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GEOMETRIC FLOWS ON WARPED PRODUCT MANIFOLD

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Abstract. We derive one unified formula for Ricci curvature tensor on arbitrary warped product manifold (WPM) by introducing a new notation for the lift vector and the Levi-Civita connection. Using well-established formula, we consider two questions on WPM related to Ricci flow (RF) and hyperbolic geometric flow (HGF). Firstly, we discuss the preserved flow-type problem which says that when the first factor (M, g) and the second factor (N, h) are solutions to the RF (or HGF), the singly WPM $M \times_\lambda N$ is still solution to the RF (or HGF). We obtain some characteristic PDEs satisfied by warping function and also construct some simple examples. Next, we discuss the evolution equations for warping function λ and Ricci curvature tensor etc. under RF/HGF. We gain some interesting results, especially adding an assumption with Einstein metric to the second factor.

1. INTRODUCTION

From Riemann's work it appears that he worked with changing metrics mostly by multiplying them by a function (conformal change). Soon after Riemmann's discoveries it was realized that in polar coordinates one can change the metric in a different way, now referred to as a warped product metric. The concept of warped product metrics was first introduced by Bishop and O'Neill [5] to construct examples of Riemannian manifolds with negative curvature. In Riemannian geometry, warped product manifolds and their generic forms have been used to construct new examples with interesting curvature properties like Einstein spaces [7, 16] or (locally) symmetric spaces [3]. In string theory, Yau in [24, P244-245] argued that "...the easiest way to partition the ten-dimensional space is to cut it cleanly, splitting it into four-dimensional spacetime and six-dimensional hidden subspace... and in the non-kähler case, the ten-dimensional spacetime is not a Cartesian product but rather a warped product."

In this paper, we shall consider the warped product metrics combining with two types of geometric flows, i.e. Ricci flow (RF) and hyperbolic geometric flow (HGF).

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As we have known, Ricci flow was introduced and studied by Hamilton [15]. This was the first means to study the geometric quantities associated to a metric $g(x,t)$, $(x,t) \in M \times \mathbb{R}$ as the metric evolves via a PDE, where M is a differentiable manifold. The Ricci flow is a powerful tool to understand the geometry and topology of some Riemann manifolds. Any solution of Ricci flow equation will help us to understand its behavior for general cases and the singularity formation, further the basic topological and geometrical properties as well as analytic properties of the underlying manifolds. On the other hand, a hyperbolic Ricci evolution is the Ricci wave, i.e. hyperbolic geometric flow (HGF) introduced by Kong and Liu [17]. In fact, both RF and HGF can be viewed as prolongations of the Einstein equation, whose left-hand side consists of what's called the modified Ricci tensor. Since the right-hand side of the RF and HGF equation also includes a key term in the famous Einstein equation–the Ricci curvature tensor which shows how matter and energy affect the geometry of spacetime, HGF, RF and Einstein equation can be unified into a single PDEs system as

$$\alpha(x,t)\frac{\partial^2}{\partial t^2}g(t) + \beta(x,t)\frac{\partial}{\partial t}g(t) + \gamma(x,t)g(t) + 2Ric_{g(t)} = 0, \tag{1.1}$$

where $\alpha(x,t)$, $\beta(x,t)$, $\gamma(x,t)$ are certain smooth functions ([17], [14]). It is easy to see from (1.1) that the above three cases correspond to "$\alpha(x,t) = 1$, $\beta(x,t) = \gamma(x,t) = 0$", "$\alpha(x,t) = 0$, $\beta(x,t) = 1$, $\gamma(x,t) = 0$" and "$\alpha(x,t) = 0$, $\beta(x,t) = 0$, $\gamma(x,t) = const$", respectively.

Recently, there has been some progress on the topic of combining geometric flow with warped product manifolds. For instance, Ma and Xu in [19] showed that the negative curvature is preserved in the deformation of hyperbolic warped product metrics under RF by such example: $\overline{M} = \mathbb{R}_+ \times N^n$ with the product metric $g(t) = \varphi(x,t)^2dx^2 + \psi(x,t)^2\hat{g}$, where $(N^n,\hat{g})$ is an Einstein manifold of dimension $n \geq 2$, $\varphi(x)$ and $\psi(x)$ are two smooth positive functions of the variable $x > 0$. Xu and Ma's work is mainly inspired from the work of Simon [23]. Das, Prabhu and Kar in their work [12] mainly considered the evolution under RF of the warped product $\mathbb{R}^1 \times M$ with line element of the form

$$ds^2 = e^{2f}(\sigma,\lambda)(-dt^2 + dx^2 + dy^2 + dz^2) + r_c^2(\sigma,\lambda)d\sigma^2$$

and the behavior of f by solving the flow equations, where M is Minkowski spacetime and $\mathbb{R}^1$ is the real line, λ is flow parameter. Especially, Simon [23] characterized the local existence of Ricci flow on the complete non-compact manifold $X = (\mathbb{R},h) \times (N^n,\gamma)$ with warped product metric $g(x,q) = h(x) \oplus r^2(x)\gamma(q)$ and showed that *if* $g_0(x,q) = h_0(x) \oplus r_0^2(x)\gamma(q)$ *is arbitrary warped product metric which satisfies some certain conditions*

$$\begin{cases} \sup\limits_{x\in\mathbb{R}}(h_0)_{xx} < \infty, \quad \inf\limits_{x\in\mathbb{R}}(h_0)_{xx} > 0, \quad \inf\limits_{x\in\mathbb{R}} r_0(x) > 0, \\ \sup\limits_{x\in\mathbb{R}}\left(|(\frac{\partial}{\partial x})^j h_0(x)| + |(\frac{\partial}{\partial x})^j \log r_0(x)|\right) < \infty, \forall j \in \{1,2,\ldots\}, \end{cases}$$

then there exists a unique warped product solution $g(x, q, t) = h(x, t) \oplus r^2(x, t)\gamma(q)$, $t \in [0, T)$ *to the Ricci flow*

$$\frac{\partial}{\partial t} g(t) = -Ric_{g(t)}, \quad g(0) = g_0, \quad t \in [0, T).$$

For more detail we refer to see Theorem 3.1 in [23].

Motivated by [19, 23] and [12], we are interested in the behavior of geometric flows associated to the general WPM $\overline{M} = (M_1, g_1) \times_\lambda (M_2, g_2)$. Combining a known fact that "*if* $(M_1, g_1(t))$ *and* $(M_2, g_2(t)$ *are solutions of the Ricci flow on a common time, then their direct product* $(M_1 \times M_2,\ g_1(t) + g_2(t))$ *is a solution to the Ricci flow*" (see Exercise 2.5 in [10], p. 99), we naturally want to generalize this result to warped product manifold and even to hyperbolic geometric flow as well. Note that the Ricci curvature tensor on WPM is of crucial role in studying the Ricci flow and hyperbolic geometric flow, we first integrate the separated Ricci curvature formula in previous academic literature, since these old formulas about Riemann curvature and Ricci curvature are split into several parts according to the horizontal lift or vertical lift of the tangent vectors attached to M_1 or M_2 (see Propositions 2.3 and 2.4). To better study the RF and HGF associated to WPM, regardless of the tangent vectors are attached to horizontal lift or vertical lift, we have to derive out one formula as a whole. By introducing a new notation for lift vector (see Proposition 2.5, Remark 2.6) and Levi-Civita connection $\bar{\nabla}$ over $\overline{M}$, we derive a unified formula (2.6) for Ricci curvature and scalar curvature (see Theorem 2.9). Using this unified Ricci curvature formula, we consider the behavior of warping function under the RF and under HGF on warped product manifold $\overline{M}$ (unnecessarily compact) and give two main results about preserved flow-type condition (Theorem 3.2, Theorem 4.2), which assert that the warping function λ should satisfy a characteristic equation when the warped product metric $\bar{g}(x, y, t) = g_1(x, t) \oplus \lambda^2(x, t) g_2(y, t)$ is also a solution to the RF (resp. HGF), where $g_1(x, t)$ and $g_2(y, t)$ are respectively solutions to the RF (resp. HGF) .

Considering one may worry about that these equations have no any solution λ, we make some appropriate illustration. We employ the following two strategies for overcoming this obstruction: one is by appealing to the known short-time existence theorem of geometric flows (in compact case, refer to [15, 13, 11]; in complete no-compact case, refer to [22]); another is to get some sense by constructing some specific examples (see Example 3.6 and Example 4.6), whose ideas mainly come from [21, 23, 1, 19].

In addition, in order to understand how the curvature on warped product manifold is evolving and behaving , using the unified Ricci curvature formula (2.6) we also consider the evolution equations along the RF and HGF. On general WPM, we derive two results: (1) the evolution equations for metric and warping function, see Proposition 5.1 and Proposition 5.3; and (2) the Ricci curvature evolution equations (5.7), (5.8) in Theorem 5.4. On a specific warped product manifold whose warped product metric is of the form $\bar{g}(x, y, t) = g_1(x, t) \oplus \lambda^2(x, t) g_2(y, t)\big)$ with a fixed Einstein metric g_2, we gain the more interesting evolution equations for Ricci curvature and special function $f(x,t)$, see (5.22) and (5.27) in Theorem 5.6 and Theorem 5.7.

The organization of this paper is below. In Section 2, we recall some preliminary results and derive out three unified formulas for Riemannian curvature, Ricci curvature and scalar curvature on WPM. Section 3 is devoted to characterize the behavior of warping function under the Ricci flow. We give a preserved Ricci flow-type condition for the warping function λ, including the elaboration on the short-time existence of warped product metric solution to the RF. Furthermore, we also construct some examples. Section 4 is parallel to Section 3. A distinction between them is in that the considered flow is HGF but not RF. In the last section, we discuss evolution equations of warping function and Ricci curvature on general and specific WPM under the RF/ HGF.

2. Unified Ricci Tensor Formula on WPM

Before studying the RF and HGF of warped product metrics, we need to deduce the crucial formula for Ricci tensor from the split form to united form on WPM. We do this by the construct of the connection. As we will see, this unified Ricci tensor formula simplifies the study of curvature tensors associated to warped product metrics, and also allows us to find explicit formulas for RF and HGF with respect to a given underlying warped product manifolds.

We first introduce background knowledge on warped product manifolds, see [5, 20] for detail.

2.1. Basics of warped products

Let M_1 and M_2 be Riemannian manifolds equipped with Riemannian metrics g_1 and g_2, respectively, and let λ be a strictly positive real function on M_1. Consider the product manifold $M_1 \times M_2$ with its natural projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$.

The warped product manifold $\overline{M} = M_1 \times_\lambda M_2$ is the manifold $M_1 \times M_2$ equipped with the Riemannian metric $\bar{g} = g_1 \oplus \lambda^2 g_2$ defined by

$$\bar{g}(X,Y) = g_1(d\pi_1(X), d\pi_1(Y)) + \lambda^2 g_2(d\pi_2(X), d\pi_2(Y))$$

for any tangent vectors $X, Y \in T_{(p,q)}(M_1 \times M_2)$. The function λ is called the warping function of the warped product. When $\lambda = 1$, $M_1 \times_\lambda M_2$ is a direct product.

For a warped product manifold $M_1 \times_\lambda M_2$, M_1 is called the base and M_2 the fiber. The fibers $p \times M_2 = \pi_1^{-1}(p)$ and the leaves $M_1 \times q = \pi_2^{-1}(q)$ are Riemannian submanifolds of $\overline{M}$. Vectors tangent to leaves are called horizontal and those tangent to fibers are called vertical. We denote by $\mathcal{H}$ the orthogonal projection of $T_{(p,q)}\overline{M}$ onto its horizontal subspace $T_{(p,q)}M_1 \times q$, and by $\mathcal{V}$ the projection onto the vertical subspace $T_{(p,q)}p \times M_2$.

