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ASYMPTOTIC BEHAVIOR FOR A VISCOELASTIC WAVE EQUATION WITH A DELAY TERM

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Abstract. The following viscoelastic wave equation with a delay term in internal feedback:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0,$$

is considered in a bounded domain. Under appropriate conditions on μ_1, μ_2 and on the kernel g, we prove the local existence result by Faedo-Galerkin method and establish the decay result by suitable Lyapunov functionals.

1. Introduction

In this paper, we consider the initial boundary value problem for a nonlinear viscoelastic equation with a linear damping and a delay term of the form:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t(x,t)$$

$$+\mu_2 u_t(x,t-\tau) = 0, \text{ in } \Omega \times (0, \infty), \tag{1.1}$$

$$u_t(x,t-\tau) = f_0(x,t-\tau), \; x \in \Omega, \; t \in (0,\tau), \tag{1.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \; x \in \Omega, \tag{1.3}$$

$$u(x,t) = 0, x \in \partial\Omega, t \geq 0, \tag{1.4}$$

where $\rho > 0$, $\Omega \subset R^N$ $(N \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$ and Δ denotes the Laplacian operator with respect to the variable x. Moreover, μ_1 and μ_2 are positive constants, $\tau > 0$ represents the time delay, g is the kernel of the memory term and the initial data $(u_{0,}u_1, f_0)$ are given functions belonging to suitable spaces.

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It is well known that delay effects, which arise in many practical problems, might induce some instabilities, see $[1, 5-7, 19, 26]$. Hence, questions related to the behavior of solutions for the PDEs with time delay effects have become active area of research in recent years. Many authors have focused on this problem and several results concerning existence, decay and instability have been obtained, see [5-7,10,19-24,26] and reference therein. In this regard, Datko et al. [7] showed that a small delay in a boundary control is a source of instability. Nicaise et al. [19] studied a system of wave equation with a linear boundary damping term with a delay as follows

$$
\begin{aligned}
u_{tt}-\Delta u &= 0, \text{ in } \Omega\times(0,\ \infty),\\
u(x,t) &= 0, x\in\Gamma_0, t\geq 0\\
\frac{\partial}{\partial\nu}(x,t) &= \mu_1 u_t(x,t)+\mu_2 u_t(x,t-\tau)=0, \text{ in } \Gamma_1\times(0,\ \infty),\\
u(x,0) &= u_0(x), u_t(x,0)=u_1(x),\ x\in\Omega,\\
u_t(x,t-\tau) &= f_0(x,t-\tau),\ x\in\Omega,\ t\in(0,\tau).
\end{aligned}
\tag{1.5}
$$

where ν is the unit outward normal to $\partial\Omega$. Under the condition

$$\mu_2<\mu_1, \tag{1.6}$$

they established a stabilization result by applying inequalities obtained from Carleman estimates for the wave equation by Lasiecka et al. [11] and by using compactness-uniqueness arguments. Conversely, if (1.6) does not hold, they showed that there exists a sequence of delays for which the corresponding solution of (1.5) is unstable. And, they also obtained the same results if both the damping and the delay act in the domain.

The case of time-varying delay in the wave equation has been studied by Nicaise et al. [22] in one space dimension, in which they obtained an exponential decay result subject to the condition

$$\mu_2\leq\sqrt{1-d}\mu_1, \tag{1.7}$$

where d is a constant such that

$$\tau'(t)\leq d<1,\ \forall t>0. \tag{1.8}$$

Later, under the same conditions (1.7)-(1.8), Nicaise et al. [23] extended this result to general space dimension. In fact, they proved exponential stability of the solution for the wave equation with a time-varying delay in the boundary condition in a bounded and smooth domain in R^N. Recently, inspired the works of Nicaise et al., M. Kirane and B. Said-Houari [10] considered the following problem

$$u_{tt}-\Delta u-\Delta u_{tt}+\int_0^t g(t-s)\Delta u(s)ds+\mu_1 u_t(x,t)+\mu_2 u_t(x,t-\tau)=0, \tag{1.9}$$

in a bounded domain with the conditions (1.2)-(1.4). In that work, they established general decay results of the energy via suitable Lyapunov functionals under the condition

$\mu_2 \leq \mu_1$. It is worth pointing out that, without imposing the viscoelastic term (i.e.$g = 0$) in (1.9), Nicasise and Pignotti [19] had proved some instabilities may occur for $\mu_2 = \mu_1$. However, due to the presence of the viscoelastic term, M. Kirane and B. Said-Houari [10] showed that the solution is still exponentially stable even for $\mu_2 = \mu_1$.

In the absence of the delay term (i.e. $\mu_2 = 0$), problems similar to (1.1) have been extensively studied and there are numerous results related to existence, asymptotic behavior and blow-up of solutions. For example, Cavalcanti et al.[3] considered the following problem:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds - \gamma \Delta u_t = 0, \tag{1.10}$$

with the same initial and boundary conditions (1.3)-(1.4), where a global existence result for $\gamma \geq 0$ and an exponential decay result for $\gamma > 0$ were established under the assumptions $0 < \rho \leq \frac{2}{N-2}$ if $N \geq 3$ or $\rho > 0$ if $N = 1,\ 2$ and $g(t)$ decays exponentially. Lately, these decay results were extended by Messaoudi and Tatar [14] to a situation where a source term is present. Recently, Messaoudi and Tatar [15] studied problem (1.10) for case of $\gamma = 0$, they showed that the solution goes to zero with an exponential or polynomial rate under some restrictions on the relaxation function. For other related works, we refer the readers to [8-9, 13, 17-18, 25] and references therein.

As $\rho = 0$ and there is no dispersion term, Cavalcanti et al. [4] considered the single viscoelastic equation as the form:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^\gamma u = 0, \text{ in } \Omega \times (0,\ \infty),$$

with the same initial and boundary conditions (1.3)-(1.4), where $a : \Omega \to R^+$ is a function whish may vanish outside a subset $\omega \subset \Omega$ of positive measure and $g(t)$ decays exponentially, they proved an exponential decay result for the energy function. This result was later extended by Berrimi and Messaoudi [2] to the nonlinear damping case by introducing a new a functional, they weakened the conditions in $a(x)$ and $g(t)$ and obtained the decay result.

Motivated by previous works, in this paper, it is interesting to investigate whether there are similar decay results as in [10] for problem (1.1)-(1.4), in which more general form than that of problem (1.9) is considered. Our proof technique closely follows the arguments of [10], with the modifications being needed for our problem. Indeed, under the hypothesis on μ_1 and μ_2, our first intention is to study the well-posedness of problem (1.1)-(1.4) by making use of Faedo-Galerkin procedure. Then, based on some estimates of the viscoelastic wave equation and some ideas developed in [10,19], our next intention is to establish the decay result for $\mu_2 \leq \mu_1$. In this way, we can extend the results of [10] where the authors considered (1.1) with $\rho = 0$. The content of this paper is organized as follows. In Section 2, we provide assumptions that will be used later, state and prove the existence result Theorem 2.3. In Section 3, we prove our stability result that is given in Theorem 3.5. Finally, we give some examples to illustrate our result.

