t_{n_i}) - \nabla f(y_{n_i}) \rangle - \langle v - t_{n_i}, \tfrac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \rangle \\
&\geq \langle v - t_{n_i}, \nabla f(t_{n_i}) - \nabla f(y_{n_i}) \rangle - \langle v - t_{n_i}, \tfrac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \rangle.
\end{aligned}$$

Hence, we obtain $\langle v - u, w - 0 \rangle = \langle v - u, w \rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $u \in T^{-1}0$ and hence $u \in \mathrm{VI}(C, \nabla f)(= \Omega)$.

Secondly, let us show $u \in F(S)$. Since $\|\hat{t}_n - x_n\| \to 0$ and $x_{n_i} \rightharpoonup u$, we have $\hat{t}_{n_i} \rightharpoonup u$. Also, since $\|\hat{t}_n - S\hat{t}_n\| \to 0$, it follows that $\|\hat{t}_{n_i} - S\hat{t}_{n_i}\| \to 0$ as $i \to \infty$. So, in terms of Lemma 2.4 (ii) we obtain $u \in F(S)$.

Next, let us show $u \in \mathrm{VI}(C, \Phi, M)$. Since Φ is α-inverse strongly monotone and M is maximal monotone, by Lemma 2.5 we know that $M + \Phi$ is maximal monotone. Take a fixed $(y, g) \in G(M + \Phi)$ arbitrarily. Then we have $g \in My + \Phi(y)$. So, we have $g - \Phi(y) \in My$. Since $\hat{t}_{n_i} = J_{M,\mu_{n_i}}(t_{n_i} - \mu_{n_i}\Phi(t_{n_i}))$ implies $\frac{1}{\mu_{n_i}}(t_{n_i} - \hat{t}_{n_i} - \mu_{n_i}\Phi(t_{n_i})) \in M\hat{t}_{n_i}$, we have

$$\langle y - \hat{t}_{n_i}, g - \Phi(y) - \frac{1}{\mu_{n_i}}(t_{n_i} - \hat{t}_{n_i} - \mu_{n_i}\Phi(t_{n_i})) \rangle \geq 0,$$

which hence yields

$$
\begin{aligned}
&\langle y-\hat{t}_{n_i}, g\rangle\\
&\geq \langle y-\hat{t}_{n_i}, \Phi(y)+\tfrac{1}{\mu_{n_i}}(t_{n_i}-\hat{t}_{n_i}-\mu_{n_i}\Phi(t_{n_i}))\rangle\\
&= \langle y-\hat{t}_{n_i}, \Phi(y)-\Phi(t_{n_i})\rangle+\langle y-\hat{t}_{n_i}, \tfrac{1}{\mu_{n_i}}(t_{n_i}-\hat{t}_{n_i})\rangle\\
&\geq \alpha\|\Phi(y)-\Phi(\hat{t}_{n_i})\|^2+\langle y-\hat{t}_{n_i}, \Phi(\hat{t}_{n_i})-\Phi(t_{n_i})\rangle+\langle y-\hat{t}_{n_i}, \tfrac{1}{\mu_{n_i}}(t_{n_i}-\hat{t}_{n_i})\rangle\\
&\geq \langle y-\hat{t}_{n_i}, \Phi(\hat{t}_{n_i})-\Phi(t_{n_i})\rangle+\langle y-\hat{t}_{n_i}, \tfrac{1}{\mu_{n_i}}(t_{n_i}-\hat{t}_{n_i})\rangle.
\end{aligned}
\tag{3.5}
$$

Observe that

$$
\begin{aligned}
&|\langle y-\hat{t}_{n_i}, \Phi(\hat{t}_{n_i})-\Phi(t_{n_i})\rangle+\langle y-\hat{t}_{n_i}, \tfrac{1}{\mu_{n_i}}(t_{n_i}-\hat{t}_{n_i})\rangle|\\
&\leq \|y-\hat{t}_{n_i}\|\|\Phi(\hat{t}_{n_i})-\Phi(t_{n_i})\|+\|y-\hat{t}_{n_i}\|\|\tfrac{1}{\mu_{n_i}}(t_{n_i}-\hat{t}_{n_i})\|\\
&\leq \tfrac{1}{\alpha}\|y-\hat{t}_{n_i}\|\|\hat{t}_{n_i}-t_{n_i}\|+\tfrac{1}{\epsilon}\|y-\hat{t}_{n_i}\|\|t_{n_i}-\hat{t}_{n_i}\|\\
&= (\tfrac{1}{\alpha}+\tfrac{1}{\epsilon})\|y-\hat{t}_{n_i}\|\|\hat{t}_{n_i}-t_{n_i}\|.
\end{aligned}
$$