If $v \in T_pM_1$, $p \in M_1$ and $q \in M_2$, then the lift $\tilde{v}$ of v to (p,q) is the unique vector in $T_{(p,q)}M_1 = T_{(p,q)}M_1 \times q \subset T_{(p,q)}\overline{M}$ such that $d\pi_1(\tilde{v}) = v$. For a vector field $X \in \mathfrak{X}(M_1)$, the lift of X to $\overline{M}$ is the vector field $\tilde{X}$ whose value at each (p,q) is the lift of X_p to (p,q). The set of all such horizontal lifts is denoted by $\mathcal{L}(M_1)$. Similarly, we denote by $\mathcal{L}(M_2)$ the set of all vertical lifts.

We state some known results below.

Proposition 2.1. (1) *If $\tilde{X}, \tilde{Y} \in \mathcal{L}(M_1)$ then*

$$[\tilde{X}, \tilde{Y}] = [X, Y]^{\sim} \in \mathcal{L}(M_1);$$

(2) *If $\tilde{U}, \tilde{V} \in \mathcal{L}(M_2)$ then*

$$[\tilde{U}, \tilde{V}] = [U, V]^{\sim} \in \mathcal{L}(M_2);$$

(3) *If $\tilde{X} \in \mathcal{L}(M_1)$ and $\tilde{V} \in \mathcal{L}(M_2)$ then $[\tilde{X}, \tilde{V}] = 0$.*

Proposition 2.2. ([20], Prop.35, P206). *On $\overline{M}$, if $X, Y \in \mathcal{L}(M_1)$ and $V, W \in \mathcal{L}(M_2)$, then*

(1) $\bar{\nabla}_X Y \in \mathcal{L}(M_1)$ *is the lift of* ${}^{M_1}\nabla_X Y$ *on* M_1*;*
(2) $\bar{\nabla}_X V = \bar{\nabla}_V X = \frac{X\lambda}{\lambda} V$.
(3) $\mathrm{nor}\bar{\nabla}_V W = II(V,W) = -\frac{<V,W>}{\lambda}\mathrm{grad}\lambda$, *where*
$$nor : \mathcal{H} \to T_{(p,q)}(M_1 \times q) = \big(T_{(p,q)}p \times M_2\big)^{\perp}.$$
(4) $\tan\bar{\nabla}_V W \in \mathcal{L}(M_2)$ *is the lift of* ${}^{M_2}\nabla_V W$ *on* M_2*, where*
$$tan : \mathcal{V} \to T_{(p,q)}(p \times M_2).$$

Let ${}^{M_1}R$ and ${}^{M_2}R$ be the lifts on $\overline{M}$ of the Riemannian curvature tensors of M_1 and M_2, respectively. Since the projection π_1 is an isometry on each leaf, ${}^{M_1}R$ gives the Riemannian curvature of each leaf. The corresponding assertion holds for ${}^{M_2}R$, since the projection π_2 is a homothety. Because leaves are totally geodesic, ${}^{M_1}R$ agrees with the curvature tensor $\bar{R}$ of $\overline{M}$ on horizontal vectors. This time the corresponding assertion fails for ${}^{M_2}R$ and $\bar{R}$, since fibers are in general only umbilic. In addition, for convenience the alternative notation $\bar{R}(X,Y)Z$ is $\bar{R}_{XY}Z$.

Proposition 2.3. ([20], Prop.42, P210). *Let $\overline{M}$ be a warped product manifold, if $X, Y, Z \in \mathcal{L}(M_1)$ and $U, V, W \in \mathcal{L}(M_2)$, then*

(1) $\bar{R}_{XY}Z \in \mathcal{L}(M_1)$ *is the lift of* ${}^{M_1}R_{XY}Z$ *on* M_1*;*
(2) $\bar{R}_{VX}Y = \big(\mathrm{Hess}(\lambda)(X,Y)/\lambda\big)V$.
(3) $\bar{R}_{XY}V = \bar{R}_{VW}X = 0$.
(4) $\bar{R}_{XV}W = \frac{\bar{g}(V,W)}{\lambda}\bar{\nabla}_X\mathrm{grad}\lambda$.
(5) $\bar{R}_{VW}U = {}^{M_2}R_{VW}U - \frac{1}{\lambda^2}\bar{g}(\mathrm{grad}\lambda, \mathrm{grad}\lambda)\big(\bar{g}(V,U)W - \bar{g}(W,U)V\big)$.

Writing ${}^{M_1}\mathrm{Ric}$ for the lift (pullback by $\pi_1 : \overline{M} \to M_1$) of the Ricci curvature of M_1, and similarly for ${}^{M_2}\mathrm{Ric}$.

Proposition 2.4. ([20],Corollary 43, P211). *On a warped product $\overline{M}$ with $m_2 = \dim M_2 > 1$, let X, Y be horizontal and V, W vertical. Then*

(1) $\overline{\mathrm{Ric}}(X,Y) = {}^{M_1}\mathrm{Ric}(X,Y) - \frac{m_2}{\lambda}\mathrm{Hess}(\lambda)(X,Y)$.

(2) $\overline{\mathrm{Ric}}(X,V) = 0$.

(3) $\overline{\mathrm{Ric}}(V,W) = {}^{M_2}\mathrm{Ric}(V,W) - \bar{g}(V,W)\lambda^{\#}$, *where*

$$\lambda^{\#} = \frac{\Delta\lambda}{\lambda} + (m_2 - 1)\frac{\bar{g}(\mathrm{grad}\lambda, \mathrm{grad}\lambda)}{\lambda^2}$$

and $\Delta\lambda = \mathrm{Tr}(Hess(\lambda))$ is the Laplacian on M_1.

2.2. The unified formulas for Ricci curvature

From the precious subsection we have seen that the formulas about Riemann curvature and Ricci curvature are split into several parts according to the horizontal lift or vertical lift of the tangent vectors attached to M_1 or M_2. To better study the RF and HGF associated to WPM, we found it is necessary to derive out one unified formula for Ricci tensor, no matter how the lift vectors are either horizontal or vertical. For this we first introduc the unified connection and unified Riemannian curvature on a general warped product manifold $\overline{M}$ (cf. [3], [4]) by introducing a new notation of lift vector.

Proposition 2.5. *Let $X = (X_1, X_2), Y = (Y_1, Y_2) \in \mathfrak{X}(\overline{M})$, where $X_1, Y_1 \in \mathfrak{X}(M_1)$ and $X_2, Y_2 \in \mathfrak{X}(M_2)$. Denote ∇ by the Levi-Civita connection on the Riemannian product $M_1 \times M_2$ with respect to the direct product metric $g = g_1 \oplus g_2$ and by R its curvature tensor field. Then the Levi-Civita connection $\bar{\nabla}$ of $\overline{M}$ is given by*

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + \frac{1}{2\lambda^2}X_1(\lambda^2)(0, Y_2) \\ &\quad + \frac{1}{2\lambda^2}Y_1(\lambda^2)(0, X_2) - \frac{1}{2}g_2(X_2, Y_2)(grad\,\lambda^2, 0) \\ &= \big({}^{M_1}\nabla_{X_1}Y_1 - \frac{1}{2}g_2(X_2, Y_2)grad\,\lambda^2, 0\big) \\ &\quad + \big(0, {}^{M_2}\nabla_{X_2}Y_2 + \frac{1}{2\lambda^2}X_1(\lambda^2)Y_2 + \frac{1}{2\lambda^2}Y_1(\lambda^2)X_2\big), \end{aligned} \tag{2.1}$$

and the relation between the curvature tensor fields of $\overline{M}$ and $M_1 \times M_2$ is

$$\begin{aligned} &\bar{R}_{XY} - R_{XY} \\ &= \frac{1}{2\lambda^2}\Big\{\Big({}^{M_1}\nabla_{Y_1}grad_{g_1}\lambda^2 - \frac{1}{2\lambda^2}Y_1(\lambda^2)grad_{g_1}\lambda^2,\, 0\Big) \wedge_{\bar{g}} (0, X_2) \\ &- \Big({}^{M_1}\nabla_{X_1}grad_{g_1}\lambda^2 - \frac{1}{2\lambda^2}X_1(\lambda^2)grad_{g_1}\lambda^2,\, 0\Big) \wedge_{\bar{g}} (0, Y_2) \\ &- \frac{1}{2\lambda^2}|grad_{g_1}\lambda^2|^2(0, X_2) \wedge_{\bar{g}} (0, Y_2)\Big\} \end{aligned} \tag{2.2}$$

where the wedge product $(X \wedge_{\bar{g}} Y)Z = \bar{g}(Y,Z)X - \bar{g}(X,Z)Y$*, for all* $X, Y, Z \in \mathfrak{X}(\overline{M})$.

Remark 2.6. We can easily show that the four cases in Proposition 2.2 can be integrated to one form as (2.1), where we denote the lifts of $X_1 \in \mathfrak{X}(M_1), X_2 \in \mathfrak{X}(M_2)$ by $(X_1, 0), (0, X_2) \in \mathfrak{X}(\overline{M})$. For example,

$$\begin{aligned}
\bar{\nabla}_{(X_1,0)}(Y_1,0) &= ({}^{M_2}\nabla_{X_1}Y_1, 0) = \text{lift of } {}^{M_1}\nabla_{X_1}Y_1,\\
\bar{\nabla}_{(X_1,0)}(0,V_2) &= \bar{\nabla}_{(0,V_2)}(X_1,0) = \frac{X_1(\lambda)}{\lambda}(0,V_2),\\
\bar{\nabla}_{(0,V_2)}(0,W_2) &= (-\frac{1}{2}g_2(V_2,W_2)\text{grad } \lambda^2, 0) + (0, {}^{M_2}\nabla_{V_2}W_2),\\
\text{nor}\bar{\nabla}_{(0,V_2)}(0,W_2) &= -\frac{1}{2}g_2(V_2,W_2)(\text{grad}\lambda^2, 0)\\
&= -\frac{\bar{g}((0,V_2),(0,W_2))}{\lambda}(\text{grad}\lambda, 0),\\
\tan\bar{\nabla}_{(0,V_2)}&(0,W_2) = (0, {}^{M_2}\nabla_{V_2}W_2) = \text{lift of } {}^{M_2}\nabla_{V_2}W_2.
\end{aligned}$$

From (2.2), we easily obtain

Proposition 2.7.

$$\begin{aligned}
&\bar{R}_{(X_1,X_2)(Y_1,Y_2)}(Z_1,Z_2)\\
&= ({}^{M_1}R_{X_1Y_1}Z_1, {}^{M_2}R_{X_2Y_2}Z_2)\\
&\quad + \frac{1}{2}g_2(X_2,Z_2)\big({}^{M_1}\nabla_{Y_1}\text{grad } \lambda^2 - \frac{1}{2\lambda^2}Y_1(\lambda^2)\text{grad } \lambda^2, 0\big)\\
&\quad - \frac{1}{2}g_2(Y_2,Z_2)\big({}^{M_1}\nabla_{X_1}\text{grad } \lambda^2 - \frac{1}{2\lambda^2}X_1(\lambda^2)\text{grad } \lambda^2, 0\big)\\
&\quad + \Big(0, \frac{1}{2\lambda^2}g_1({}^{M_1}\nabla_{X_1}\text{grad } \lambda^2 - \frac{1}{2\lambda^2}X_1(\lambda^2)\text{grad } \lambda^2, Z_1)Y_2\Big)\\
&\quad - \Big(0, \frac{1}{2\lambda^2}g_1({}^{M_1}\nabla_{Y_1}\text{grad } \lambda^2 - \frac{1}{2\lambda^2}Y_1(\lambda^2)\text{grad } \lambda^2, Z_1)X_2\Big)\\
&\quad + \Big(0, \frac{1}{4\lambda^2} \mid \text{grad } \lambda^2 \mid^2 g_2(X_2,Z_2)Y_2\Big)\\
&\quad - \Big(0, \frac{1}{4\lambda^2} \mid \text{grad } \lambda^2 \mid^2 g_2(Y_2,Z_2)X_2\Big).
\end{aligned} \tag{2.3}$$