2. Preliminaries Results

In this section, we shall give some lemmas and assumptions which will be used throughout this work. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual products and norms.

Lemma 2.1. (Sobolev-Poincaré inequality). *Let* $2 \le p \le \frac{2N}{N-2}$, *the inequality*

$$\|u\|_p \le c_s \|\nabla u\|_2 \quad \text{for } u \in H_0^1(\Omega),$$

holds with some positive constant c_s.

Assume that ρ satisfies

$$0 < \rho \le \frac{2}{N-2} \text{ if } N \ge 3 \text{ or } \rho > 0 \text{ if } N = 1, 2. \tag{2.1}$$

Regarding the relaxation function $g(t)$, we assume that it verifies:
(A1) $g : R^+ \to R^+$ is a bounded C^1 function satisfying

$$1 - \int_0^\infty g(s)ds = l > 0, \tag{2.2}$$

and there exists a positive nonincreasing function ξ such that, for $t \ge 0$,

$$g'(t) \le -\xi(t)g(t) \text{ and } \int_0^\infty \xi(s)ds = \infty. \tag{2.3}$$

We also need the following technical Lemma in the course of the investigation.

Lemma 2.2. [10]. *For any* $g \in C^1(R)$ *and* $\phi \in H^1(0,T)$, *we have*

$$\begin{aligned} -2\int_0^t \int_\Omega g(t-s)\phi\phi_t dxds = {} & \frac{d}{dt}\left((g \circ \phi)(t) - \int_0^t g(s)ds \|\phi\|_2^2 \right) \\ & + g(t)\|\phi\|_2^2 - \left(g' \circ \phi\right)(t), \end{aligned}$$

where

$$(g \circ \phi)(t) = \int_0^t g(t-s) \int_\Omega |\phi(s) - \phi(t)|^2 dxds.$$

In order to prove the existence of solutions of problem (1.1)-(1.4), we introduced the new variable z as in [19],

$$z(x,\kappa,t) = u_t(x, t-\tau\kappa), \; x \in \Omega, \; \kappa \in (0,1),$$

which implies that

$$\tau z_t(x,\kappa,t) + z_\kappa(x,\kappa,t) = 0 \text{ in } \Omega \times (0,1) \times (0,\infty).$$

Therefore, problem (1.1)-(1.4) can be transformed as follows

$$(2.4)\quad \begin{aligned} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t(x,t) \\ +\mu_2 z(x,1,t) &= 0, \text{ in } \Omega \times (0,\ \infty), \\ \tau z_t(x,\kappa,t) + z_\kappa(x,\kappa,t) &= 0,\ x \in \Omega,\ \kappa \in (0,1),\ t > 0, \\ z(x,0,t) &= u_t(x,t),\ x \in \Omega,\ t > 0, \\ z(x,\kappa,0) &= f_0(x,-\tau\kappa),\ x \in \Omega, \\ u(x,0) &= u_0(x), u_t(x,0) = u_1(x),\ x \in \Omega, \\ u(x,t) &= 0, x \in \partial\Omega, t \geq 0, \end{aligned}$$

In the following, we will give sufficient conditions that guarantee the well-posedness of problem (2.4) by using the Fadeo-Galerkin procedure.

Theorem 2.3. *Suppose that $\mu_2 < \mu_1$, (A1) and (2.1) hold. Assume that $u_0, u_1 \in H_0^1(\Omega)$ and $f_0 \in L^2(\Omega \times (0,1))$. Then there exists a unique solution (u,z) of (2.4) satisfying*

$$u,\ u_t \in C\left([0,\ T);\ H_0^1(\Omega)\right),$$
$$z \in C\left([0,\ T);\ L^2(\Omega \times (0,1))\right),$$

for $T > 0$.

Proof. Let $(w_n)_{n\in N}$ be a basis in $H_0^1(\Omega)$ and W_n be the space generated by $w_1,\cdots, w_n$, $n = 1,2,3,\cdots$. Now, we define for $1 \leq i \leq n$, the sequence $\varphi_i(x,\kappa)$ as follows $\varphi_i(x,0) = w_i(x)$. Then, we may extend $\varphi_i(x,0)$ by $\varphi_i(x,\kappa)$ over $L^2(\Omega \times [0,1])$ and denote V_n to be the space generated by $\varphi_1,\cdots,\varphi_n$, $n = 1,2,3,\cdots$. Let us consider

$$u_n(t) = \sum_{i=1}^n c_{in}(t) w_i(x)$$

and

$$z_n(t) = \sum_{i=1}^n r_{in}(t)\varphi_i(x,\kappa),$$

where $(u_n(t),\ z_n(t))$ are the solutions of the following approximate problem corresponding to (2.4)

$$\int_\Omega |u_n'|^\rho u_n''(t) w_i dx + \int_\Omega \nabla u_n(t)\cdot\nabla w_i dx$$
$$-\int_0^t g(t-\tau)\int_\Omega \nabla u_n(\tau)\cdot\nabla w_i dx d\tau + \int_\Omega \nabla u_n''(t)\cdot\nabla w_i dx$$

$$+\int_{\Omega}\left(\mu_1 u_n'(t,x)+\mu_2 z_n(x,1,t)\right)w_i dx=0, \tag{2.5}$$

$$u_n(0)=u_{0n}\to u_0 \text{ in } H_0^1(\Omega),\ u_n'(0)=u_{1n}\to u_1 \text{ in } H_0^1(\Omega), \tag{2.6}$$

and

$$\int_{\Omega}\left(\tau z_n'(x,\kappa,t)+z_{n\kappa}(x,\kappa,t)\right)\varphi_i dx=0, \tag{2.7}$$

$$z_n(0)=z_{0n}\to f_0 \text{ in } L^2(\Omega\times(0,1)), \tag{2.8}$$

where $i=1,2,\cdots,n$. In view of the assumption (2.1), from Hölder inequality, the nonlinear term $\int_\Omega |u_n'|^\rho u_n''(t)w_i dx$ makes sense in (2.5). Then, by standard methods in ordinary differential equations, we infer the existence of solutions to $(2.5)-(2.8)$ on some interval $[0,t_n), 0<t_n<T$ for some arbitrary $T>0$. And the solution can be extended to the whole interval $[0,T)$ by the the first estimate below.