It follows from $\|t_n-\hat{t}_n\|\to 0$ that

$$\lim_{i\to\infty}|\langle y-\hat{t}_{n_i}, \Phi(\hat{t}_{n_i})-\Phi(t_{n_i})\rangle+\langle y-\hat{t}_{n_i}, \frac{1}{\mu_{n_i}}(t_{n_i}-\hat{t}_{n_i})\rangle|=0.$$

Letting $i\to\infty$, we get from (3.5)

$$\langle y-u, g\rangle\geq 0.$$

This shows that $0\in\Phi(u)+Mu$. Hence, $u\in \mathrm{VI}(C,\Phi,M)$. Therefore, $u\in F(S)\cap\Omega\cap\mathrm{VI}(C,\Phi,M)$.

Step 5. We claim that

$$\lim_{n\to\infty}\|x_n-l_0\|=\lim_{n\to\infty}\|y_n-l_0\|=\lim_{n\to\infty}\|z_n-l_0\|=0,$$

where $l_0=P_{F(S)\cap\Omega\cap\mathrm{VI}(C,\Phi,M)}x$.

Indeed, from $l_0=P_{F(S)\cap\Omega\cap\mathrm{VI}(C,\Phi,M)}x,\ u\in F(S)\cap\Omega\cap\mathrm{VI}(C,\Phi,M)$, and (3.3), we have

$$\|l_0-x\|\leq\|u-x\|\leq\liminf_{i\to\infty}\|x_{n_i}-x\|\leq\limsup_{i\to\infty}\|x_{n_i}-x\|\leq\|l_0-x\|.$$

So, we obtain

$$\lim_{i\to\infty}\|x_{n_i}-x\|=\|u-x\|.$$

From $x_{n_i}-x\rightharpoonup u-x$ we have $x_{n_i}-x\to u-x$ (since $\|x_{n_i}-u\|^2=\|(x_{n_i}-x)-(u-x)\|^2=\|x_{n_i}-x\|^2-2\langle x_{n_i}-x,u-x\rangle+\|u-x\|^2\to 0$) and hence $x_{n_i}\to u$. Since $x_n=P_{Q_n}x$ and $l_0\in F(S)\cap\Omega\cap\mathrm{VI}(C,\Phi,M)\subset C_n\cap Q_n\subset Q_n$, we have

$$-\|l_0-x_{n_i}\|^2=\langle l_0-x_{n_i},x_{n_i}-x\rangle+\langle l_0-x_{n_i},x-l_0\rangle\geq\langle l_0-x_{n_i},x-l_0\rangle.$$

As $i \to \infty$, we obtain $-\|l_0 - u\|^2 \geq \langle l_0 - u, x - l_0 \rangle \geq 0$ by $l_0 = P_{F(S) \cap \Omega \cap \text{VI}(C, \Phi, M)} x$ and $u \in F(S) \cap \Omega \cap \text{VI}(C, \Phi, M)$. Hence we have $u = l_0$. This implies that $x_n \to l_0$. It is easy to see that $y_n \to l_0$ and $z_n \to l_0$. This completes the proof. ∎

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume the minimization (1.6) is consistent and let Ω be its solution set. Let $S : C \to C$ be a κ-strictly pseudocontractive mapping such that $F(S) \cap \Omega \neq \emptyset$. Assume the gradient ∇f satisfies the Lipschitz condition (1.10). For $x_0 = x \in C$ chosen arbitrarily, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences generated by*

$$(1.14) \qquad \begin{cases} y_n = P_C(x_n - \lambda_n \nabla f(x_n)), \\ t_n = P_C(x_n - \lambda_n \nabla f(y_n)), \\ z_n = (1 - \alpha_n - \hat{\alpha}_n) x_n + \alpha_n J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) \\ \qquad + \hat{\alpha}_n S J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0, 1, 2, \ldots$, where the following conditions hold:

(i) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, \frac{1}{L})$;

(ii) $\alpha_n + \hat{\alpha}_n \leq 1$ *for every* $n = 0, 1, 2, \ldots$;

(iii) $\{\alpha_n\} \subset [c, 1]$ *for some* $c \in (\kappa, 1]$ *and* $\{\hat{\alpha}_n\} \subset [\hat{c}, 1]$ *for some* $\hat{c} \in (0, 1]$.

Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to $P_{F(S) \cap \Omega} x$.

Proof. Putting $\Phi = M = 0$ in Theorem 3.1, we have $\text{VI}(C, 0, 0) = C$ and $F(S) \cap \Omega \cap \text{VI}(C, 0, 0) = F(S) \cap \Omega$. Let α be any positive number in the interval $(0, \infty)$ and take any sequence $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$. Then Φ is α-inverse strongly monotone and we have

$$\begin{cases} t_n = P_C(x_n - \lambda_n \nabla f(y_n)), \\ \hat{t}_n = J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) = (I + \mu_n M)^{-1} t_n = t_n, \\ z_n = (1 - \alpha_n - \hat{\alpha}_n) x_n + \alpha_n \hat{t}_n + \hat{\alpha}_n S \hat{t}_n \\ \quad = (1 - \alpha_n - \hat{\alpha}_n) x_n + \alpha_n t_n + \hat{\alpha}_n S t_n \\ \quad = (1 - \alpha_n - \hat{\alpha}_n) x_n + \alpha_n P_C(x_n - \lambda_n \nabla f(y_n)) + \hat{\alpha}_n S P_C(x_n - \lambda_n \nabla f(y_n)). \end{cases}$$

Therefore, by Theorem 3.1 we obtain the desired result. ∎

Remark 3.1. Compared with Theorem 4.4 in Xu [17], our Theorem 3.1 improves and extends Xu [17, Theorem 4.4] in the following aspects:

(i) Xu's gradient-projection method in [17, Theorem 4.4] is extended to develop the relaxed hybrid-extragradient method in our Theorem 3.1.

(ii) the technique of proving strong convergence in our Theorem 3.1 is very different from that in Xu [17, Theorem 4.4] because our technique depends on the properties for maximal monotone mappings and their resolvent operators (see, e.g., Lemmas 2.1, 2.3 and 2.5), the demiclosedness principle for strict pseudocontractions and the geometric properties for Hilbert spaces (see, e.g., Kadec-Klee's property [34]).

(iii) our problem of finding an element of $\mathrm{Fix}(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M)$ is more general than Xu's problem of finding an element of Ω in [17, Theorem 4.4].

4. Applications

Utilizing Theorem 3.1, we prove some strong convergence theorems in a real Hilbert space.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume the minimization (1.6) is consistent and let Ω be its solution set. Let $\Phi : C \to H$ be an α-inverse strongly monotone mapping and M be a maximal monotone mapping with $D(M) = C$ such that $\Omega \cap \mathrm{VI}(C, \Phi, M) \neq \emptyset$. Assume the gradient ∇f satisfies the Lipschitz condition (1.10). For $x_0 = x \in C$ chosen arbitrarily, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{cases} y_n = P_C(x_n - \lambda_n \nabla f(x_n)), \\ t_n = P_C(x_n - \lambda_n \nabla f(y_n)), \\ z_n = (1-\beta_n)x_n + \beta_n J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\ C_n = \{z \in C : \|z_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0, 1, 2, ...$, where the following conditions hold:

(i) $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{L})$;

(ii) $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$;

(iii) $\{\beta_n\} \subset [c, 1]$ for some $c \in (0, 1]$.

Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to $P_{\Omega \cap \mathrm{VI}(C,\Phi,M)} x$.

