TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 3, pp. 999-1030, June 2013
DOI: 10.11650/tjm.17.2013.2496
This paper is available online at http://journal.taiwanmathsoc.org.tw

EXISTENCE AND ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS OF FOURTH ORDER QUASILINEAR DIFFERENTIAL EQUATIONS

Kusano Takaŝi, Jelena Manojlović and Tomoyuki Tanigawa*

Abstract. The feature of the present work is to demonstrate that the method of regular variation can be effectively applied to fourth order quasilinear differential equations of the forms

$$(|x''|^{\alpha-1}x'')'' + q(t)|x|^{\beta-1}x = 0,$$

under the assumptions that $\alpha > \beta$ and $q(t) : [a, \infty) \to (0, \infty)$ is regularly varying function, providing full information about the existence and the precise asymptotic behavior of all possible positive solutions.

1. Introduction

We consider fourth order quasilinear differential equation of the type

$$\text{(A)} \qquad (|x''|^{\alpha-1}x'')'' + q(t)|x|^{\beta-1}x = 0,$$

where α and β are positive constants such that $\alpha > \beta$ and $q : [a, \infty) \to (0, \infty)$, $a > 0$ is a continuous function. Equation (A) is said to be *half-linear*, *sub-half-linear* or *super-half-linear* according as $\alpha = \beta$, $\alpha > \beta$ or $\alpha > \beta$.

Beginning with the papers [5] and [6] of Kiguradze, oscillation theory of higher order nonlinear differential equations of Emden-Fowler type has been the subject of intensive investigations in recent years.

Received October 2, 2012, accepted December 5, 2012.
Communicated by Eiji Yanagida.
2010 *Mathematics Subject Classification*: 34C11, 26A12.
Key words and phrases: Fourth-order differential equations, Regularly varying solutions, Slowly varying solutions, Asymptotic behavior of solutions, Positive solutions.
The second author is supported by the Research project OI-174007 of the Ministry of Education and Science of Republic of Serbia.
The third author is supported by Grant-in-Aid for Scientific Research (C) (No. 23540218), the Ministry of Education, Culture, Sports, Science and Technology, Japan.
*Corresponding auhtor.

A solution $x(t)$ of (A) existing in an infinite interval of the form $[T_x, \infty)$ is said to be *proper* if

$$\sup\{|x(t)| : t \geq T\} > 0 \quad \text{for any} \ \ T \geq T_x.$$

A proper solution is called *oscillatory* if it has an infinite sequence of zeros clustering at infinity and *nonoscillatory* otherwise. Thus, a nonoscillatory solution is eventually positive or eventually negative. It is clear that if $x(t)$ satisfies (A), then so does $-x(t)$, and hence it suffices for us to restrict our attention to *positive* solutions of (A).

The oscillation situation of sub-half-linear equation (A) can be completely characterized as the following theorem (see [17]) asserts:

Theorem 1. *Let* $q(t) \in C[a, \infty)$. *All solutions of equation* (A) *are oscillatory if and only if*

$$\int_a^\infty t^{\left(2+\frac{1}{\alpha}\right)\beta} q(t)dt = \infty. \tag{1}$$

The question that naturally arises from Theorem 1 is: *if* (1) *does not hold, is it possible to characterize the existence of all possible positive solutions of* (A) *and moreover to determine their asymptotic behavior at infinity accurately*?

The oscillatory and asymptotic behavior of solutions of quasilinear differential equation of the form (A) has been considered by Naito and Wu in [16], [17] and Wu [19]. The aim of this paper is to obtain a more detailed information on the existence and asymptotic behavior of nonoscillatory solutions of (A). So, we are interested in those solutions $x(t)$ of equation (A) which exist and are positive on infinite intervals of the form $[t_0, \infty)$. Let $x(t)$ be one such solution. We use notations:

$$D_3x(t) = (|x''|^{\alpha-1}x'')', \ \ \lim_{t\to\infty} D_3x(t) = D_3x(\infty), \ \ \lim_{t\to\infty} x^{(i)}(t) = x^{(i)}(\infty), \ \ i = 0, 1, 2,$$

the symbols $\sim$ and $\prec$ to denote the asymptotic equivalence and asymptotic dominance of two positive functions $f(t)$, $g(t)$ defined in a neighborhood of infinity:

$$f(t) \sim g(t), \quad t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1,$$

and

$$f(t) \prec g(t) \ \text{ or } \ g(t) \succ f(t), \quad t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{g(t)}{f(t)} = \infty.$$