The first estimate: Multiplying (2.5) by $c_{in}'(t)$ and summing with respect to i, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{\rho+2}\left\|u_n'(t)\right\|_{\rho+2}^{\rho+2}+\frac{1}{2}\left\|\nabla u_n(t)\right\|_2^2+\frac{1}{2}\left\|\nabla u_n'(t)\right\|_2^2\right)\\&+\mu_1\left\|u_n'(t)\right\|_2^2+\int_\Omega \mu_2 z_n(x,1,t)u_n'(t)dx\\&-\int_0^t g(t-s)\int_\Omega \nabla u_n(s)\cdot\nabla u_n'(t)dxds=0.\end{aligned} \tag{2.9}$$

Using Lemma 2.2 on the last term of the left hand side of (2.9), we find

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{\rho+2}\left\|u_n'(t)\right\|_{\rho+2}^{\rho+2}+\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\left\|\nabla u_n(t)\right\|_2^2\right.\\&\left.+\frac{1}{2}\left\|\nabla u_n'(t)\right\|_2^2+\frac{1}{2}(g\circ\nabla u_n)(t)\right)+\mu_1\left\|u_n'(t)\right\|_2^2+\int_\Omega \mu_2 z_n(x,1,t)u_n'(t)dx\\&+\frac{1}{2}g(t)\left\|\nabla u_n(t)\right\|_2^2-\frac{1}{2}(g'\circ\nabla u_n)(t)=0.\end{aligned} \tag{2.10}$$

Integrating (2.10) over $(0,t)$, we arrive at

$$\begin{aligned}&\frac{1}{\rho+2}\left\|u_n'(t)\right\|_{\rho+2}^{\rho+2}+\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\left\|\nabla u_n(t)\right\|_2^2+\frac{1}{2}\left\|\nabla u_n'(t)\right\|_2^2\\&+\frac{1}{2}(g\circ\nabla u_n)(t)+\mu_1\int_0^t\left\|u_n'(s)\right\|_2^2ds+\mu_2\int_0^t\int_\Omega z_n(x,1,s)u_n'(s)dxds\\&+\frac{1}{2}\int_0^t g(s)\left\|\nabla u_n(s)\right\|_2^2ds-\int_0^t\frac{1}{2}(g'\circ\nabla u_n)(s)ds\\=&\frac{1}{\rho+2}\left\|u_{1n}\right\|_{\rho+2}^{\rho+2}+\frac{1}{2}\left\|\nabla u_{0n}\right\|_2^2+\frac{1}{2}\left\|\nabla u_{1n}\right\|_2^2.\end{aligned} \tag{2.11}$$

Letting $\zeta > 0$ be chosen later and multiplying (2.7) by $\frac{\zeta}{\tau} r_{in}(t)$, summing with respect to i and integrating over $(0,t)\times(0,1)$, we obtain

$$\begin{aligned} &\frac{\zeta}{2}\int_\Omega\int_0^1 z_n^2(x,\kappa,t)d\kappa dx + \frac{\zeta}{\tau}\int_0^t\int_\Omega\int_0^1 z_{n\kappa}(x,\kappa,s)z_n(x,\kappa,s)d\kappa dxds \\ =&\ \frac{\zeta}{2}\|z_{0n}\|^2_{L^2(\Omega\times(0,1))}. \end{aligned} \tag{2.12}$$

Additionally, we note that

$$\begin{aligned} &\int_0^t\int_\Omega\int_0^1 z_{n\kappa}(x,\kappa,s)z_n(x,\kappa,s)d\kappa dxds \\ =&\ \frac{1}{2}\int_0^t\int_\Omega \left(z_n^2(x,1,s)-z_n^2(x,0,s)\right)dxds. \end{aligned} \tag{2.13}$$

Then, combining (2.12) and (2.11) together and taking (2.13) into account, we obtain

$$\begin{aligned} &E_n(t) + \left(\mu_1 - \frac{\zeta}{2\tau}\right)\int_0^t \|u_n'(s)\|_2^2\,ds + \frac{\zeta}{2\tau}\int_0^t\int_\Omega z_n^2(x,1,s)dxds \\ &+\mu_2\int_0^t\int_\Omega z_n(x,1,s)u_n'(s)dxds \\ &+\frac{1}{2}\int_0^t g(s)\|\nabla u_n(s)\|_2^2\,ds - \int_0^t \frac{1}{2}(g'\circ\nabla u_n)(s)ds \\ =&\ E_n(0), \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} E_n(t) =&\ \frac{1}{\rho+2}\|u_n'(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u_n(t)\|_2^2 \\ &+\frac{1}{2}\|\nabla u_n'(t)\|_2^2 + \frac{1}{2}(g\circ\nabla u_n)(t) + \frac{\zeta}{2}\int_\Omega\int_0^1 z_n^2(x,\kappa,t)d\kappa dx. \end{aligned}$$

Making use of the inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ on the fourth term of the left hand side of (2.14), we deduce that

$$\begin{aligned} &E_n(t) + \left(\mu_1 - \frac{\zeta}{2\tau} - \frac{\mu_2}{2}\right)\int_0^t \|u_n'(s)\|_2^2\,ds \\ &+\left(\frac{\zeta}{2\tau} - \frac{\mu_2}{2}\right)\int_0^t\int_\Omega z_n^2(x,1,s)dxds \\ &+\frac{1}{2}\int_0^t g(s)\|\nabla u_n(s)\|_2^2\,ds - \int_0^t \frac{1}{2}(g'\circ\nabla u_n)(s)ds \\ =&\ E_n(0). \end{aligned} \tag{2.15}$$

Now, we choose ζ such that

$$\tau\mu_2 < \zeta < \tau\left(2\mu_1 - \mu_2\right), \tag{2.16}$$

which implies that

$$c_1 = \mu_1 - \frac{\zeta}{2\tau} - \frac{\mu_2}{2} > 0 \text{ and } c_2 = \frac{\zeta}{2\tau} - \frac{\mu_2}{2} > 0,$$

due to $\mu_1 > \mu_2$. Hence, from (A1) and (2.15), we obtain

$$\begin{aligned} &\left\|u_n'\right\|_{\rho+2}^{\rho+2} + \|\nabla u_n\|_2^2 + \left\|\nabla u_n'\right\|_2^2 + \int_0^t \int_\Omega z_n^2(x,1,s)dxds \\ &+(g\circ\nabla u_n)(t) + \int_\Omega \int_0^1 z_n^2(x,\kappa,t)d\kappa dx \le L_1, \end{aligned} \tag{2.17}$$

where L_1 is a positive constant independent of $n \in N$ and $t \in [0,T)$.