Proof. In Theorem 3.1, putting $S = I$, $\kappa = 0$ and $\alpha_n = \hat{\alpha}_n = \frac{1}{2}\beta_n$ for every $n = 0, 1, 2, ...$, we have

$$\begin{cases} \hat{t}_n = J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\ z_n = (1 - \alpha_n - \hat{\alpha}_n)x_n + \alpha_n \hat{t}_n + \hat{\alpha}_n S \hat{t}_n \\ \quad = (1 - \alpha_n - \hat{\alpha}_n)x_n + (\alpha_n + \hat{\alpha}_n)\hat{t}_n \\ \quad = (1 - \beta_n)x_n + \beta_n \hat{t}_n \\ \quad = (1 - \beta_n)x_n + \beta_n J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)). \end{cases}$$

In this case, we know that $F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M) = \Omega \cap \mathrm{VI}(C, \Phi, M)$. Therefore, by Theorem 3.1 we obtain the desired result. ■

Theorem 4.2. (see [22, Theorem 4.2]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping such that $F(S)$ is nonempty. For $x_0 = x \in C$ chosen arbitrarily, let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{cases} z_n = (1 - \hat{\alpha}_n)x_n + \hat{\alpha}_n S x_n, \\ C_n = \{z \in C : \|z_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0, 1, 2, \ldots$, where $\{\hat{\alpha}_n\} \subset [c, 1]$ for some $c \in (0, 1]$. Then the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{F(S)}x$.

Proof. Putting $\nabla f = \Phi = M = 0$ in Theorem 3.1, we let L and α be any positive numbers in the interval $(0, \infty)$ and take any sequence $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{L})$ and any sequence $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$. Then ∇f is L-Lipschitz-continuous and Φ is α-inverse strongly monotone. In this case, we know that $F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M) = F(S)$ and

$$\begin{cases} y_n = P_C(x_n - \lambda_n \nabla f(x_n)) = x_n, \\ t_n = P_C(x_n - \lambda_n \nabla f(y_n)) = x_n, \\ \hat{t}_n = J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) = t_n = x_n, \\ z_n = (1 - \alpha_n - \hat{\alpha}_n)x_n + \alpha_n \hat{t}_n + \hat{\alpha}_n S \hat{t}_n = (1 - \hat{\alpha}_n)x_n + \hat{\alpha}_n S x_n. \end{cases}$$

Therefore, by Theorem 3.1 we obtain the desired result. ■

Remark 4.1. Originally Theorem 4.2 is the result of Nakajo and Takahashi [18].

Theorem 4.3. *Let H be a real Hilbert space. Assume the minimization (1.6) is consistent with $C = H$ and let Ω be its solution set. Let $\Phi : H \to H$ be an α-inverse strongly monotone mapping, $M : H \to 2^H$ be a maximal monotone mapping and $S : H \to H$ be a κ-strictly pseudocontractive mapping such that $F(S) \cap (\nabla f)^{-1}0 \cap \mathrm{VI}(H, \Phi, M) \neq \emptyset$. Assume the gradient $\nabla f : H \to H$ satisfies the Lipschitz condition (1.10). For $x_0 = x \in H$ chosen arbitrarily, let $\{x_n\}$ and $\{z_n\}$ be the sequences* *generated by*

$$\begin{cases} t_n = x_n - \lambda_n \nabla f(x_n - \lambda_n \nabla f(x_n)), \\ z_n = (1 - \alpha_n - \hat{\alpha}_n)x_n + \alpha_n J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) + \hat{\alpha}_n S J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\ C_n = \{z \in H : \|z_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0, 1, 2, ...$, where the following conditions hold:

(*i*) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, \frac{1}{L})$*;*

(*ii*) $\{\mu_n\} \subset [\epsilon, 2\alpha]$ *for some* $\epsilon \in (0, 2\alpha]$*;*

(*iii*) $\alpha_n + \hat{\alpha}_n \le 1$ *for every* $n = 0, 1, 2, ...$*;*

(*iv*) $\{\alpha_n\} \subset [c, 1]$ *for some* $c \in (\kappa, 1]$ *and* $\{\hat{\alpha}_n\} \subset [\hat{c}, 1]$ *for some* $\hat{c} \in (0, 1]$.

Then the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{F(S)\cap(\nabla f)^{-1}0\cap \mathrm{VI}(H,\Phi,M)}x$.

Proof. Putting $C = H$ in Theorem 3.1, we have $(\nabla f)^{-1}0 = \mathrm{VI}(H, \nabla f) = \Omega$ and $P_C = P_H = I$. In this case, we know that

$$\begin{cases} y_n = P_C(x_n - \lambda_n \nabla f(x_n)) = x_n - \lambda_n \nabla f(x_n), \\ t_n = P_C(x_n - \lambda_n \nabla f(y_n)) = x_n - \lambda_n \nabla f(x_n - \lambda_n \nabla f(x_n)), \\ z_n = (1 - \alpha_n - \hat{\alpha}_n)x_n + \alpha_n J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) + \hat{\alpha}_n S J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)). \end{cases}$$

Therefore, by Theorem 3.1 we obtain the desired result. ■

Let $B : H \to 2^H$ be a maximal monotone mapping. Then, for any $x \in H$ and $r > 0$, consider $J_{B,r}x = (I + rB)^{-1}x$. It is known that such a $J_{B,r}$ is the resolvent of B.