Corollary 2.8.

$$\begin{aligned}
&\bar{R}_{(X_1,X_2)(Y_1,Y_2)}(Z_1,Z_2)\\
&= ({}^{M_1}R_{X_1Y_1}Z_1, {}^{M_2}R_{X_2Y_2}Z_2)\\
&\quad + \lambda g_2(X_2,Z_2)\big({}^{M_1}\nabla_{Y_1}\text{grad } \lambda, 0\big) - \lambda g_2(Y_2,Z_2)\big({}^{M_1}\nabla_{X_1}\text{grad } \lambda, 0\big)\\
&\quad + \frac{1}{\lambda}\text{Hess}(\lambda)(X_1,Z_1)(0,Y_2) - \frac{1}{\lambda}\text{Hess}(\lambda)(Y_1,Z_1)(0,X_2)\\
&\quad + \mid \text{grad } \lambda \mid^2 g_2(X_2,Z_2)(0,Y_2) - \mid \text{grad } \lambda \mid^2 g_2(Y_2,Z_2)(0,X_2).
\end{aligned} \tag{2.4}$$

Proof. Note that

$$\begin{aligned}
&{}^{M_1}\nabla_{X_1}\text{grad }\lambda^2 - \frac{1}{2\lambda^2}X_1(\lambda^2)\text{grad }\lambda^2 \\
&= {}^{M_1}\nabla_{X_1}(2\lambda\text{grad }\lambda) - \frac{1}{2\lambda^2}2\lambda X_1(\lambda)2\lambda\text{grad }\lambda \\
&= 2\lambda\,{}^{M_1}\nabla_{X_1}\text{grad }\lambda
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\lambda^2}g_1\left({}^{M_1}\nabla_{X_1}\text{grad }\lambda^2 - \frac{1}{2\lambda^2}X_1(\lambda^2)\text{grad }\lambda^2,\ Z_1\right) \\
&= \frac{1}{\lambda}g_1\left({}^{M_1}\nabla_{X_1}\text{grad }\lambda,\ Z_1\right) \\
&= \frac{1}{\lambda}\text{Hess}(\lambda)(X_1, Z_1).
\end{aligned}$$

Exchanging X_1 with Y_1, we obtain

$$\begin{aligned}
&{}^{M_1}\nabla_{Y_1}\text{grad }\lambda^2 - \frac{1}{2\lambda^2}Y_1(\lambda^2)\text{grad }\lambda^2 = 2\lambda\,{}^{M_1}\nabla_{Y_1}\text{grad }\lambda, \\
&\frac{1}{2\lambda^2}g_1\left({}^{M_1}\nabla_{Y_1}\text{grad }\lambda^2 - \frac{1}{2\lambda^2}Y_1(\lambda^2)\text{grad }\lambda^2,\ Z_1\right) = \frac{1}{\lambda}\text{Hess}(\lambda)(Y_1, Z_1).
\end{aligned}$$

Putting these facts together, (2.3) can reduce to

$$\begin{aligned}
&\bar{R}_{(X_1,X_2)(Y_1,Y_2)}(Z_1, Z_2) \\
&= ({}^{M_1}R_{X_1Y_1}Z_1,\ {}^{M_2}R_{X_2Y_2}Z_2) \\
&\quad + \lambda g_2(X_2, Z_2)\big({}^{M_1}\nabla_{Y_1}\text{grad }\lambda, 0\big) - \lambda g_2(Y_2, Z_2)\big({}^{M_1}\nabla_{X_1}\text{grad }\lambda, 0\big) \\
&\quad + \frac{1}{\lambda}\text{Hess}(\lambda)(X_1, Z_1)(0, Y_2) - \frac{1}{\lambda}\text{Hess}(\lambda)(Y_1, Z_1)(0, X_2) \\
&\quad + \mid \text{grad }\lambda \mid^2 g_2(X_2, Z_2)(0, Y_2) - \mid \text{grad }\lambda \mid^2 g_2(Y_2, Z_2)(0, X_2),
\end{aligned}$$

as claimed (2.4). ■

Thus we have the unified formulas for (0,4)-type Riemannian curvature tensor $\overline{Rm}$, Ricci curvature $\overline{\text{Ric}}$ and saclar curvature $\overline{\text{Scal}}$.

Theorem 2.9. *On a warped product* $\overline{M}$ *with* $m_2 = \text{dim}M_2 \geq 2$. *Let* (X_1, X_2), $(Y_1, Y_2), (Z_1, Z_2), (W_1, W_2) \in \mathfrak{X}(\overline{M})$. *Then* (*i*) *(0,4)-type Riemannian curvature tensor* $\overline{Rm}$ *satisfies*

$$\begin{aligned}&\overline{\mathrm{Rm}}\big((W_1,W_2),(Z_1,Z_2),(X_1,X_2),(Y_1,Y_2)\big)\\&= {}^{M_1}\mathrm{Rm}(W_1,Z_1,X_1,Y_1)+\lambda^2\,{}^{M_2}\mathrm{Rm}(W_2,Z_2,X_2,Y_2)\\&\quad+\lambda\mathrm{Hess}(\lambda)(Y_1,W_1)g_2(X_2,Z_2)-\lambda\mathrm{Hess}(\lambda)(X_1,W_1)g_2(Y_2,Z_2)\\&\quad+\lambda\mathrm{Hess}(\lambda)(X_1,Z_1)g_2(W_2,Y_2)-\lambda\mathrm{Hess}(\lambda)(Y_1,Z_1)g_2(W_2,X_2)\\&\quad+\lambda^2\mid\mathrm{grad}\ \lambda\mid^2 g_2(X_2,Z_2)g_2(W_2,Y_2)\\&\quad-\lambda^2\mid\mathrm{grad}\ \lambda\mid^2 g_2(Y_2,Z_2)g_2(W_2,X_2).\end{aligned}\tag{2.5}$$

(*ii*) *The Ricci curvature tensor* $\overline{Ric}$ *satisfies*

$$\begin{aligned}\overline{\mathrm{Ric}}\big((X_1,X_2),(Y_1,Y_2)\big)=&\ {}^{M_1}\mathrm{Ric}(X_1,Y_1)+{}^{M_2}\mathrm{Ric}(X_2,Y_2)\\&-\lambda g_2(X_2,Y_2)\Delta_{M_1}\lambda-\frac{m_2}{\lambda}\mathrm{Hess}(\lambda)(X_1,Y_1)\\&-(m_2-1)\mid\mathrm{grad}\ \lambda\mid^2 g_2(X_2,Y_2).\end{aligned}\tag{2.6}$$

(*iii*) *The scalar curvature* $\overline{Scal}$ *is*

$$\begin{aligned}\overline{\mathrm{Scal}}=&\ {}^{M_1}\mathrm{Scal}+\frac{1}{\lambda^2}\,{}^{M_2}\mathrm{Scal}\\&-\frac{2m_2}{\lambda}\Delta_{M_1}\lambda-\frac{m_2(m_2-1)}{\lambda^2}\mid\mathrm{grad}\ \lambda\mid^2.\end{aligned}\tag{2.7}$$

Proof. (i) Note that

$$\overline{\mathrm{Rm}}\big((W_1,W_2),(Z_1,Z_2),(X_1,X_2),(Y_1,Y_2)\big)=\bar g\big((W_1,W_2),\bar R_{(X_1,X_2)(Y_1,Y_2)}(Z_1,Z_2)\big)$$

and $\bar g=g_1\oplus\lambda^2 g_2$. By (2.4) and the property of $\mathrm{Hess}(\lambda)$, we immediately obtain (2.5).

As to the assertions (ii) and (iii), let $\{e_j\}_{j=1}^{m_1}$ be a local orthonormal frame on (M_1,g_1) and $\{\bar e_\alpha\}_{\alpha=1}^{m_2}$ on (M_2,g_2). Then $\{(e_j,0),(0,\frac{1}{\lambda}\bar e_\alpha)\}_{j=1,\dots,m_1,\alpha=1,\dots,m_2}$ forms a local orthonormal frame on $\overline{M}$. By the definition of Ricci curvature, we have

$$\begin{aligned}\overline{\mathrm{Ric}}\big((X_1,X_2),(Y_1,Y_2)\big)=&\sum_{i=1}^{m_1}\overline{\mathrm{Rm}}\big((e_i,0),(X_1,X_2),(e_i,0),(Y_1,Y_2)\big)\\&+\sum_{\alpha=1}^{m_2}\overline{\mathrm{Rm}}\big((0,\frac{1}{\lambda}\bar e_\alpha),(X_1,X_2),(0,\frac{1}{\lambda}\bar e_\alpha).(Y_1,Y_2)\big)\end{aligned}\tag{2.8}$$

By substituting (2.5) into (2.8) and keeping in mind the relation $\sum_\alpha g_2(\bar e_\alpha,X_2)g_2(\bar e_\alpha,Y_2)=g_2(\sum_\alpha g_2(\bar e_\alpha,X_2)\bar e_\alpha,Y_2)=g_2(X_2,Y_2)$, (2.6) follows.

Furthermore, since the scalar curvature $\overline{\mathrm{Scal}}$ satisfies

$$\overline{\mathrm{Scal}}=\sum_{i=1}^{m_1}\overline{\mathrm{Ric}}\big((e_i,0),(e_i,0)\big)+\sum_{\alpha=1}^{m_2}\overline{\mathrm{Ric}}\big((0,\frac{1}{\lambda}\bar e_\alpha),(0,\frac{1}{\lambda}\bar e_\alpha)\big),\tag{2.9}$$

substituting (2.6) into (2.9) gives (2.7). ■

Remark 2.10. It is not hard to verify that the three cases in Theorem (2.9) agree with the results in Propositions 2.3 and 2.4. For instance, by (2.6), we have

$$\begin{aligned}
&\overline{\mathrm{Ric}}\big((0,V),(0,W)\big)\\
&= {}^{M_2}\mathrm{Ric}(V,W) - \lambda g_2(V,W)\Delta_{M_1}\lambda - (m_2-1)\mid \mathrm{grad}\ \lambda \mid_{g_1}^2 g_2(V,W)\\
&= {}^{M_2}\mathrm{Ric}(V,W) - \left(\frac{1}{\lambda}\Delta_{M_1}\lambda + \frac{m_2-1}{\lambda^2}\mid \mathrm{grad}\ \lambda \mid_{\bar g}^2\right)\bar g(V,W),
\end{aligned}$$

which is consistent with the third case (3) in Proposition 2.4.

Remark 2.11. Since (2.6) and (2.7) contain a term with factor (m_2-1), to avoid trivial case, the dimension of M_2 is restricted to $m_2 \geq 2$.

3. The Behavior of Warping Function Under Ricci Flow

In this section, we shall use the unified version of Ricci curvature formula in the previous section to characterize the behavior of warping function under Ricci flow. More specifically, we wish to determine a certain condition which a smooth warping function satisfies such that the warped product metric is the solution to the corresponding Ricci flow.

Before we launch the issue, let us state the definition of the Ricci flow [15, 8].

Definition 3.1. Let M be a manifold, and let $g(t),\ t\in[0,T)$, be a one-parameter family of Riemannian metrics on M. We say that $g(t)$ is a solution to the Rici flow if

$$\frac{\partial}{\partial t}g(t) = -2Ric. \tag{3.1}$$

For the warped product metrics, the Ricci flow is the evolution equation

$$\frac{\partial \bar g(x,y,t)}{\partial t} = -2\overline{\mathrm{Ric}} \tag{3.2}$$

for a one-parameter family of Riemannian metrics $\bar g(t),\ t\in[0,\bar T)$ on $\overline{M}$.

For behavior of the warping function on warped product manifold under RF, we have the following main result.