The second estimate: Multiplying (2.5) by $c_{in}''(t)$ and summing with respect to i, it holds that

$$\begin{aligned} &\int_\Omega \left|u_n'\right|^\rho \left|u_n''(t)\right|^2 dx + \left\|\nabla u_n''\right\|_2^2 + \frac{\mu_1}{2}\frac{d}{dt}\left\|u_n'(t)\right\|_2^2 \\ &= -\int_\Omega \nabla u_n(t)\cdot\nabla u_n''(t)dx + \int_0^t g(t-\tau)\int_\Omega \nabla u_n(\tau)\cdot\nabla u_n''(t)dxd\tau \\ &-\mu_2\int_\Omega z_n(x,1,t)u_n''(t)dx. \end{aligned} \tag{2.18}$$

Exploiting Hölder inequality, Young's inequality, (A1) and Lemma 2.1, for $\eta > 0$, we have

$$\left|-\int_\Omega \nabla u_n(t)\cdot\nabla u_n''(t)dx\right| \le \eta\left\|\nabla u_n''(t)\right\|_2^2 + \frac{1}{4\eta}\|\nabla u_n(t)\|_2^2, \tag{2.19}$$

$$\begin{aligned} &\left|\int_0^t g(t-s)\int_\Omega \nabla u_n(s)\cdot\nabla u_n''(t)dxd\tau\right| \\ &\le \eta\left\|\nabla u_n''(t)\right\|_2^2 + \frac{(1-l)g(0)}{4\eta}\int_0^t \|\nabla u_n(s)\|_2^2 ds, \end{aligned} \tag{2.20}$$

and

$$\left|-\int_\Omega z_n(x,1,t)u_n''(t)dx\right| \le \frac{\eta}{\mu_2}\left\|\nabla u_n''(t)\right\|_2^2 + \frac{\mu_2 c_s^2}{4\eta}\int_\Omega z_n^2(x,1,t)dx. \tag{2.21}$$

Substituting these estimates (2.19)-(2.21) into (2.18), then integrating the obtained inequality over $(0,t)$ and using (2.17), we deduce that

$$\begin{aligned} &\int_0^t \int_\Omega \left|u_n'\right|^\rho \left|u_n''(t)\right|^2 dxds + (1-3\eta)\int_0^t \left\|\nabla u_n''\right\|_2^2 ds + \frac{\mu_1}{2}\left\|u_n'(t)\right\|_2^2 \\ &\le \frac{L_1}{4\eta}\left(\mu_2^2 c_s^2 + (1+(1-l)g(0)T)T\right) + c_3, \end{aligned} \tag{2.22}$$

where c_3 is a positive constant depending only on $\|u_1\|_2^2$. Choosing $\eta > 0$ small enough in (2.22), we obtain the second estimate

$$\left\|u_n'(t)\right\|_2^2 + \int_0^t \left\|\nabla u_n''(t)\right\|_2^2 dt \leq L_2, \tag{2.23}$$

where L_2 is a positive constant independent of $n \in N$ and $t \in [0, T)$.

We observe that estimates (2.17) and (2.23) imply that there exists a subsequence (u_i, z_i) of (u_n, z_n) and a function (u, z) such that

$$u_i \rightharpoonup u \text{ weak star in } L^\infty(0, T; H_0^1(\Omega)), \tag{2.24}$$

$$u_i' \rightharpoonup u' \text{ weak star in } L^\infty(0, T; H_0^1(\Omega)), \tag{2.25}$$

$$u_i'' \rightharpoonup u'' \text{ weakly in } L^2(0, T; H_0^1(\Omega)) \tag{2.26}$$

$$z_i \rightharpoonup z \text{ weak star in } L^\infty(0, T; L^2(\Omega \times (0, 1))). \tag{2.27}$$

Further, by Aubin's Lemma [12], it follows from (2.25) and (2.26) that there exists a subsequence (u_i), still represented by the same notation, such that

$$u_i' \to u' \text{ strongly in } L^2(0, T; L^2(\Omega)),$$

which implies $u_i' \to u'$ a.e. on $\Omega \times (0, T)$. Hence

$$\left|u_i'\right|^\rho u_i' \rightharpoonup \left|u'\right|^\rho u' \text{ a.e. on } \Omega \times (0, T). \tag{2.28}$$

On the other hand, from the first estimate and Lemma 2.1, we deduce that

$$\begin{aligned} \||u_i'|^\rho u_i'\|_{L^2(0,T;L^2(\Omega))} &= \int_0^T \int_\Omega \left|u_i'\right|^{2(\rho+1)} dxdt \\ &\leq c_s^{2(\rho+1)} \int_0^T \left\|\nabla u_i'\right\|_2^{2(\rho+1)} dt \\ &\leq c_s^{2(\rho+1)} T L_1^{\rho+1}. \end{aligned} \tag{2.29}$$

Combining (2.28) and (2.29) and owing to Lion's Lemma [12], we derive that

$$\left|u_i'\right|^\rho u_i' \rightharpoonup \left|u'\right|^\rho u' \text{ weakly in } L^2(0, T; L^2(\Omega)).$$

The proof now can be completed arguing as in [12, Theorem 3.1].

3. Asymptotic Behavior

In this section, we shall investigate the asymptotic behavior of problem (1.1)-(1.4) for $\mu_2 \leq \mu_1$. To achieve this, we will use the energy method combined with the choice of a suitable functional as in the work of M. Kirane and B. Said-Houari [10]. First, we define the energy function of problem (1.1)-(1.4) as

$$\begin{aligned} E(t) = & \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) \\ & + \frac{1}{2}\|\nabla u_t(t)\|_2^2 + \frac{\zeta}{2}\int_\Omega\int_0^1 u_t^2(x,t-\tau\kappa)d\kappa dx, \end{aligned} \tag{3.1}$$

where ζ is a positive constant such that

$$\tau\mu_2 \le \zeta \le \tau(2\mu_1-\mu_2). \tag{3.2}$$

Remark 3.1. (i) It is clear that this energy function $E(t)$ by (3.1) is larger than the usual one

$$\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) + \frac{1}{2}\|\nabla u_t(t)\|_2^2$$

and contains an additional term that comes from the delay term.
(ii) The local existence theorem 2.3 does not include the case $\mu_1=\mu_2$, however, we find our decay result also hold for $\mu_1=\mu_2$. For this reason, the existence of local solution for $\mu_1=\mu_2$ is hypothesized in this work.

Lemma 3.2. *$E(t)$ is a nonincreasing function on $[0,T]$ and*

$$\begin{aligned} E'(t) = & -c_1\|u_t\|_2^2 - c_2\int_\Omega u_t^2(x,t-\tau)dx + \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \\ \le & \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \le 0, \ \forall t\ge 0. \end{aligned}$$

Proof. As in deriving (2.15), we see that

$$\begin{aligned} \frac{d}{dt}E(t) \le & -c_1\|u_t\|_2^2 - c_2\int_\Omega u_t^2(x,t-\tau)dx + \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \\ \le & \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \le 0, \ \forall t\ge 0, \end{aligned}$$

where

$$c_1 = \begin{cases} \mu_1 - \dfrac{\zeta}{2\tau} - \dfrac{\mu_2}{2} > 0, \text{ if } \mu_2<\mu_1 \\ 0, \text{ if } \mu_2=\mu_1, \end{cases},$$

and

$$c_2 = \begin{cases} \dfrac{\zeta}{2\tau} - \dfrac{\mu_2}{2} > 0, \text{ if } \mu_2<\mu_1 \\ 0, \text{ if } \mu_2=\mu_1, \end{cases},$$

since ζ is chosen satisfying assumption (3.2).