Theorem 4.4. *Let H be a real Hilbert space. Assume the minimization (1.6) is consistent with $C = H$ and let Ω be its solution set. Let $\Phi : H \to H$ be an α-inverse strongly monotone mapping and $B, M : H \to 2^H$ be two maximal monotone mappings such that $(\nabla f)^{-1}0 \cap B^{-1}0 \cap \mathrm{VI}(H, \Phi, M) \ne \emptyset$. Let $J_{B,r}$ be the resolvent of B for each $r > 0$. Assume the gradient $\nabla f : H \to H$ satisfies the Lipschitz condition (1.10). For $x_0 = x \in H$ chosen arbitrarily, let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{cases} t_n = x_n - \lambda_n \nabla f(x_n - \lambda_n \nabla f(x_n)), \\ z_n = (1 - \alpha_n - \hat{\alpha}_n)x_n + \alpha_n J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) + \hat{\alpha}_n J_{B,r} J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\ C_n = \{z \in H : \|z_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0, 1, 2, ...,$ *where the following conditions hold:*

(*i*) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, \frac{1}{L})$;

(*ii*) $\{\mu_n\} \subset [\epsilon, 2\alpha]$ *for some* $\epsilon \in (0, 2\alpha]$;

(*iii*) $\alpha_n + \hat{\alpha}_n \leq 1$ *for every* $n = 0, 1, 2, ...$;

(*iv*) $\{\alpha_n\} \subset [c, 1]$ *and* $\{\hat{\alpha}_n\} \subset [\hat{c}, 1]$ *for some* $c, \hat{c} \in (0, 1]$.

Then the sequences $\{x_n\}$ *and* $\{z_n\}$ *converge strongly to* $P_{(\nabla f)^{-1}0 \cap B^{-1}0 \cap \mathrm{VI}(H,\Phi,M)} x$.

Proof. Putting $C = H$, $S = J_{B,r}$ and $\kappa = 0$ in Theorem 3.1, we know that $P_C = P_H = I$, $(\nabla f)^{-1}0 = \mathrm{VI}(H, \nabla f) = \Omega$ and $F(J_{B,r}) = B^{-1}0$. In this case, we have

$$\begin{cases} y_n = P_C(x_n - \lambda_n \nabla f(x_n)) = x_n - \lambda_n \nabla f(x_n), \\ t_n = P_C(x_n - \lambda_n \nabla f(y_n)) = x_n - \lambda_n \nabla f(x_n - \lambda_n \nabla f(x_n)), \\ \hat{t}_n = J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\ z_n = (1 - \alpha_n - \hat{\alpha}_n) x_n + \alpha_n \hat{t}_n + \hat{\alpha}_n S \hat{t}_n \\ \quad = (1 - \alpha_n - \hat{\alpha}_n) x_n + \alpha_n J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) + \hat{\alpha}_n J_{B,r} J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)). \end{cases}$$

Therefore, by Theorem 3.1 we obtain the desired result. ∎

It is well known that a mapping $T : C \to C$ is called pseudocontractive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2$ for all $x, y \in C$. It is easy to see that the definition of a pseudocontractive mapping is equivalent to the one that a mapping $T : C \to C$ is called pseudocontractive if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 \tag{4.1}$$

for all $x, y \in C$; see [22]. Obviously, the class of pseudocontractive mappings is more general than the class of nonexpansive mappings. In the meantime, we also know one more definition of a κ-strictly pseudocontractive mapping, which is equivalent to the definition given in the introduction. A mapping $T : C \to C$ is called κ-strictly pseudocontractive if there exists a constant $0 \leq \kappa < 1$ such that

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2} \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$. It is clear that in this case the mapping $I - T$ is $\frac{1-\kappa}{2}$-inverse strongly monotone. From [27], we know that if T is a κ-strictly pseudocontractive mapping, then T is Lipschitz continuous with constant $\frac{1+\kappa}{1-\kappa}$, i.e., $\|Tx - Ty\| \leq \frac{1+\kappa}{1-\kappa} \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the fixed point set of T. It is obvious that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings and the class of pseudocontractions strictly includes the class of strict pseudocontractions.