Theorem 3.2. *Suppose that Riemannian manifold (M_1, g_1) is compact (or complete non-compact) and (M_2, g_2) is compact. Let $(M_1, g_1(t))$ and $(M_2, g_2(t)$ be solutions to the Ricci flow on a common time interval $[0,\bar T)$. Then the warped product metric $\bar g(t) = g_1(x,t)\oplus\lambda^2(x,t)g_2(y,t)$ is a solution to the Ricci flow (3.2) if and only if the warping function $\lambda=\lambda(x,t), t\in[0,\bar T)$ satisfies*

$$\begin{aligned}
\frac{\partial\lambda^2(x,t)}{\partial t}g_2(X_2,Y_2) &= 2(\lambda^2-1)\,{}^{M_2}\mathrm{Ric}(X_2,Y_2) + 2\lambda g_2(X_2,Y_2)\Delta_{M_1}\lambda\\
&\quad + 2\frac{m_2}{\lambda}\mathrm{Hess}(\lambda)(X_1,Y_1) + 2(m_2-1)\mid \mathrm{grad}\ \lambda\mid^2 g_2(X_2,Y_2),
\end{aligned} \tag{3.3}$$

for all $X_1, Y_1 \in \mathfrak{X}(M_1)$, $X_2, Y_2 \in \mathfrak{X}(M_2)$. *In particular, at least a necessary condition is*

$$\frac{\partial \lambda(x,t)}{\partial t} - \left(1 + \frac{m_2}{m_1 \lambda^2}\right) \Delta_{M_1} \lambda - \frac{m_2 - 1}{\lambda} \mid \text{grad } \lambda \mid^2 = \frac{\lambda^2 - 1}{m_2 \lambda} {}^{M_2}\text{Scal}, \tag{3.4}$$

where $m_i = dim M_i$.

Proof. Since $g_i(t), i = 1, 2$ satisfy

$$\frac{\partial g_1(t)}{\partial t} = -2\, {}^{M_1}\text{Ric}, \quad t \in [0, T_1),$$
$$\frac{\partial g_2(t)}{\partial t} = -2\, {}^{M_2}\text{Ric}, \quad t \in [0, T_2),$$

we have the derivative of $\bar{g}(t)$ with respect to the flow parameter t

$$\begin{aligned} &\frac{\partial}{\partial t}(\bar{g}(t))\big((X_1, X_2), (Y_1, Y_2)\big) \\ &= \Big(\frac{\partial g_1(t)}{\partial t} \oplus \big(\lambda^2 \frac{\partial g_2(t)}{\partial t} + \frac{\partial \lambda^2}{\partial t} g_2(t)\big)\Big)\big((X_1, X_2), (Y_1, Y_2)\big) \\ &= -2\, {}^{M_1}\text{Ric}(X_1, Y_1) - 2\lambda^2\, {}^{M_2}\text{Ric}(X_2, Y_2) + \frac{\partial \lambda^2}{\partial t} g_2(t)(X_2, Y_2). \end{aligned} \tag{3.5}$$

Putting this with (2.6) together, we can easily show that $\bar{g}(x, y, t)$ is a solution to (3.2), $t \in [0, \bar{T} = \min(T_1, T_2)\ )$ if and only if $\lambda(x, t)$ satisfies

$$\begin{aligned} &\frac{\partial \lambda^2(x,t)}{\partial t} g_2(X_2, Y_2) = 2(\lambda^2 - 1)\, {}^{M_2}\text{Ric}(X_2, Y_2) + 2\lambda g_2(X_2, Y_2) \Delta_{M_1} \lambda \\ &\quad + 2\frac{m_2}{\lambda} \text{Hess}(\lambda)(X_1, Y_1) + 2(m_2 - 1) \mid \text{grad } \lambda \mid^2 g_2(X_2, Y_2). \end{aligned} \tag{3.6}$$

On one hand, by the symmetry of ${}^{M_2}\text{Ric}$, we can choose an orthonormal basis $\{\bar{e}_\alpha\}$ on M_2 such that ${}^{M_2}\text{Ric}(\bar{e}_\alpha, \bar{e}_\beta) = 0$, $\alpha \neq \beta$. Thus (3.6) reduces to

$$\frac{m_2}{\lambda} \text{Hess}(\lambda)(X_1, Y_1) = 0. \tag{3.7}$$

As only necessary condition, on the other hand, by taking trace in both sides of (3.6) with respect to g_1 and g_2, and noting that $\Delta_{M_1} \lambda = Tr_{g_1} \text{Hess}(\lambda)$, ${}^{M_2}\text{Scal} = Tr_{g_2}\, {}^{M_2}\text{Ric}$, we conclude that (3.6) is equivalent to

$$\begin{aligned} &2m_1 m_2 \lambda \frac{\partial \lambda}{\partial t} = 2m_1(\lambda^2 - 1)\, {}^{M_2}\text{Scal} + 2\lambda m_1 m_2 \Delta_{M_1} \lambda \\ &\quad + 2\frac{m_2^2}{\lambda} \Delta_{M_1} \lambda + 2m_1 m_2 (m_2 - 1) \mid \text{grad } \lambda \mid^2, \end{aligned} \tag{3.8}$$

Clearly, (3.7) is included in (3.8). Thus, we complete the proof. ■

Remark 3.3. (1) If M_1 is compact, then from (3.3) or (3.4), then we immediately see that λ is a constant function in term to M_1.
(2) In Theorem 3.2, we don't stress that $\overline{M}$ is compact or complete non-compact. Assume that M_1 is non-compact complete manifold and M_2 is compact, then $\overline{M}$ is complete non-compact. At this point we need to add a initial metric $\bar{g}_0 = (g_1)_0(x) \oplus \lambda^2(x,0)(g_2)_0(y)$ such that $\bar{R}_{g_0}$ has a boundary.

Now we concern about two questions: 1. Does the PDE (3.3) have any solution? 2. How many degrees of freedom for the warping function λ are there?

In deed, it is easily seen that (3.4) doesn't follow from standard PDE theory. (3.4) tells us that the terms on its left-hand side only consist of the points in the first factor manifold M_1 and flow parameter t whereas those on its right-hand side consist of the points in the second factor manifold M_2 besides t, thus one worries that such complicated nonlinear PDE (3.4) may have no any solution λ.

As for degrees of freedom for the warping function λ, we first consider the simplest cases:

(i) If λ is constant, then from (3.3) or (3.4), then we easily observe that $\lambda = \pm 1$. Since λ is positive, thus $\lambda = 1$, which implies that $\overline{M}$ is exactly a direct product manifold. This is true.

(ii) Assume M_2 has constant scalar curvature, then (3.4) no longer involves the point of M_2. This should be a kernel heat equation, of course it must have solution.

The next two theorems naturally give a guarantee for existence of solution to (3.3) or (3.4) as long as there exists a warped product solution $\bar{g}(t)$ to the RF. From the short-time existence and uniqueness result for Ricci flow on a compact manifold [15, 13], we give the corresponding version for WPM.

Theorem 3.4. *Let* $\big(M_1 \times M_2, (\bar{g}_0)_{ij\alpha\beta}(x,y) := (g_1)^0_{ij}(x,t) + \lambda^2(x)(g_2)^0_{\alpha\beta}(y,t)\big)$ *be a compact Riemannian manifold. Then there exists a constant* $\bar{T} > 0$ *such that the initial value problem*

$$\begin{cases} \dfrac{\partial}{\partial t}(\bar{g}_{ij\alpha\beta}(x,y,t)) = -2\overline{\mathrm{Ric}}_{ij\alpha\beta}(x,y,t) \\ \bar{g}_{ij\alpha\beta}(x,y,0) = (\bar{g}_0)_{ij\alpha\beta}(x,y) \end{cases}$$

has a unique smooth solution $\bar{g}_{ij\alpha\beta}(x,y,t) = (g_1)_{ij}(x,t) \oplus \lambda^2(x,t)(g_2)_{\alpha\beta}(y,t)$ *on* $\overline{M} \times [0,\bar{T})$*, where* $\overline{\mathrm{Ric}}_{ij\alpha\beta}(x,y,t) := {}^{M_1}\mathrm{Ric}_{ij}(x,t) + \lambda^2(x,t)\,{}^{M_2}\mathrm{Ric}_{\alpha\beta}(y,t)$.

On a non-compact complete manifold $\overline{M}$, we only require the short-time existence established by Shi [22]. The following result is modified to the warped product case according to the version of Shi.

Theorem 3.5. *Let $\big(M_1 \times M_2, \bar{g}_0(x,y) = (g_1)^0(x) \oplus \lambda_0^2(x)(g_2)^0(y)\big)$ be a complete noncompact Riemannian manifold of dimension $m_1 + m_2$ with bounded curvature. Then there exists a constant $\bar{T} > 0$ such that the initial value problem*

$$\begin{cases} \dfrac{\partial}{\partial t}(\bar{g}_{ij\alpha\beta}(x,y,t)) = -2\overline{\mathrm{Ric}}_{ij\alpha\beta}(x,y,t), \\ \bar{g}_{ij\alpha\beta}(x,y,0) = (g_1)^0_{ij}(x) \oplus \lambda_0^2(x)(g_2)^0_{\alpha\beta}(y) \end{cases}$$

has a smooth solution $\bar{g}_{ij\alpha\beta}(x,y,t) = (g_1)_{ij}(x,t) \oplus \lambda^2(x,t)(g_2)_{\alpha\beta}(y,t)$ on $\overline{M} \times [0,\bar{T}]$ with uniformly bounded curvature.

Now we construct a relatively simple example.

Example 3.6. Let $M_1 = \mathbb{R}$ with flat metric $g_1 = h(x) = \mu^2(x)dx^2$ ($\mu(x)$ is a smooth positive function) and $M_2 = S^n$ $(n \geq 2)$ with the standard metric which implies M_2 admits an Einstein metric $g_2 = \lambda^2(x)g_{S^n}$. By the main result in [23], under some constraints for initial values, there exists warping functions $\lambda(x,t)$ and a maximal constant T such that warped product solution

$$\bar{g}(x,y) = h(x,t) \oplus \lambda^2(x,t)g_{S^n}(y), \quad t \in [0,T)$$

to the RF (3.2). Of course, we don't write $\lambda(x,t)$ as explicit form. On $\overline{M} = \mathbb{R} \times S^n$, the warped product metric $\bar{g} = \mu^2(x)dx^2 \oplus \lambda^2(x)g_{S^n}$ can be read as

$$\bar{g}(s,y) = ds^2 \oplus \lambda^2(s)g_{S^n}(y),$$

where $s = \int_0^x \mu(x)dx$ is the arc-length parameter. Then the sectional curvatures of planes containing or perpendicular to the radical vector $\frac{\partial}{\partial s} = \frac{1}{\mu(x)}\frac{\partial}{\partial x}$ are respectively (cf. Chap.3 in [21], or [1, 19])

$$K_{rad} = -\frac{\lambda_{ss}}{\lambda}, \quad K_{sph} = \frac{1-\lambda_s^2}{\lambda^2},$$

and the Ricci tensor is

$$\begin{aligned} \overline{\mathrm{Ric}} &= -n\frac{\mu\lambda_{xx}-\lambda_x\mu_x}{\lambda\mu}dx^2 \oplus \left(-\frac{\lambda\mu\lambda_{xx}+(n-1)\mu\lambda_x^2-\lambda\lambda_x\mu_x}{\mu^3}+n-1\right)g_{S^n} \\ &= -n\frac{\lambda_{ss}}{\lambda}ds^2 \oplus \Big((n-1)\big(1-\lambda_s^2\big) - \lambda\lambda_{ss}\Big)g_{S^n} \\ &= nK_{rad}ds^2 \oplus \left(K_{rad} + (n-1)K_{sph}\right)\lambda^2(s)g_{S^n}. \end{aligned} \tag{3.9}$$