Remark 3.3. It follows from the definition of $E(t)$ by (3.1) and Lemma 3.2 that the energy function is uniformly bounded and decreasing in t, which implies that

$$l\left\|\nabla u\right\|_2^2+\left\|\nabla u_t\right\|_2^2\le 2E(t)\le 2E(0),\ \forall t\ge 0. \tag{3.3}$$

This infers that the solution of problem (1.1) is bounded and global in time.

Now, we define

$$G(t)=ME(t)+\varepsilon_1\Phi(t)+\varepsilon_2 I(t)+\Psi(t), \tag{3.4}$$

where M, ε_1 and ε_2 are positive constants which will be specified later and

$$\Phi(t)=\frac{1}{\rho+1}\int_\Omega\left|u_t\right|^\rho u_t u dx+\int_\Omega\nabla u_t(t)\cdot\nabla u(t)dx, \tag{3.5}$$

$$I(t)=\int_\Omega\int_0^1 e^{-2\tau\kappa}u_t^2(x,t-\tau\kappa)d\kappa dx, \tag{3.6}$$

$$\Psi(t)=\int_\Omega\left(\Delta u_t-\frac{1}{\rho+1}\left|u_t\right|^\rho u_t\right)\int_0^t g(t-s)\left(u(t)-u(s)\right)dsdx. \tag{3.7}$$

The following lemma tells us that $G(t)$ and $E(t)$ are equivalent.

Lemma 3.4. *Let* u *be a solution of problem* (1.1)-(1.4), *then there exists two positive constants* β_1 and β_2 *such that*

$$\beta_1 E(t)\le G(t)\le\beta_2 E(t),\ \forall\ t\ge 0, \tag{3.8}$$

for M *sufficiently large*.

Proof. By Young's inequality, Lemma 2.1 and (3.3), we have

$$\begin{aligned}&\left|\frac{1}{\rho+1}\int_\Omega\left|u_t\right|^\rho u_t u dx\right|\\ &\le\frac{1}{\rho+2}\left\|u_t\right\|_{\rho+2}^{\rho+2}+\frac{c_s^{\rho+2}}{(\rho+2)(\rho+1)}\left(\frac{2E(0)}{l}\right)^{\frac{\rho}{2}}\left\|\nabla u\right\|_2^2\end{aligned} \tag{3.9}$$

and

$$\left|\int_\Omega\nabla u_t(t)\cdot\nabla u(t)dx\right|\le\frac{1}{2}\left\|\nabla u_t\right\|_2^2+\frac{1}{2}\left\|\nabla u\right\|_2^2. \tag{3.10}$$

It follows from (3.6) that

$$\left|I(t)\right|\le c_3\int_\Omega\int_0^1 u_t^2(x,t-\tau\kappa)d\kappa dx, \tag{3.11}$$

where c_3 is a positive constant. Further, from (3.7), we have

$$\Psi(t) = -\int_\Omega \nabla u_t \cdot \int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right) dsdx \\ -\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g(t-s)\left(u(t)-u(s)\right) dsdx. \tag{3.12}$$

By Young's inequality, Hölder inequality and (3.8), we see that

$$\begin{aligned} &\left| -\int_\Omega \nabla u_t \cdot \int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right) dsdx \right| \\ &\le \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right) ds\right)^2 dx \\ &\le \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1-l}{2}\left(g\circ\nabla u\right)(t), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\left| -\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g(t-s)\left(u(t)-u(s)\right) dsdx \right| \\ &\le \frac{1}{\rho+2}\left(\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{\rho+1}\int_\Omega \left(\int_0^t g(t-s)\left(u(t)-u(s)\right) ds\right)^{\rho+2} dx\right) \\ &\le \frac{1}{\rho+2}\left(\|u_t\|_{\rho+2}^{\rho+2} + \frac{(1-l)^{\rho+1} c_s^{\rho+2}}{\rho+1}\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|_2^{\rho+2} ds\right) \\ &\le \frac{1}{\rho+2}\left(\|u_t\|_{\rho+2}^{\rho+2} + \frac{(1-l)^{\rho+1} c_s^{\rho+2}}{\rho+1}\left(\frac{8E(0)}{l}\right)^{\frac{\rho}{2}}\left(g\circ\nabla u\right)(t)\right). \end{aligned} \tag{3.14}$$

Hence, combining $(3.9) - (3.14)$ with (3.4) yields

$$\begin{aligned} |G(t) - ME(t)| &= \varepsilon_1\Phi(t) + \varepsilon_2 I(t) + \Psi(t) \\ &\le c_4\|u_t\|_{\rho+2}^{\rho+2} + c_5\|\nabla u\|_2^2 + c_6\|\nabla u_t\|_2^2 \\ &\quad + c_7\left(g\circ\nabla u\right)(t) + c_3\varepsilon_2\int_\Omega\int_0^1 u_t^2(x,t-\tau\kappa)d\kappa dx \\ &\le c_8 E(t), \end{aligned}$$

where $c_4 = \frac{1+\varepsilon_1}{\rho+2}$, $c_5 = \varepsilon_1\left(\frac{c_s^{\rho+2}}{(\rho+2)(\rho+1)}\left(\frac{2E(0)}{l}\right)^{\frac{\rho}{2}} + \frac{1}{2}\right)$, $c_6 = \frac{\varepsilon_1+1}{2}$, $c_7 = \frac{1-l}{2} + \frac{(1-l)^{\rho+1}c_s^{\rho+2}}{(\rho+2)(\rho+1)}\left(\frac{8E(0)}{l}\right)^{\frac{\rho}{2}}$, and $c_8 = \max(c_3\varepsilon_2, c_4, c_5, c_6, c_7)$. Thus, from the definition of $E(t)$ by (3.1) and selecting M sufficiently large, there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \le G(t) \le \beta_2 E(t).$$

Theorem 3.5. *Let* $u_0, u_1 \in H_0^1(\Omega)$ *be given. Suppose that* $(A1), (2.1), (3.2)$ *and* $\mu_2 \le \mu_1$ *hold. Then for each* $t_0 > 0$ *the solution energy of problem* $(1.1)-(1.4)$ *satisfies*

$$E(t) \le K e^{-\alpha \int_{t_0}^{t} \xi(s) ds}, \ t \ge t_0, \tag{3.15}$$

where α *and* K *are some positive constants given in the proof.*

Proof. In order to obtain the decay result of $E(t)$, it is sufficient to prove that of $G(t)$. To this end, we need to estimate the derivative of $G(t)$. It follows from (3.5) that

$$\begin{aligned} &\Phi'(t) \\ &= -\|\nabla u\|_2^2 + \int_\Omega \nabla u(t) \cdot \int_0^t g(t-s)\nabla u(s) ds dx - \mu_1 \int_\Omega u_t(x,t)u(t)dx \\ &\quad -\mu_2 \int_\Omega u_t(x,t-\tau)u(t)dx + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 . \end{aligned} \tag{3.16}$$