In the following theorem we introduce an iterative algorithm that converges strongly to a solution of the minimization (1.6), which is also a common fixed point of two mappings taken from the more general class of strictly pseudocontractive mappings.

Theorem 4.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume the minimization (1.6) is consistent and let Ω be its solution set. Let $\Gamma : C \to C$ be a γ-strictly pseudocontractive mapping and $S : C \to C$ be a κ-strictly pseudocontractive mapping such that $F(S) \cap F(\Gamma) \cap \Omega \neq \emptyset$. Assume the gradient ∇f satisfies the Lipschitz condition (1.10). For $x_0 = x \in C$ chosen arbitrarily, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{cases} y_n = P_C(x_n - \lambda_n \nabla f(x_n)), \\ t_n = P_C(x_n - \lambda_n \nabla f(y_n)), \\ z_n = (1-\alpha_n-\hat{\alpha}_n)x_n + \alpha_n(t_n - \mu_n(t_n - \Gamma t_n)) + \hat{\alpha}_n S(t_n - \mu_n(t_n - \Gamma t_n)), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0, 1, 2, ...$, where the following conditions hold:

(*i*) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, \frac{1}{L})$*;*

(*ii*) $\{\mu_n\} \subset [\epsilon, 1-\gamma]$ *for some* $\epsilon \in (0, 1-\gamma]$*;*

(*iii*) $\alpha_n + \hat{\alpha}_n \leq 1$ *for every* $n = 0, 1, 2, ...$*;*

(*iv*) $\{\alpha_n\} \subset [c, 1]$ *for some* $c \in (\kappa, 1]$ *and* $\{\hat{\alpha}_n\} \subset [\hat{c}, 1]$ *for some* $\hat{c} \in (0, 1]$.

Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to $P_{F(S)\cap F(\Gamma)\cap\Omega}x$.

Proof. Putting $\Phi = I - \Gamma$ and $M = 0$ in Theorem 3.1, we know that Φ is α-inverse strongly monotone with $\alpha = \frac{1-\gamma}{2}$. Noticing that $\{\mu_n\} \subset [\epsilon, 1-\gamma] \subset (0, 1-\gamma]$, we know that $\{\mu_n\} \subset (0, 1]$ and hence $(1-\mu_n)t_n + \mu_n \Gamma x_n \in C$. This implies that

$$\begin{cases} \hat{t}_n = J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) = t_n - \mu_n(t_n - \Gamma t_n), \\ z_n = (1-\alpha_n-\hat{\alpha}_n)x_n + \alpha_n \hat{t}_n + \hat{\alpha}_n S\hat{t}_n \\ \quad = (1-\alpha_n-\hat{\alpha}_n)x_n + \alpha_n(t_n - \mu_n(t_n - \Gamma t_n)) + \hat{\alpha}_n S(t_n - \mu_n(t_n - \Gamma t_n)). \end{cases}$$

Now let us show that $\mathrm{VI}(C, \Phi, M) = F(\Gamma)$. In fact, noticing that $M = 0$ and $\Phi = I - \Gamma$ we have

$$\begin{aligned} u \in \mathrm{VI}(C, \Phi, M) &\Leftrightarrow 0 \in \Phi(u) + Mu \\ &\Leftrightarrow 0 = \Phi(u) = u - \Gamma u \\ &\Leftrightarrow u \in F(\Gamma). \end{aligned}$$

Consequently,

$$F(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M) = F(S) \cap F(\Gamma) \cap \Omega.$$

Therefore, by Theorem 3.1 we obtain the desired result. ■

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L. C. Ceng
Department of Mathematics
Shanghai Normal University
Shanghai 200234
and
Scientific Computing Key Laboratory of Shanghai Universities
Shanghai 200234
P. R. China
E-mail: zenglc@hotmail.com

C. Y. Chou
Department of Applied Mathematics
A330 Science and Engineering Building
National Dong Hwa University
Hualien 97401, Taiwan
E-mail: choucy@mail.ndhu.edu.tw