It is known (see [19]) that for $x(t)$ one of the following two cases holds:

$$\text{(i)} \quad x'(t) > 0, \ \ x''(t) > 0, \ \ D_3x(t) > 0 \quad \text{for all large } t,$$

(2)

$$\text{(ii)} \quad x'(t) > 0, \ \ x''(t) < 0, \ \ D_3x(t) > 0 \quad \text{for all large } t.$$

Then, since (A) implies that $D_3x(t)$ is decreasing, there exists a finite limit $D_3x(\infty) \geq 0$. First suppose that solution $x(t)$ is of type (i). If $D_3x(\infty) > 0$, then clearly $x^{(i)}(\infty) = \infty$, $i = 0, 1, 2$, so that from L'Hospital's rule it follows that

$$\lim_{t\to\infty} \frac{x(t)}{t^{2+\frac{1}{\alpha}}} = c > 0,$$

which means that

$$(P_3) \qquad x(t) \sim c\, t^{2+\frac{1}{\alpha}}, \quad \text{as} \quad t \to \infty.$$

On the other hand, let $D_3x(\infty) = 0$. Since $x''(t)$ is positive and increasing, $x''(\infty) \in (0, \infty]$. Repeated integration of the inequality $x''(t) \geq x''(t_1) > 0$ for $t \geq t_1 > t_0$ yields that $x(\infty) = x'(\infty) = \infty$. Thus, L'Hospital's rule gives

$$\lim_{t\to\infty} \frac{x(t)}{t^2} = x''(\infty) \in (0, \infty].$$

In the case $x''(\infty) = \infty$, solution $x(t)$ satisfies

$$(I_2) \qquad t^2 \prec x(t) \prec t^{2+\frac{1}{\alpha}}, \quad \text{as} \quad t \to \infty.$$

while in the case $x''(\infty)$ is finite, we have that

$$(P_2) \qquad x(t) \sim c\, t^2, \quad \text{as} \quad t \to \infty.$$

Next, suppose that solution $x(t)$ is of type (ii). Then, $x''(t)$ is negative, increasing and must tend to 0 as $t \to \infty$, because otherwise, integration of inequality $x''(t) \leq x''(t_1) < 0$, holding for $t \geq t_1 > t_0$, would lead to $x'(t) \to -\infty$, $t \to \infty$, contradicting positivity of $x'(t)$. Moreover, $x'(t)$ is positive and decreasing, so that $x'(\infty) \in [0, \infty)$. If $x'(\infty) > 0$, then

$$\lim_{t\to\infty} \frac{x(t)}{t} = c > 0,$$

i.e.

$$(P_1) \qquad x(t) \sim c\, t, \quad \text{as} \quad t \to \infty.$$

If $x'(\infty) = 0$, then $x(t)/t \to 0$ as $t \to \infty$, and there are two possibilities for $x(t)$: either $x(\infty) = \infty$ or $x(\infty) = c > 0$. Therefore, solution $x(t)$ satisfies either

$$(I_1) \qquad 1 \prec x(t) \prec t, \quad \text{as} \quad t \to \infty,$$

or

$$(P_0) \qquad x(t) \sim c, \quad \text{as} \quad t \to \infty.$$

Solutions of type (P_j), $j \in \{0,1,2,3\}$ are collectively referred to as the *primitive positive solutions* of (A) and the existence of such solutions for the equation with continuous coefficient $q(t)$ has been completely characterized in [16], [17] and [19] for both sub-half-linear and super-half-linear cases.