We estimate the second term in the right hand side of (3.16) as follows, for $\eta > 0$,

$$\begin{aligned} &\left| \int_\Omega \nabla u(t) \cdot \int_0^t g(t-s)\nabla u(s) ds dx \right| \\ &\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)\left(|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|\right) ds \right)^2 dx \\ &\le \frac{1+(1+\eta)(1-l)^2}{2}\|\nabla u\|_2^2 + \frac{\left(1+\frac{1}{\eta}\right)(1-l)}{2}(g \circ \nabla u)(t). \end{aligned} \tag{3.17}$$

For the third term and the fourth term, Young's inequality and Lemma 2.1 imply that, for $\delta_1 > 0$,

$$\left| \int_\Omega u_t(x,t)u(t)dx \right| \le \delta_1 c_s^2 \|\nabla u\|_2^2 + \frac{c_s^2}{4\delta_1}\|\nabla u_t\|_2^2, \tag{3.18}$$

and

$$\left| \int_\Omega u_t(x,t-\tau)u(t)dx \right| \le \delta_1 c_s^2 \|\nabla u\|_2^2 + \frac{1}{4\delta_1}\int_\Omega u_t^2(x,t-\tau)dx. \tag{3.19}$$

Letting $\eta = \frac{l}{1-l}$ in (3.17) and using (3.18)-(3.19), we derive from (3.16) that

$$\begin{aligned} &\Phi'(t) \\ &\le -\left(\frac{l}{2} - \delta_1 c_s^2 (\mu_1 + \mu_2) \right)\|\nabla u\|_2^2 + \frac{1-l}{2l}(g \circ \nabla u)(t) + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} \\ &\quad + \frac{\mu_2}{4\delta_1}\int_\Omega u_t^2(x,t-\tau)dx + \left(\frac{\mu_1 c_s^2}{4\delta_1} + 1 \right)\|\nabla u_t\|_2^2 . \end{aligned} \tag{3.20}$$

As in [10], the derivative of $I(t)$ can be estimated as

$$\begin{aligned}\frac{d}{dt}I(t) &\leq -\kappa I(t)+\frac{1}{2\tau}\|u_t\|_2^2-\frac{c_9}{2\tau}\int_\Omega u_t^2(x,t-\tau)dx \\ &\leq -\kappa I(t)+\frac{c_s^2}{2\tau}\|\nabla u_t\|_2^2-\frac{c_9}{2\tau}\int_\Omega u_t^2(x,t-\tau)dx,\end{aligned} \tag{3.21}$$

where c_9 is a positive constant. Taking the derivative of $\Psi(t)$ in (3.7) and using Eq. (1.1), we get

$$\begin{aligned}&\Psi'(t)\\ =&\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)dsdx\\ &-\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right)\cdot\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right)dx\\ &+\mu_1\int_\Omega u_t(t)\int_0^t g(t-s)\left(u(t)-u(s)\right)dsdx\\ &+\mu_2\int_\Omega u_t(x,t-\tau)\int_0^t g(t-s)\left(u(t)-u(s)\right)dsdx\\ &-\int_\Omega \nabla u_t(t)\cdot\int_0^t g'(t-s)\left(\nabla u(t)-\nabla u(s)\right)dsdx\\ &-\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t g'(t-s)\left(u(t)-u(s)\right)dsdx\\ &-\left(\int_0^t g(s)ds\right)\|\nabla u_t\|_2^2-\frac{1}{\rho+1}\left(\int_0^t g(s)ds\right)\|u_t\|_{\rho+2}^{\rho+2}.\end{aligned} \tag{3.22}$$

In what follows we will estimate the right hand side of (3.22). Using Hölder inequality, Young's inequality and (2.2), for $\delta>0$, we have

$$\begin{aligned}&\left|\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)dsdx\right|\\ &\leq \delta\|\nabla u\|_2^2+\frac{1-l}{4\delta}\left(g\circ\nabla u\right)(t).\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}&\left|\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right)\cdot\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right)dx\right|\\ &\leq \delta\int_\Omega\left(\int_0^t g(t-s)\left|\nabla u(s)\right|ds\right)^2dx\\ &+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-s)\left|\nabla u(t)-\nabla u(s)\right|ds\right)^2dx.\end{aligned} \tag{3.24}$$

Similar to the estimate of (3.17), for $\eta_1 > 0$, we have

$$\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-s)\,|\nabla u(s)|\,ds \right)^2 dx \\
&\le \int_\Omega \left(\int_0^t g(t-s)\left(|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|\right) ds \right)^2 dx \\
&\le (1+\eta_1)(1-l)^2 \|\nabla u\|_2^2 + \left(1 + \frac{1}{\eta_1}\right)(1-l)\,(g \circ \nabla u)(t).
\end{aligned} \tag{3.25}$$

Taking $\eta_1 = 1$ in (3.25), we then get from (3.24) that

$$\begin{aligned}
&\left| \int_\Omega \left(\int_0^t g(t-s)\nabla u(s) ds \right) \cdot \left(\int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right) ds \right) dx \right| \\
&\le 2\delta (1-l)^2 \|\nabla u\|_2^2 + \left(2\delta + \frac{1}{4\delta}\right)(1-l)\,(g \circ \nabla u)(t).
\end{aligned} \tag{3.26}$$

By Young's inequality and Lemma 2.1, the third term and the fourth term on the right hand side of (3.22) can be estimated as

$$\begin{aligned}
&\left| \mu_1 \int_\Omega u_t(t) \int_0^t g(t-s)\left(u(t) - u(s)\right) ds dx \right| \\
&\le \delta_2 \mu_1 c_s^2 \|\nabla u_t\|_2^2 + \frac{\mu_1 c_s^2 (1-l)}{4\delta_2}(g \circ \nabla u)(t), \ \delta_2 > 0,
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
&\left| \mu_2 \int_\Omega u_t(x, t-\tau) \int_0^t g(t-s)\left(u(t) - u(s)\right) ds dx \right| \\
&\le \mu_2 \delta_3 \int_\Omega u_t^2(x, t-\tau) dx + \frac{\mu_2 (1-l) c_s^2}{4\delta_3}(g \circ \nabla u)(t),
\end{aligned} \tag{3.28}$$

for $\delta_3 > 0$. Using Young's inequality and (A1) to deal with the fifth term

$$\begin{aligned}
&\left| \int_\Omega \nabla u_t(t) \int_0^t g'(t-s)\left(\nabla u(t) - \nabla u(s)\right) ds dx \right| \\
&\le \delta_4 \|\nabla u_t\|_2^2 + \frac{1}{4\delta_4} \int_\Omega \left(\int_0^t g'(t-s)\left(\nabla u(t) - \nabla u(s)\right) ds \right)^2 dx \\
&\le \delta_4 \|\nabla u_t\|_2^2 - \frac{g(0)}{4\delta_4}\left(g' \circ \nabla u\right)(t), \ \delta_4 > 0.
\end{aligned} \tag{3.29}$$