Since dx^2 and g_{S^n} are independent of t, a direct computation gives

$$\begin{aligned} &\frac{\partial}{\partial t}\big(\mu^2(x,t)dx^2 \oplus \lambda^2(x,t)g_{S^n}(y)\big) \\ &= 2\mu\mu_t dx^2 \oplus 2\lambda\lambda_t g_{S^n} \\ &= 2\frac{\mu_t}{\mu}ds^2 \oplus 2\lambda\lambda_t g_{S^n}. \end{aligned} \tag{3.10}$$

Hence if the warped product metrics $\bar{g}(x, y, t) = \mu^2(x, t)dx^2 \oplus \lambda^2(x, t)g_{S^n}(y)$ is a solution to the Ricci flow (3.2), then substituting (3.9) and (3.10) into (3.2) immediately yields

$$\begin{cases} \dfrac{\lambda_t}{\lambda} = -\big(K_{rad} + (n-1)K_{sph}\big), \\ \dfrac{\mu_t}{\mu} = -nK_{rad}, \end{cases} \tag{3.11}$$

which happens to be

$$\begin{cases} \dfrac{\partial \log \lambda}{\partial t} = -\big(K_{rad} + (n-1)K_{sph}\big), \\ \dfrac{\partial \log \mu}{\partial t} = -nK_{rad}. \end{cases} \tag{3.12}$$

Since the sectional curvature functions K_{rad} and K_{sph} are uniform bound (see the proof of Theorem 1.2 in [19]), we integrate (3.12) over the time interval $[0, t], t < T$ and get the functions

$$\begin{cases} \lambda(x, t) = \lambda(x, 0)e^{-\int_0^t \big(K_{rad}+(n-1)K_{sph}\big)dt}, \\ \mu(x, t) = \mu(x, 0)e^{-n\int_0^t K_{rad}dt}. \end{cases}$$

4. The Behavior of Warping Function Under HGF

We now investigate the behavior of warping function under the hyperbolic geometric flow.

Recall that Kong and Liu [17] introduced a geometric flow called hyperbolic geometric flow (HGF) whose definition is as follows.

Definition 4.1. Let M be a Riemannian manifold. The hyperbolic geometric flow is the evolution equation

$$\frac{\partial^2}{\partial t^2} g(t) = -2Ric \tag{4.1}$$

for a one-parameter family of Riemannian metrics $g(t),\ t \in [0, T)$ on M. We say that $g(t)$ is a solution to the hyperbolic geometric flow if it satisfies (4.1).

When M is exchanged for our warped product manifold $\overline{M}$, the corresponding HGF is

$$\frac{\partial^2}{\partial t^2} \bar{g}(t) = -2\overline{\mathrm{Ric}}. \tag{4.2}$$

In this case, similar to Theorem 3.2 we have a result with preserved flow-type condition as follows.

Theorem 4.2. *Suppose that Riemannian manifold (M_1, g) is compact (or complete non-compact) and M_2 is compact. If $(M_1, g_1(t))$ and $(M_2, g_2(t)$ are the solution to the HGF on a common time interval I, respectively, then the warped product metric $\bar{g}(x, y, t) = g_1(x, t) \oplus \lambda^2(x, t) g_2(y, t)$ is a solution to the HGF (4.2) if and only if the warped product function $\lambda = \lambda(x, t), t \in I$ satisfies*

$$\begin{aligned} &\frac{m_2}{2}\frac{\partial^2\lambda^2}{\partial t^2} - \frac{(\lambda^2+m_2)m_2}{\lambda}\Delta_{M_1}\lambda - m_2(m_2-1) \mid \mathrm{grad}\ \lambda \mid^2 \\ &= (\lambda^2 - 1)\ {}^{M_2}\mathrm{Scal} - Tr_{g_2}(\frac{\partial g_2}{\partial t})\frac{\partial \lambda^2}{\partial t} \end{aligned} \tag{4.3}$$

and

$$m_1 \frac{\partial \lambda^2}{\partial t}\frac{\partial g_2(t)}{\partial t}(\bar{e}_\alpha, \bar{e}_\beta) + \frac{m_2}{\lambda}\Delta_{M_1}\lambda = 0, \quad \alpha \neq \beta, \tag{4.4}$$

where $\{\bar{e}_\alpha\}$ is an orthonormal basis on M_2 such that ${}^{M_2}\mathrm{Ric}(\bar{e}_\alpha, \bar{e}_\beta) = 0$.

Proof. Since $g_i(t), i = 1, 2$ satisfy

$$\begin{aligned} \frac{\partial^2 g_1(t)}{\partial t^2} &= -2\ {}^{M_1}\mathrm{Ric}, \\ \frac{\partial^2 g_2(t)}{\partial t^2} &= -2\ {}^{M_2}\mathrm{Ric}, \quad t \in I, \end{aligned}$$

we have

$$\begin{aligned} &\frac{\partial^2 \bar{g}(t)}{\partial t^2}\big((X_1, X_2), (Y_1, Y_2)\big) \\ &= \frac{\partial^2 g_1(t)}{\partial t^2}(X_1, Y_1) + \lambda^2 \frac{\partial^2 g_2(t)}{\partial t^2}(X_2, Y_2) \\ &+ 2\frac{\partial \lambda^2}{\partial t}\frac{\partial g_2(t)}{\partial t}(X_2, Y_2) + \frac{\partial^2 \lambda^2}{\partial t^2} g_2(t)(X_2, Y_2) \\ &= -2\ {}^{M_1}\mathrm{Ric}(X_1, Y_1) - 2\lambda^2\ {}^{M_2}\mathrm{Ric}(X_2, Y_2) \\ &+ 2\frac{\partial \lambda^2}{\partial t}\frac{\partial g_2(t)}{\partial t}(X_2, Y_2) + \frac{\partial^2 \lambda^2}{\partial t^2} g_2(t)(X_2, Y_2). \end{aligned}$$

Combining this and (2.6), we easily see that $\bar{g}(x, y, t)$ is the solution to (4.2) if and only if $\lambda = \lambda(x, y, t)$ satisfies

$$\begin{aligned} &\frac{\partial^2 \lambda^2}{\partial t^2} g_2(t)(X_2, Y_2) + 2\frac{\partial \lambda^2}{\partial t}\frac{\partial g_2(t)}{\partial t}(X_2, Y_2) \\ &= (2\lambda^2 - 2)\ {}^{M_2}\mathrm{Ric}(X_2, Y_2) + 2\lambda\Delta_{M_1}\lambda g_2(X_2, Y_2) \\ &+ \frac{2m_2}{\lambda}\mathrm{Hess}(\lambda)(X_1, Y_1) + 2(m_2 - 1) \mid \mathrm{grad}\lambda \mid^2 g_2(X_2, Y_2). \end{aligned} \tag{4.5}$$

After we choose an orthonormal basis $\{\bar{e}_\alpha\}$ on M_2 such that ${}^{M_2}\mathrm{Ric}(\bar{e}_\alpha, \bar{e}_\beta) = 0$, $\alpha \neq \beta$, (4.5) reduces to

$$\frac{\partial \lambda^2}{\partial t} \frac{\partial g_2(t)}{\partial t} (\bar{e}_\alpha, \bar{e}_\beta) = \frac{m_2}{\lambda} \mathrm{Hess}(\lambda)(X_1, Y_1). \tag{4.6}$$

Further trace it with respect to g_1, we get

$$m_1 \frac{\partial \lambda^2}{\partial t} \frac{\partial g_2(t)}{\partial t} (\bar{e}_\alpha, \bar{e}_\beta) = \frac{m_2}{\lambda} \Delta_{M_1} \lambda, \quad \alpha \neq \beta,$$

which is just (4.4).

On the other hand, by taking trace in both sides of (4.5) with respect to g_1 and g_2, we can reduce (4.5) to

$$\begin{aligned} & m_1 m_2 \frac{\partial^2 \lambda^2}{\partial t^2} + 2m_1 \frac{\partial \lambda^2}{\partial t} Tr_{g_2} (\frac{\partial g_2(t)}{\partial t}) \\ = & 2m_1 (\lambda^2 - 1)\, {}^{M_2}\mathrm{Scal} + 2\lambda m_1 m_2 \Delta_{M_1} \lambda \\ + & \frac{2m_1 m_2^2}{\lambda} \Delta_{M_1} \lambda + 2m_1 m_2 (m_2 - 1) \mid \mathrm{grad}\ \lambda \mid^2, \end{aligned}$$

which implies (4.3). ■

Obviously, the equation (4.3) is analogous to the previous equation (3.3) or (3.4) but much more complicated. This is manifested chiefly by the second-order derivative term $\frac{\partial^2 \lambda^2}{\partial t^2}$ and the extra term $Tr_{g_2}(\frac{\partial g_2(t)}{\partial t})$ without carrying given information. Therefore one may worry about the equation (4.3) has no any solution and makes no any sense. Indeed, it need't worry, because the short-time existence result for HGF on a compact manifold (see Theorem 1.1 in [11]) can provide us an evidence. We give its version related to WPM as follows.

Theorem 4.3. *Let* $\big(M_1 \times M_2, \bar{g}^0(x, y) = (g_1)^0(x) \oplus \lambda_0^2(x)(g_2)^0(y)\big)$ *be a compact Riemannian manifold. Then there exists a constant* $\bar{T} > 0$ *such that the initial value problem*

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} (\bar{g}_{ij\alpha\beta}(x, y, t)) = -2\overline{\mathrm{Ric}}_{ij\alpha\beta}(x, y, t) \\ \bar{g}_{ij\alpha\beta}(x, y, 0) = \bar{g}^0_{ij\alpha\beta}(x, y), \quad \dfrac{\partial}{\partial t} \bar{g}_{ij\alpha\beta}(x, y, 0) = h^0_{ij\alpha\beta}(x, y) \end{cases}$$

has a unique smooth solution $\bar{g}_{ij\alpha\beta}(x, y, t) = (g_1)_{ij}(x, t) \oplus \lambda^2 (g_2)_{\alpha\beta}(y, t)$ *on* $\overline{M} \times [0, \bar{T}]$*, where* $h^0_{ij\alpha\beta}(x, y)$ *is a symmetric tensor on* $M_1 \times M_2$*.*

In non-compact complete manifold $\overline{M}$, framing Theorem 3.5 and Theorem 4.3 and combining Theorem 3.1 in [23] (see Introduction section), we present an analogous result ro Theorem 3.5 without proof.

Proposition 4.4. *Let* $\big(M_1 \times M_2, \bar{g}_0(x,y) = (g_1)^0(x) \oplus \lambda_0^2(x)(g_2)^0(y)\big)$ *and* $\big(M_1 \times M_2, \bar{h}_0(x,y)\big)$ *be complete noncompact Riemannian manifolds of dimension* $m_1 + m_2$ *with bounded curvature and* λ_0 *be imposed certain constrains. Then there exists a constant* $\bar{T} > 0$ *such that the initial value problem*

$$\begin{cases} \dfrac{\partial}{\partial t}(\bar{g}_{ij\alpha\beta}(x,y,t)) = -2\overline{\mathrm{Ric}}_{ij\alpha\beta}(x,y,t), \\ \bar{g}_{ij\alpha\beta}(x,y,0) = \bar{g}^0_{ij\alpha\beta}(x,y) = (g_1)^0_{ij}(x) \oplus \lambda_0^2(x)(g_2)^0_{\alpha\beta}(y), \\ \dfrac{\partial}{\partial t}\bar{g}_{ij\alpha\beta}(x,y,0) = h^0_{ij\alpha\beta}(x,y) \end{cases}$$

has a smooth solution $\bar{g}_{ij\alpha\beta}(x,y,t) = (g_1)_{ij}(x,t) \oplus \lambda^2 (g_2)_{\alpha\beta}(y,t)$ *on* $\overline{M} \times [0, \bar{T}]$ *with uniformly bounded curvature.*

In order to gain a sense of (4.3), we present several special examples.