Employing Young's inequality, (2.1), Lemma 2.1 and (3.3), we have, for $\delta_5 > 0$,

$$\begin{aligned}
&\left|\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g'(t-s)\left(u(t)-u(s)\right)dsdx\right| \\
&\le \frac{1}{\rho+1}\left(\delta_5 \|u_t\|_{2(\rho+1)}^{2(\rho+1)} + \frac{1}{4\delta_5}\int_\Omega \left(\int_0^t g'(t-s)\left(u(t)-u(s)\right)ds\right)^2 dx\right) \\
&\le \frac{1}{\rho+1}\left(\delta_5 \|u_t\|_{2(\rho+1)}^{2(\rho+1)} - \frac{g(0)c_s^2}{4\delta_5}\int_\Omega\int_0^t g'(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 dsdx\right) \\
&\le \frac{\delta_5 c_s^{2(\rho+1)}}{\rho+1}\left(2E(0)\right)^\rho \|\nabla u_t\|_2^2 - \frac{g(0)c_s^2}{4\delta_5(\rho+1)}\left(g'\circ\nabla u\right)(t).
\end{aligned} \tag{3.30}$$

A substitution of (3.23)-(3.30) into (3.22) yields

$$\begin{aligned}
\Psi'(t) \le\ & \delta c_{10}\|\nabla u\|_2^2 + c_{11}\left(g\circ\nabla u\right)(t) - c_{12}\left(g'\circ\nabla u\right)(t) \\
& + \left(c_{13} - \int_0^t g(s)ds\right)\|\nabla u_t\|_2^2 + \mu_2\delta_3\int_\Omega u_t^2(x,t-\tau)dx \\
& - \frac{1}{\rho+1}\left(\int_0^t g(s)ds\right)\|u_t\|_{\rho+2}^{\rho+2},
\end{aligned} \tag{3.31}$$

where $c_{10} = 1 + 2(1-l)^2$, $c_{11} = \left(2\delta + \frac{1}{2\delta} + \frac{\mu_1 c_s^2}{4\delta_2} + \frac{\mu_2 c_s^2}{4\delta_3}\right)(1-l)$, $c_{12} = \frac{g(0)c_s^2}{4\delta_5(\rho+1)} + \frac{g(0)}{4\delta_4}$, and $c_{13} = \delta_2\mu_1 c_s^2 + \delta_4 + \frac{\delta_5 c_s^{2(\rho+1)}}{\rho+1}\left(2E(0)\right)^\rho$. Since g is positive, continuous and $g(0) > 0$, then for any $t_0 > 0$, we have

$$\int_0^t g(s)ds \ge \int_0^{t_0} g(s)ds = g_0,\ \forall\ t \ge t_0. \tag{3.32}$$

Hence, we conclude from (3.4), Lemma 3.2, (3.20), (3.21), (3.31) and (3.32) that for any $t \ge t_0 > 0$,

$$\begin{aligned}
G'(t) =\ & ME'(t) + \varepsilon_1\Phi'(t) + \varepsilon_2 I'(t) + \Psi'(t) \\
\le\ & \left(\frac{M}{2} - c_{12}\right)\left(g'\circ\nabla u\right)(t) - \frac{g_0 - \varepsilon_1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} \\
& - \left(\varepsilon_1\left(\frac{l}{2} - \delta_1 c_s^2(\mu_1+\mu_2)\right) - \delta c_{10}\right)\|\nabla u\|_2^2 \\
& - \left(g_0 - \varepsilon_1\left(\frac{\mu_1 c_s^2}{4\delta_1} + 1\right) - \frac{\varepsilon_2 c_s^2}{2\tau} - c_{13}\right)\|\nabla u_t\|_2^2 \\
& - \varepsilon_2\kappa I(t) - \left(\frac{c_9\varepsilon_2}{2\tau} - \frac{\varepsilon_1\mu_2}{4\delta_1} - \mu_2\delta_3\right)\int_\Omega u_t^2(x,t-\tau)dx \\
& + \left(\frac{\varepsilon_1(1-l)}{2l} + c_{11}\right)\left(g\circ\nabla u\right)(t).
\end{aligned}$$

At this point, we choose δ_1 such that

$$\delta_1 c_s^2 (\mu_1 + \mu_2) \leq \frac{l}{4},$$

and let $\delta_2 = \delta_4 = \delta_5$ satisfying

$$c_{13} = \delta_2 \left(\mu_1 c_s^2 + 1 + \frac{c_s^{2(\rho+1)}}{\rho+1} (2E(0))^\rho \right) \leq \frac{g_0}{2},$$

After that, we select ε_2 so that

$$\frac{\varepsilon_2 c_s^2}{2\tau} \leq \frac{g_0}{8}.$$

Once ε_2 is fixed, we choose δ_3 to satisfy

$$\mu_2 \delta_3 \leq \frac{c_9 \varepsilon_2}{4\tau}.$$

Further, we take ε_1 such that

$$\varepsilon_1 < \min \left\{ \frac{\frac{g_0}{8}}{\frac{\mu_1 c_s^2}{4\delta_1} + 1}, \ \frac{\delta_1 c_9 \varepsilon_2}{2\mu_2 \tau}, \ g_0 \right\}.$$

Also let δ small so that

$$\delta < \frac{\frac{\varepsilon_1 l}{8}}{c_{10}} = \frac{\varepsilon_1 l}{8\left(1 + 2(1-l)^2\right)}.$$

Finally, we pick M sufficiently large such that

$$M > 4c_{12} = \frac{g(0)}{\delta_2} \left(\frac{c_s^2}{\rho+1} + 1 \right).$$

Consequently, there exist two positive constants λ_1 and λ_2 satisfying

$$G'(t) \leq -\lambda_1 E(t) + \lambda_2 (g \circ \nabla u)(t), \text{ for all } t \geq t_0. \tag{3.33}$$

Multiplying (3.33) by $\xi(t)$, we have

$$\xi(t) G'(t) \leq -\lambda_1 \xi(t) E(t) + \lambda_2 \xi(t) (g \circ \nabla u)(t).$$

Then, employing the assumption $g'(t) \leq -\xi(t) g(t)$ by (2.3) and using the fact that $-(g' \circ \nabla u)(t) \leq -2E'(t)$ by Lemma 3.2, we get

$$\begin{aligned} \xi(t) G'(t) &\leq -\lambda_1 \xi(t) E(t) - \lambda_2 \left(g' \circ \nabla u\right)(t) \\ &\leq -\lambda_1 \xi(t) E(t) - 2\lambda_2 E'(t), \text{ for all } t \geq t_0. \end{aligned} \tag{3.34}$$

Now, we define

$$F(t) = \xi(t) G(t) + 2\lambda_2 E(t),$$

which is equivalent to $E(t)$ by Lemma 3.4. Using (3.34) and the assumption $\xi'(t) \le 0$, $\forall t \ge 0$ by (A1), we obtain

$$F'(t) \le \xi'(t)G(t) - \lambda_1\xi(t)E(t) \\ \le -\lambda_1\xi(t)E(t) \le -\lambda_3\xi(t)F(t), \ \forall t \ge t_0. \tag{3.35}$$

An integration of (3.35) over (t_0, t) gives

$$F(t) \le F(0)e^{-\lambda_3 \int_{t_0}^t \xi(s)ds}, \ \forall t \ge t_0, \tag{3.36}$$

Therefore, the equivalent relation between $F(t)$ and $E(t)$ yields

$$E(t) \le Ke^{-\alpha \int_{t_0}^t \xi(s)ds}, \ \forall t \ge t_0, \tag{3.37}$$

where α and K are some positive constants. This completes the proof. ■

Remark 3.6. We illustrate the energy decay rate given by Theorem 3.5 through the following examples which are introduced in [10,16].

(1) If $g(t) = \frac{a}{(1+t)^\nu}$, for $a > 0$ and $\nu > 1$, then $\xi(t) = \frac{\nu}{1+t}$ satisfies the condition (2.3). Thus (3.15) gives the estimate

$$E(t) \le K(1+t)^{-\alpha}.$$

(2) If $g(t) = ae^{-b(1+t)^\nu}$, for $a, b > 0$ and $0 < \nu \le 1$, then $\xi(t) = b\nu(1+t)^{\nu-1}$ satisfies the condition (2.3). Thus (3.15) gives the estimate

$$E(t) \le Ke^{-\alpha(1+t)^\nu}.$$

(3) If $g(t) = ae^{-b\ln^\nu(1+t)}$, for $a, b > 0$ and $\nu > 1$, then $\xi(t) = \frac{b\nu\ln^{\nu-1}(1+t)}{1+t}$ satisfies the condition (2.3). Thus (3.15) gives the estimate

$$E(t) \le Ke^{-\alpha\ln^\nu(1+t)}.$$

(4) If $g(t) = \frac{a}{(1+t)\ln^\nu(1+t)}$, for $a > 0$ and $\nu > 1$, then $\xi(t) = \frac{\ln(1+t)+\nu}{(1+t)\ln^\nu(1+t)}$ satisfies the condition (2.3). Thus (3.15) gives the estimate

$$E(t) \le K\left((1+t)\ln^\nu(1+t)\right)^{-\alpha}.$$

References

1. C. Abdallah, P. Dorato, J. Benitez-Read and R. Byrne, Delayed positive feedback can stabilize oscillatory system, *ACC. San Francisco*, (1993), 3106-3107.

2. S. Berrimi and S. A. Messaoudi, Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping, *Electronic J. Diff. Eqns.*, **88** (2004), 1-10.

3. M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. Ferreira, Existence and uniform decay of nonlinear viscoelastic equation with strong damping, *Mathematical Methods in Applied Sciences*, **24** (2001), 1043-1053.

4. M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, Exponential decay for the solution of semilinear viscoelastic wave equation with localized damping, *Electronic J. Diff. Eqns.*, **44** (2002), 1-14.

5. R. Datko, Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks, *SIAM J. Control Optim.*, **26** (1988), 697-713.

6. R. Datko, Two examples of ill-posedness with respect to time delays revisited, *IEEE Trans. Automatic Control*, **42** (1997), 511-515.

7. R. Datko, J. Lagnese and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, *SIAM J. Control Optim.*, **24(1)** (1986), 152-156.

8. Han Xiaosen and Wang Mingxin, General decay of energy for a viscoelastic equation with nonlinear damping, *Mathematical Methods in Applied Sciences*, **32** (2009), 346-358.

9. Han Xiaosen and Wang Mingxin, Global existence and uniform decay for a nonlinear viscoelastic equation with damping, *Nonlinear Anal. Theory, Methods & Applications*, **70(9)** (2009), 3090-3098.

10. M. Kirane and S. H. Belkacem, Existence and asymptotic stability of a viscoelastic wave equation with a delay, *Z. Angew. Math. Phys.*, DOI 10.1007/s00033-011-0145-0.

11. I. Lasiecka, R. Triggiani and P. F. Yao, Inverse/observability estimates for second-order hyperbolic equations with variable coefficients, *Journal of Mathematical Analysis and Applications*, **235** (1999), 13-57.

12. J.-L. Lions, *Quelques méthodes de résolution des probl'emes aux limites non linéaires*, Dunod, 1969.

13. W. J. Liu, General decay rate estimates for a viscoelastic equation with weakly nonlinear time-dependent dissipation and source terms, *J. Math. Phys.*, **50** (2009), 113506.

14. S. A. Messaoudi and N.-e. Tatar, Global existence and asymptotic behavior for a nonlinear viscoelastic problem, *Mathematical Science Research Journal*, **7(4)** (2003), 136-149.

15. S. A. Messaoudi and N-e. Tatar, Global existence and uniform stability of solutions for a quasilinear viscoelastic problem, *Mathematical Methods in Applied Sciences*, **30** (2007), 665-680.

16. S. A. Messaoudi, General decay of solutions of a viscoelastic equation, *Journal of Mathematical Analysis and Applications*, **341** (2008), 1457-1467.

17. S. A. Messaoudi and N-e. Tatar, Global existence and asymptotic behavior for a nonlinear viscoelastic problem, *Mathematical Science Research Journal*, **7(4)** (2003), 136-149.

18. S. A. Messaoudi and N-e. Tatar, Exponential decay for a quasilinear viscoelastic equation, *Math. Nachr.*, **282(10)** (2009), 1443-1450.

19. S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.*, **45(5)** (2006), 1561-1585.

20. S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, *Diff. Int. Equs.*, **21(9-10)** (2008), 935-958.

21. S. Nicaise and J. Valein, Stabilization of the wave equation on 1-d networks with a delay term in the nodal feedbacks, *Netw. Heterog. Media.*, **2(3)** (2007), 425-479.

22. S. Nicaise, J. Valein and E. Fridman, Stabilization of the heat and the wave equations with boundary time-varying delays, *DCDS-S*, **2(3)** (2009), 559-581.

23. S. Nicaise, C. Pignotti and J. Valein, Exponential stability of the wave equation with boundary time-varying delay, *DCDS-S*, **4(3)** (2011), 693-722.

24. I. H. Suh and Z. Bien, Use of time delay action in the controller design, *IEEE Trans. Automat. Control.*, **25** (1980), 600-603.

25. Shun-Tang Wu, General decay of solutions for a viscoelastic equation with nonlinear damping and source terms, *Acta Mathematica Scientia*, **31(B)(4)** (2011), 1436-1448.

26. G. Q. Xu, S. P. Yung and L. K. Li, Stabilization of the wave system with input delay in the boundary control, *ESAIM: Control Optim. Calc. Var.*, **12** (2006), 770-785.

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