Example 4.5. (Trivial example). If λ is constant, then we easily observe from (4.3) and (4.4) that $\lambda = \pm 1$. Since λ is positive, thus $\lambda = 1$, which implies that $\overline{M}$ is exactly a direct product manifold. This is a fact.

Example 4.6. For simplicity sake, we manage to let the unknown term $\frac{\partial}{\partial t}\bar{g}_2(x,y,t) = 0$ in (4.3). Take $M_2 = S^n$ $(n \geq 2)$ with the standard metric which implies M_2 admits an Einstein metric $g_2 = g_{S^n}$. Like the previous Example 3.6, let $M_1 = \mathbb{R}$ with flat metric $g_1 = \mu(x)dx^2$. On $\overline{M} = \mathbb{R} \times S^n$, the warped product metric $\bar{g} = \mu^2(x)dx^2 \oplus \lambda^2(x) g_{S^n}$ can be read as

$$\bar{g}(s,y) = ds^2 \oplus \lambda^2(s) g_{S^n}(y),$$

where $s = \int_0^x \mu(x)dx$ is the arc-length parameter.

Remembering dx^2 and g_{S^n} are independent of t, we get

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\big(\mu^2(x,t)dx^2 \oplus \lambda^2(x,t) g_{S^n}(y)\big) \\ &= 2(\mu\mu_{tt} + \mu_t^2)dx^2 \oplus 2(\lambda\lambda_{tt} + \lambda_t^2) g_{S^n} \\ &= 2\frac{\mu\mu_{tt} + \mu_t^2}{\mu^2} ds^2 \oplus 2(\lambda\lambda_{tt} + \lambda_t^2) g_{S^n}. \end{aligned} \tag{4.7}$$

Therefore, if the warped product metrics $\bar{g}(x,y,t) = \mu^2(x,t)dx^2 \oplus \lambda^2(x,t) g_{S^n}(y)$ is a solution to the HGF (4.2), then substituting (2.6) and (4.7) into (4.2), we obtain

$$\begin{cases} \dfrac{\lambda\lambda_{tt} + \lambda_t^2}{\lambda^2} = -\big(K_{rad} + (n-1)K_{sph}\big), \\ \dfrac{\mu\mu_{tt} + \mu_t^2}{\mu^2} = -nK_{rad}, \end{cases} \tag{4.8}$$

which happens to be

$$
\left\{
\begin{aligned}
&\frac{\partial^2 \bar{\lambda}}{\partial t^2} = -\big(K_{rad} + (n-1)K_{sph}\big), \\
&\frac{\partial^2 \bar{\mu}}{\partial t^2} = -nK_{rad},
\end{aligned}
\right. \tag{4.9}
$$

where we assume that there are exactly the relations

$$
\bar{\lambda}_{tt} = \frac{\lambda\lambda_{tt} + \lambda_t^2}{\lambda^2} \tag{4.10}
$$

and

$$
\bar{\mu}_{tt} = \frac{\mu\mu_{tt} + \mu_t^2}{\mu^2}. \tag{4.11}
$$

Since the sectional curvature functions K_{rad} and K_{sph} are of uniform bound, we can integrate (4.9) over the time interval $[0, t], t < T$ for twice and get

$$
\left\{
\begin{aligned}
&\bar{\lambda}(x,t) = \bar{\lambda}(x,0) + t\bar{\lambda}_t(x,0) - \int_0^t \int_0^u \big(K_{rad} + (n-1)K_{sph}\big) du dt, \\
&\bar{\mu}(x,t) = \bar{\mu}(x,0) + t\bar{\mu}_t(x,0) - n\int_0^t \int_0^u K_{rad} du dt.
\end{aligned}
\right.
$$

Further we locally re-solve the original functions $\lambda(x,t)$ and $\mu(x,t)$.

Remark 4.7. Although (4.8) may look simple and has a local solution, we have to remind it is a set of nonlinear weakly hyperbolic PDEs

$$
\left\{
\begin{aligned}
&\lambda_{tt} - \lambda_{ss} = \frac{1}{\lambda}\lambda_t^2 + \frac{1}{\lambda}\lambda_s^2 - (n-1)\lambda, \\
&\frac{1}{\mu}\mu_{tt} + \frac{1}{\mu^2}\mu_t^2 = \frac{n}{\lambda}\lambda_{ss},
\end{aligned}
\right. \tag{4.12}
$$

which is almost never easy to solve. (4.10) and (4.11) may be only our own wishful thinking or be taken for granted.

5. Evolution Equations of Warping Function and Ricci Curvature

In this section, we present evolution equations for an arbitrary family of warped product metrics $\bar{g}(x, y, t) = g_1(x, t) \oplus \lambda^2(x, t)g_2(y, t), t \in [0, T]$ with Einstein metric g_2 that is evolving by RF and by HGF as well. We also present evolution equations for the Ricci curvatures of such an evolving metric. Our idea mainly comes from Simon's strategy [23].

From now on, we make informal convention for some notations on $\overline{M} = M_1 \times_\lambda M_2$:

$$
(5.1)\quad
\begin{aligned}
&\partial_i := \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, m_1; \quad \partial_\alpha := \frac{\partial}{\partial y^\alpha}, \quad \alpha = 1, \ldots, m_2; \\
&\bar{g}_{ij} = \bar{g}_{(i0)(j0)} := \bar{g}\big((\partial_i, 0), (\partial_j, 0)\big); \quad \bar{g}_{\alpha\beta} = \bar{g}_{(0\alpha)(0\beta)} := \bar{g}\big((0, \partial_\alpha), (0, \partial_\beta)\big); \\
&\bar{g}_{ij\alpha\beta} = \bar{g}_{(i\alpha)(j\beta)} := \bar{g}\big((\partial_i, \partial_\alpha), (\partial_j, \partial_\beta)\big); \quad (\bar{g}^{ij\alpha\beta}) := (\bar{g}_{ij\alpha\beta})^{-1}; \\
&\overline{\mathrm{Rm}}_{(i\alpha)(j\beta)(k\sigma)(l\tau)} := \overline{\mathrm{Rm}}\left((\partial_i, \partial_\alpha), (\partial_j, \partial_\beta), (\partial_k, \partial_\sigma), (\partial_l, \partial_\tau)\right); \\
&\overline{\mathrm{Ric}}_{ij} = \overline{\mathrm{Ric}}_{(i0)(j0)} := \overline{\mathrm{Ric}}\big((\partial_i, 0), (\partial_j, 0)\big); \\
&\overline{\mathrm{Ric}}_{ij\alpha\beta} = \overline{\mathrm{Ric}}_{(i\alpha)(j\beta)} := \overline{\mathrm{Ric}}\big((\partial_i, \partial_\alpha), (\partial_j, \partial_\beta)\big).
\end{aligned}
$$

5.1. Metric and warping function evolution equations

Since the cross terms of $\bar{g}$ are zero, we need only to consider the evolution equations of $\bar{g}_{ij}$ and $\bar{g}_{\alpha\beta}$. Meanwhile we make an assumption that g_2 has a fixed Einstein metric of the form ${}^{M_2}\mathrm{Ric} = cg_2$ (c is some constant) and derive on evolution equation of warping function.

Proposition 5.1. *Let the smooth warped product metric*

$$\bar{g}(x, y, t) = g_1(x, t) \oplus \lambda^2(x, t) g_2(y, t), t \in [0, \bar{T})$$

be a solution to the Ricci flow (3.2) on the manifold $M_1 \times M_2$. Then the metrics g_1 and g_2 satisfy the evolution equations

$$(5.2)\quad \frac{\partial}{\partial t}(g_1)_{ij} = -2\,{}^{M_1}\mathrm{Ric}_{ij} + \frac{2m_2}{\lambda}\mathrm{Hess}(\lambda)(\partial_i, \partial_j),$$

$$(5.3)\quad \frac{\partial}{\partial t}\left(\lambda^2 (g_2)_{\alpha\beta}\right) = -2\,{}^{M_2}\mathrm{Ric}_{\alpha\beta} + \left(\Delta_{M_1}\lambda^2 + (2m_2 - 4)|\mathrm{grad}\lambda|^2\right)(g_2)_{\alpha\beta}$$

Proof. Since we see that $(g_1)_{ij} = \bar{g}_{ij}$ and $\lambda^2 (g_2)_{\alpha\beta} = \bar{g}_{\alpha\beta}$, by using the Ricci flow (3.2) and Ricci curvature formula (2.6), we immediately get the desired identities (5.2) and (5.3). ■

Corollary 5.2. *Soppose that g_2 is a fixed Einstein metric with constant c. Then under the Ricci flow (3.2), the warping function λ satisfies the following evolution equation*

$$(5.4)\quad \frac{\partial}{\partial t}\lambda^2 = -2c + \Delta_{M_1}\lambda^2 + (2m_2 - 4)|\mathrm{grad}\lambda|^2$$

Proof. By already assumption, g_2 is independent of t. Combining this and ${}^{M_2}\mathrm{Ric}_{\alpha\beta} = c(g_2)_{\alpha\beta}$, (5.4) follows from (5.3). ■

Similar to the above results, we have some parallel conclusions under the HGF.

Proposition 5.3. *Let*

$$\bar{g}(x,y,t) = g_1(x,t) \oplus \lambda^2(x,t) g_2(y,t), t \in [0,\bar{T})$$

be a solution to the hyperbolic geometric flow (4.2) on the manifold $M_1 \times M_2$, where g_2 is a fixed Einstein metric with constant c. Then the metrics g_1 and the warping function λ satisfy the evolution equation

$$\frac{\partial^2}{\partial t^2}(g_1)_{ij} = -2\,{}^{M_1}\mathrm{Ric}_{ij} + \frac{2m_2}{\lambda}\mathrm{Hess}(\lambda)(\partial_i, \partial_j), \tag{5.5}$$

$$\frac{\partial^2}{\partial t^2}\lambda^2 = -2c + \Delta_{M_1}\lambda^2 + (2m_2 - 4)|\mathrm{grad}\lambda|^2 \tag{5.6}$$

5.2. Ricci curvature evolution equations

From (2.6) or Proposition 2.4, we see that the cross terms of $\overline{\mathrm{Ric}}$ are zero. Hence we only consider the evolution equations for $\overline{\mathrm{Ric}}_{ij}$ and $\overline{\mathrm{Ric}}_{\alpha\beta}$.

Theorem 5.4. *Under the Ricci flow (3.2) on the manifold $\overline{M}$, the Ricci curvature $\overline{\mathrm{Ric}}_{ij}$ and $\overline{\mathrm{Ric}}_{\alpha\beta}$ satisfy the following evolution equations*

$$\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{ij} = \bar{\Delta}\overline{\mathrm{Ric}}_{ij} + \frac{2}{m_2}\bar{g}^{\alpha\beta}\overline{\mathrm{Ric}}_{\alpha\beta}\left(\overline{\mathrm{Ric}}_{ij} - {}^{M_1}\mathrm{Ric}_{ij}\right) - 2\bar{g}^{kl}\overline{\mathrm{Ric}}_{ik}\overline{\mathrm{Ric}}_{jl}, \tag{5.7}$$

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{\alpha\beta} = {}& \bar{\Delta}\overline{\mathrm{Ric}}_{\alpha\beta} + \frac{2}{m_2}\bar{g}^{kl}\bar{g}^{pq}\overline{\mathrm{Ric}}_{lq}\left(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\right)\bar{g}_{\alpha\beta} - 2\bar{g}^{\gamma\delta}\overline{\mathrm{Ric}}_{\gamma\alpha}\overline{\mathrm{Ric}}_{\delta\beta} \\ & + 2\lambda^2\bar{g}^{\gamma\delta}\bar{g}^{\sigma\tau}\overline{\mathrm{Ric}}_{\delta\tau}\left({}^{M_2}\mathrm{Rm}_{\alpha\gamma\beta\sigma} + |\mathrm{grad}\lambda|^2\left((g_2)_{\alpha\sigma}(g_2)_{\beta\gamma} - (g_2)_{\alpha\beta}(g_2)_{\gamma\sigma}\right)\right).\end{aligned} \tag{5.8}$$

Proof. According to our notational convention (5.1) on $\overline{M}$, the evolution equation of the Ricci curvature in [15] is transformed into such form as

$$\begin{aligned}&\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{\bar{\imath}\bar{\jmath}\bar{\alpha}\bar{\beta}} \\ &= \bar{\Delta}\overline{\mathrm{Ric}}_{\bar{\imath}\bar{\jmath}\bar{\alpha}\bar{\beta}} + 2\bar{g}^{\bar{k}\bar{l}\bar{\gamma}\bar{\delta}}\bar{g}^{\bar{p}\bar{q}\bar{\sigma}\bar{\tau}}\overline{\mathrm{Rm}}_{(\bar{k}\bar{\gamma})(\bar{\imath}\bar{\alpha})(\bar{p}\bar{\sigma})(\bar{\jmath}\bar{\beta})}\overline{\mathrm{Ric}}_{\bar{l}\bar{q}\bar{\delta}\bar{\tau}} - 2\bar{g}^{\bar{k}\bar{l}\bar{\gamma}\bar{\delta}}\overline{\mathrm{Ric}}_{\bar{k}\bar{\imath}\bar{\gamma}\bar{\alpha}}\overline{\mathrm{Ric}}_{\bar{l}\bar{\jmath}\bar{\delta}\bar{\beta}}.\end{aligned} \tag{5.9}$$

where $\bar{\imath}, \bar{\jmath} = 0, 1, \ldots, m_1, \quad \bar{\alpha}, \bar{\beta} = 0, 1, \ldots, m_2$, etc. Hence we have

$$\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{ij} = \bar{\Delta}\overline{\mathrm{Ric}}_{ij} + 2\bar{g}^{\bar{k}\bar{l}\bar{\gamma}\bar{\delta}}\bar{g}^{\bar{p}\bar{q}\bar{\sigma}\bar{\tau}}\overline{\mathrm{Rm}}_{(\bar{k}\bar{\gamma})(i0)(\bar{p}\bar{\sigma})(j0)}\overline{\mathrm{Ric}}_{\bar{l}\bar{q}\bar{\delta}\bar{\tau}} - 2\bar{g}^{\bar{k}\bar{l}\bar{\gamma}\bar{\delta}}\overline{\mathrm{Ric}}_{\bar{k}i\bar{\gamma}0}\overline{\mathrm{Ric}}_{\bar{l}j\bar{\delta}0}. \tag{5.10}$$

(2.4) and (2.5) tell us that the only non-zero $\overline{\mathrm{Rm}}_{(\bar{k}\bar{\gamma})(i0)(\bar{p}\bar{\sigma})(j0)}$ are of the form $\overline{\mathrm{Rm}}_{(0\gamma)(i0)(0\sigma)(j0)}$. On the other hand, we also see that $\bar{g}^{i00\alpha} = \bar{g}^{i\alpha} = 0$ and $\overline{\mathrm{Ric}}_{0i\alpha0} = \overline{\mathrm{Ric}}_{\alpha i} = 0$. Putting these facts together, (5.10) can be reduced to

$$\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{ij} = \bar{\Delta}\overline{\mathrm{Ric}}_{ij} + 2\bar{g}^{\gamma\delta}\bar{g}^{\sigma\tau}\overline{\mathrm{Rm}}_{(0\gamma)(i0)(0\sigma)(j0)}\overline{\mathrm{Ric}}_{\delta\tau} - 2\bar{g}^{kl}\overline{\mathrm{Ric}}_{ki}\overline{\mathrm{Ric}}_{lj}. \tag{5.11}$$

Since (2.5) gives

$$\overline{\mathrm{Rm}}_{(0\gamma)(i0)(0\sigma)(j0)} = -\lambda \mathrm{Hess}(\lambda)(\partial_i, \partial_j)(g_2)_{\gamma\sigma} = -\frac{1}{\lambda}\mathrm{Hess}(\lambda)(\partial_i, \partial_j)\bar{g}_{\gamma\sigma}, \tag{5.12}$$

again (2.6) gives

$$\overline{\mathrm{Ric}}_{ij} = {}^{M_1}\mathrm{Ric}_{ij} - \frac{m_2}{\lambda}\mathrm{Hess}(\lambda)(\partial_i, \partial_j), \tag{5.13}$$

combining (5.12) and (5.13) gives

$$\overline{\mathrm{Rm}}_{(0\gamma)(i0)(0\sigma)(j0)} = \frac{1}{m_2}\big(\overline{\mathrm{Ric}}_{ij} - {}^{M_1}\mathrm{Ric}_{ij}\big)\bar{g}_{\gamma\sigma}. \tag{5.14}$$

Substituting (5.14) into (5.11) yields

$$\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{ij} = \bar{\Delta}\overline{\mathrm{Ric}}_{ij} + \frac{2}{m_2}\bar{g}^{\delta\tau}\overline{\mathrm{Ric}}_{\delta\tau}\big(\overline{\mathrm{Ric}}_{ij} - {}^{M_1}\mathrm{Ric}_{ij}\big) - 2\bar{g}^{kl}\overline{\mathrm{Ric}}_{ki}\overline{\mathrm{Ric}}_{lj},$$

which is (5.7).

Now we calculate the evolution of $\overline{\mathrm{Ric}}_{\alpha\beta}$. From (5.9) we get

$$\begin{aligned}&\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{\alpha\beta}\\ &= \bar{\Delta}\overline{\mathrm{Ric}}_{\alpha\beta} + 2\bar{g}^{\bar{k}\bar{l}\bar{\gamma}\bar{\delta}}\bar{g}^{\bar{p}\bar{q}\bar{\sigma}\bar{\tau}}\overline{\mathrm{Rm}}_{(\bar{k}\bar{\gamma})(0\alpha)(\bar{p}\bar{\sigma})(0\beta)}\overline{\mathrm{Ric}}_{\bar{l}\bar{q}\bar{\delta}\bar{\tau}} - 2\bar{g}^{\bar{k}\bar{l}\bar{\gamma}\bar{\delta}}\overline{\mathrm{Ric}}_{\bar{k}0\bar{\gamma}\alpha}\overline{\mathrm{Ric}}_{\bar{l}0\bar{\delta}\beta}.\end{aligned} \tag{5.15}$$

Once again using that $\bar{g}^{i\alpha} = 0$ and $\overline{\mathrm{Ric}}_{\alpha i} = 0$, and remembering the only non-zero terms $\overline{\mathrm{Rm}}_{(k0)(0\alpha)(p0)(0\beta)}$ and $\overline{\mathrm{Rm}}_{(0\gamma)(0\alpha)(0\sigma)(0\beta)}$, (5.15) becomes

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{\alpha\beta} &= \bar{\Delta}\overline{\mathrm{Ric}}_{\alpha\beta} + 2\bar{g}^{kl}\bar{g}^{pq}\overline{\mathrm{Rm}}_{(k0)(0\alpha)(p0)(0\beta)}\overline{\mathrm{Ric}}_{lq}\\ &\quad + 2\bar{g}^{\gamma\delta}\bar{g}^{\sigma\tau}\overline{\mathrm{Rm}}_{(0\gamma)(0\alpha)(0\sigma)(0\beta)}\overline{\mathrm{Ric}}_{\delta\tau} - 2\bar{g}^{\gamma\delta}\overline{\mathrm{Ric}}_{\gamma\alpha}\overline{\mathrm{Ric}}_{\delta\beta}.\end{aligned} \tag{5.16}$$

Since (5.14) gives

$$\overline{\mathrm{Rm}}_{(k0)(0\alpha)(p0)(0\beta)} = \frac{1}{m_2}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)\bar{g}_{\alpha\beta} \tag{5.17}$$

and (2.6) gives

$$\begin{aligned}&\overline{\mathrm{Rm}}_{(0\gamma)(0\alpha)(0\sigma)(0\beta)}\\ &= \lambda^2\, {}^{M_2}\mathrm{Rm}_{\alpha\gamma\beta\sigma} + \lambda^2|\mathrm{grad}\lambda|^2\left((g_2)_{\alpha\sigma}(g_2)_{\beta\gamma} - (g_2)_{\alpha\beta}(g_2)_{\gamma\sigma}\right),\end{aligned} \tag{5.18}$$

substituting (5.17) and (5.18) into (5.16) gives

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{\alpha\beta} &= \bar{\Delta}\overline{\mathrm{Ric}}_{\alpha\beta} + \frac{2}{m_2}\bar{g}^{kl}\bar{g}^{pq}\overline{\mathrm{Ric}}_{lq}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)\bar{g}_{\alpha\beta}\\ &\quad - 2\bar{g}^{\gamma\delta}\overline{\mathrm{Ric}}_{\gamma\alpha}\overline{\mathrm{Ric}}_{\delta\beta} + 2\lambda^2\bar{g}^{\gamma\delta}\bar{g}^{\sigma\tau}\overline{\mathrm{Ric}}_{\delta\tau}\\ &\qquad \big({}^{M_2}\mathrm{Rm}_{\alpha\gamma\beta\sigma} + |\mathrm{grad}\lambda|^2\big(((g_2)_{\alpha\sigma}(g_2)_{\beta\gamma} - (g_2)_{\alpha\beta}(g_2)_{\gamma\sigma}\big)\big),\end{aligned} \tag{5.19}$$

which is (5.8). ■

To further simplify the evolution (5.8) and consider perhaps significant implication for physics, like previous subsection we assume that g_2 is a fixed Einstein metric with constant c. We first give a lemma.

Lemma 5.5. *Let*

$$\bar{g}(x,y,t) = g_1(x,t) \oplus \lambda^2(x,t) g_2(y,t)$$

be a smooth warped product metric on the manifold $M_1 \times M_2$, where g_2 is an Einstein metric with ${}^{M_2}\mathrm{Ric} = cg_2$. Then

$$\overline{\mathrm{Ric}}_{\alpha\beta} = f(x,t)\bar{g}_{\alpha\beta}, \tag{5.20}$$

where

$$\begin{aligned} f &= \frac{1}{m_2}\bar{g}^{\alpha\beta}\overline{\mathrm{Ric}}_{\alpha\beta} \\ &= \frac{1}{2\lambda^2}\left((4-2m_2)|\mathrm{grad}\lambda|^2 - \Delta_{M_1}\lambda^2 + 2c\right). \end{aligned} \tag{5.21}$$

Proof. By (2.6) and ${}^{M_2}\mathrm{Ric} = cg_2$, we get

$$\begin{aligned} \overline{\mathrm{Ric}}_{\alpha\beta} &= c(g_2)_{\alpha\beta} - \left(\lambda\Delta_{M_1}\lambda + (m_2-1)|\mathrm{grad}\lambda|^2\right)(g_2)_{\alpha\beta} \\ &= \frac{1}{\lambda^2}\left(c - \lambda\Delta_{M_1}\lambda - (m_2-1)|\mathrm{grad}\lambda|^2\right)\bar{g}_{\alpha\beta}. \end{aligned}$$

Note that

$$\lambda\Delta_{M_1}\lambda = \frac{1}{2}\Delta_{M_1}\lambda^2 - |\mathrm{grad}\lambda|^2.$$

Putting these with (5.20), we obtain

$$f = \frac{1}{2\lambda^2}\left(-\Delta_{M_1}\lambda^2 - 2(m_2-2)|\mathrm{grad}\lambda|^2 + 2c\right),$$

which is the second "=" in (5.21).

As to the first "=" in (5.21), note that $\sum\limits_{\alpha,\beta=1}^{m_2} \bar{g}_{\alpha\beta}\bar{g}^{\alpha\beta} = m_2$, it quickly follows from (5.20). ■

Applying this Lemma, we can simplify (5.8) to a better expression.

Theorem 5.6. *Assume that g_2 is a fixed Einstein metric with ${}^{M_2}\mathrm{Ric} = cg_2$. Then under the Ricci flow (3.2), the Ricci curvature evolution equation (5.8) has another form :*

$$\begin{aligned} \frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{\alpha\beta} &= \bar{\Delta}\overline{\mathrm{Ric}}_{\alpha\beta} - \frac{2}{m_2}\bar{g}^{kp}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)\overline{\mathrm{Ric}}_{\alpha\beta} \\ &\quad + \frac{2}{m_2}\bar{g}^{kl}\bar{g}^{pq}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)\overline{\mathrm{Ric}}_{lq}\bar{g}_{\alpha\beta}. \end{aligned} \tag{5.22}$$

Proof. By (5.20) and (5.18), the last two terms on the left-hand side of (5.8) can be transformed to

$$
\begin{aligned}
(I) := & -2\bar{g}^{\gamma\delta}\overline{\mathrm{Ric}}_{\gamma\alpha}\overline{\mathrm{Ric}}_{\delta\beta} + 2\lambda^2\bar{g}^{\gamma\delta}\bar{g}^{\sigma\tau}\overline{\mathrm{Ric}}_{\delta\tau} \\
& \left({}^{M_2}\mathrm{Rm}_{\alpha\gamma\beta\sigma} + |\mathrm{grad}\lambda|^2\big(((g_2)_{\alpha\sigma}(g_2)_{\beta\gamma} - (g_2)_{\alpha\beta}(g_2)_{\gamma\sigma})\big)\right) \\
= & -2\bar{g}^{\gamma\delta} f\bar{g}_{\gamma\alpha}\overline{\mathrm{Ric}}_{\delta\beta} + 2\bar{g}^{\gamma\delta}\bar{g}^{\sigma\tau} f\bar{g}_{\delta\tau}\overline{\mathrm{Rm}}_{(0\gamma)(0\alpha)(0\sigma)(0\beta)} \\
= & \, 2f\left(-\overline{\mathrm{Ric}}_{\alpha\beta} + \bar{g}^{\gamma\sigma}\overline{\mathrm{Rm}}_{(0\gamma)(0\alpha)(0\sigma)(0\beta)}\right).
\end{aligned}
\tag{5.23}
$$

By the definition of the Ricci curvature and the only non-zero terms $\overline{\mathrm{Rm}}_{(k0)(0\alpha)(p0)(0\beta)}$ and $\overline{\mathrm{Rm}}_{(0\gamma)(0\alpha)(0\sigma)(0\beta)}$, we have

$$
\begin{aligned}
\overline{\mathrm{Ric}}_{\alpha\beta} : &= \bar{g}^{\bar{k}\bar{p}\bar{\gamma}\bar{\sigma}}\overline{\mathrm{Rm}}_{(\bar{k}\bar{\gamma})(0\alpha)(\bar{p}\bar{\sigma})(0\beta)} \\
&= \bar{g}^{kp}\overline{\mathrm{Rm}}_{(k0)(0\alpha)(p0)(0\beta)} + \bar{g}^{\gamma\sigma}\overline{\mathrm{Rm}}_{(0\gamma)(0\alpha)(0\sigma)(0\beta)}.
\end{aligned}
\tag{5.24}
$$

Combining (5.24) and (5.17), we get

$$
\bar{g}^{\gamma\sigma}\overline{\mathrm{Rm}}_{(0\gamma)(0\alpha)(0\sigma)(0\beta)} = \overline{\mathrm{Ric}}_{\alpha\beta} - \frac{1}{m_2}\bar{g}^{kp}\bar{g}_{\alpha\beta}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big).
\tag{5.25}
$$

Substituting (5.25) into (5.23) and using the relation (5.20), we obtain

$$
\begin{aligned}
(I) &= -\frac{2}{m_2}(f\bar{g}_{\alpha\beta})\bar{g}^{kp}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big) \\
&= -\frac{2}{m_2}\bar{g}^{kp}\overline{\mathrm{Ric}}_{\alpha\beta}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big).
\end{aligned}
\tag{5.26}
$$

Finally, substituting (5.26) into (5.8) gives

$$
\begin{aligned}
\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{\alpha\beta} = \bar{\Delta}\overline{\mathrm{Ric}}_{\alpha\beta} &+ \frac{2}{m_2}\bar{g}^{kl}\bar{g}^{pq}\overline{\mathrm{Ric}}_{lq}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)\bar{g}_{\alpha\beta} \\
&- \frac{2}{m_2}\bar{g}^{kp}\overline{\mathrm{Ric}}_{\alpha\beta}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big),
\end{aligned}
$$

which is (5.22). ■

Now we return to find out the interesting evolution equation of $f(x,t)$.

Theorem 5.7. *Assume that g_2 is a fixed Einstein metric with ${}^{M_2}\mathrm{Ric} = cg_2$. Then under the Ricci flow (3.2), $f(x,t)$ satisfies the evolution equation*

$$
\begin{aligned}
\frac{\partial}{\partial t}f = \bar{\Delta}\, f + 2f^2 &- \frac{2}{m_2}\bar{g}^{kp}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)f \\
&+ \frac{2}{m_2}\bar{g}^{kl}\bar{g}^{pq}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)\overline{\mathrm{Ric}}_{lq}.
\end{aligned}
\tag{5.27}
$$

Proof. Since $\bar{g}^{i\alpha} = 0$ and $\overline{\mathrm{Ric}}_{\alpha i} = 0$, by (5.21) we have

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial}{\partial t}\Big(\frac{1}{m_2}\bar{g}^{\alpha\beta}\overline{\mathrm{Ric}}_{\alpha\beta}\Big) \\ &= \frac{1}{m_2}\left(\frac{\partial}{\partial t}\bar{g}^{\alpha\beta}\right)\overline{\mathrm{Ric}}_{\alpha\beta} + \frac{1}{m_2}\bar{g}^{\alpha\beta}\left(\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{\alpha\beta}\right). \end{aligned} \tag{5.28}$$

Note that

$$\begin{aligned} 0 = \frac{\partial}{\partial t}(\delta_{\alpha\gamma}) &= \frac{\partial}{\partial t}\left(\bar{g}^{\alpha\beta}\bar{g}_{\beta\gamma}\right) \\ &= \frac{\partial}{\partial t}\left(\bar{g}^{\alpha\beta}\right)\bar{g}_{\beta\gamma} + \bar{g}^{\alpha\beta}\frac{\partial}{\partial t}\left(\bar{g}_{\beta\gamma}\right). \end{aligned}$$

Combining this and (3.2), yields

$$\frac{\partial}{\partial t}\bar{g}^{\alpha\beta} = 2\bar{g}^{\alpha\tau}\bar{g}^{\beta\sigma}\overline{\mathrm{Ric}}_{\tau\sigma}.$$

Thus substituting this and (5.22) into (5.28) gives

$$\begin{aligned} \frac{\partial}{\partial t}f &= \frac{2}{m_2}\bar{g}^{\alpha\tau}\bar{g}^{\beta\sigma}\overline{\mathrm{Ric}}_{\tau\sigma}\overline{\mathrm{Ric}}_{\alpha\beta} \\ &\quad + \frac{1}{m_2}\bar{g}^{\alpha\beta}\bigg(\bar{\Delta}\overline{\mathrm{Ric}}_{\alpha\beta} - \frac{2}{m_2}\bar{g}^{kp}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)\overline{\mathrm{Ric}}_{\alpha\beta} \\ &\quad + \frac{2}{m_2}\bar{g}^{kl}\bar{g}^{pq}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)\overline{\mathrm{Ric}}_{lq}\bar{g}_{\alpha\beta}\bigg) \\ &= \frac{2}{m_2}\bar{g}^{\alpha\tau}\bar{g}^{\beta\sigma}(f\bar{g}_{\tau\sigma})(f\bar{g}_{\alpha\beta}) \\ &\quad + \frac{1}{m_2}\bar{\Delta}\left(\bar{g}^{\alpha\beta}(f\bar{g}_{\alpha\beta})\right) - \frac{2}{m_2^2}\bar{g}^{kp}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)\bar{g}^{\alpha\beta}(f\bar{g}_{\alpha\beta}) \\ &\quad + \frac{2}{m_2^2}\bar{g}^{kl}\bar{g}^{pq}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)\overline{\mathrm{Ric}}_{lq}(\bar{g}^{\alpha\beta}\bar{g}_{\alpha\beta}). \\ &= \frac{2}{m_2}f^2 m_2 + \frac{1}{m_2}\bar{\Delta}(m_2 f) - \frac{2}{m_2^2}\bar{g}^{kp}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)(fm_2) \\ &\quad + \frac{2}{m_2^2}\bar{g}^{kl}\bar{g}^{pq}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)\overline{\mathrm{Ric}}_{lq}\cdot m_2, \end{aligned} \tag{5.29}$$

that is

$$\begin{aligned} \frac{\partial}{\partial t}f = 2f^2 + \bar{\Delta}\ f - \frac{2}{m_2}\bar{g}^{kp}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)f \\ + \frac{2}{m_2}\bar{g}^{kl}\bar{g}^{pq}\big(\overline{\mathrm{Ric}}_{kp} - {}^{M_1}\mathrm{Ric}_{kp}\big)\overline{\mathrm{Ric}}_{lq}. \end{aligned}$$

This is the desired equality (5.27). ∎

Remark 5.8. In above theorems, if we take $M_1 = \mathbb{R}$, then ${}^{M_1}\mathrm{Ric} = 0$. This time, since $m_1 = 1$, by changing the indices i, j, k, l, p, q into a same notation "x", then (5.7), (5.22) and (5.27) are respectively rewritten as

$$
\begin{aligned}
&\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{xx} = \bar{\Delta}\overline{\mathrm{Ric}}_{xx} + \frac{2}{m_2}\bar{g}^{\alpha\beta}\overline{\mathrm{Ric}}_{\alpha\beta}\overline{\mathrm{Ric}}_{xx} - 2\bar{g}^{xx}(\overline{\mathrm{Ric}}_{xx})^2,\\
&\frac{\partial}{\partial t}\overline{\mathrm{Ric}}_{\alpha\beta} = \bar{\Delta}\overline{\mathrm{Ric}}_{\alpha\beta} - \frac{2}{m_2}\bar{g}^{xx}\overline{\mathrm{Ric}}_{xx}\overline{\mathrm{Ric}}_{\alpha\beta} + \frac{2}{m_2}\bar{g}^{xx}\bar{g}^{xx}(\overline{\mathrm{Ric}}_{xx})^2\bar{g}_{\alpha\beta},\\
&\frac{\partial}{\partial t}f = 2f^2 + \bar{\Delta}\ f - \frac{2}{m_2}\bar{g}^{xx}\overline{\mathrm{Ric}}_{xx}f + \frac{2}{m_2}\bar{g}^{xx}\bar{g}^{xx}(\overline{\mathrm{Ric}}_{xx})^2,
\end{aligned}
$$

which are exactly (4.3)-(4.4) in Proposition 4.1 in [23].

Remark 5.9. Under HGF, the evolution equations for Ricci curvature on a single manifold are much more complicated when compared with the case under RF, because they involve some complex terms such as $B\big(X, B(X,Y)\big) := \frac{\partial}{\partial t}\big(\nabla_X(\frac{\partial}{\partial t}\nabla_Y Z)\big)$ and the unknown term $\frac{\partial g(t)}{\partial t}$(see Theorem 1.4 or Theorem 1.1 in [18], or Theorem 5.2 in [11]), let alone on the warped product manifold. Therefore, it is very hard to gain some novel evolution equations for Ricci curvature on warped product manifold $\bar{M}$ when we still want to follow the introduced approach under the RF. Taking into account the just mentioned reason and our present technique, in this paper we put this issue aside for a moment.

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