Theorem 2. *Let* $q(t) \in C[a,\infty)$. *Equation* (A) *has a positive solution* $x(t)$ *satisfying* (P_3) *if and only if*

$$\int_a^\infty t^{(2+\frac{1}{\alpha})\beta} q(t)dt < \infty. \tag{3}$$

Theorem 3. *Let* $q(t) \in C[a,\infty)$. *Equation* (A) *has a positive solution* $x(t)$ *satisfying* (P_2) *if and only if*

$$\int_a^\infty t^{1+2\beta} q(t)dt < \infty. \tag{4}$$

Theorem 4. *Let* $q(t) \in C[a,\infty)$. *Equation* (A) *has a positive solution* $x(t)$ *satisfying* (P_1) *if and only if*

$$\int_a^\infty \left(\int_t^\infty (s-t)s^\beta q(s)ds\right)^{\frac{1}{\alpha}} dt < \infty. \tag{5}$$

Theorem 5. *Let* $q(t) \in C[a,\infty)$. *Equation* (A) *has a positive solution* $x(t)$ *satisfying* (P_0) *if and only if*

$$\int_a^\infty t\left(\int_t^\infty (s-t)q(s)ds\right)^{\frac{1}{\alpha}} dt < \infty. \tag{6}$$

Unlike solutions of type (P_j), $j \in \{0,1,2,3\}$ almost nothing is known about the existence and the asymptotic behavior of solutions of other two types of positive solutions which is often called *intermediate solutions*. Facing the difficulty of providing solution of this problem, the recent development of asymptotic analysis of differential equations by means of regular variation (see the monograph [15] and the papers [2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14]) suggests investigating the problem in the framework of regularly varying functions (or Karamata functions). Therefore, focusing our attention on equation (A) with *regularly varying coefficient* $q(t)$ and to *regularly varying solutions* of (A), it will turn out that the theory of regular variation could provide a complete information about the structure of positive solutions of equation (A). In fact, in virtue of the theory of regular variation one can determine the complete solution to the problem of the existence and the asymptotic behavior of regularly varying solutions of type (I_1) and (I_2).

Our main results are presented in Sections 3 and 4 which are devoted, respectively, to solutions of type (I_2) and (I_1). In each section, we first partition the class of regularly varying solutions of type (I_i), $i \in \{1, 2\}$ into three different subclasses, and then characterize the membership of each subclass completely, establishing necessary and sufficient condition for the existence and showing that all members of each subclass enjoy one and the same accurate asymptotic behavior as $t \to \infty$.

2. Basic Results of Theory of Regular Variation

At the beginning, let us recall the definition of regularly varying functions. A measurable function $f : (a, \infty) \to (0, \infty)$ for some $a > 0$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho, \quad \forall \lambda > 0, \tag{7}$$

or equivalently, if $f(t)$ is expressed in the form

$$f(t) = c(t) \exp\left\{ \int_{t_0}^t \frac{\delta(s)}{s} ds \right\}. \quad t \geq t_0,$$

some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty), \quad \lim_{t\to\infty} \delta(t) = \rho.$$

In what follows the symbol RV(ρ) is used to denote the set of all regularly varying functions of index ρ. If in particular $\rho = 0$, we often use SV instead of RV(0) and refer to members of SV as *slowly varying* functions. It is clear that an RV(ρ)-function $f(t)$ is expressed as $f(t) = t^\rho g(t)$ with $g(t) \in$ SV. So, the class SV of slowly varying functions is of particular importance in the theory of regular variation. If

$$\lim_{t\to\infty} \frac{f(t)}{t^\rho} = \lim_{t\to\infty} g(t) = \text{const} > 0,$$

then $f(t)$ is said to be a *trivial* RV(ρ)-function of index ρ (denoted by $f(t) \in \mathrm{tr} - \mathrm{RV}(\rho)$). Otherwise $f(t)$ is said to be a *nontrivial* RV(ρ)-function (denoted by $f(t) \in \mathrm{ntr} - \mathrm{RV}(\rho)$). According to these terminologies solutions $x(t)$ of type (P_i), $i = 0, 1, 2, 3$ are trivial regularly varying of indices $0, 1, 2, 2 + 1/\alpha$, respectively.

Typical examples of slowly varying functions are: all functions tending to positive constants as $t \to \infty$,

$$\prod_{n=1}^N (\log_n t)^{\alpha_n}, \quad \alpha_n \in \mathbb{R}, \quad \text{and} \quad \exp\left\{ \prod_{n=1}^N (\log_n t)^{\beta_n} \right\}, \quad \beta_n \in (0, 1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm. Note that a slowly varying function $L(t)$ may oscillate in the sense that

$$\limsup_{t\to\infty} L(t) = \infty \quad \text{and} \quad \liminf_{t\to\infty} L(t) = 0,$$

which is exemplified by the function

$$L(t) = \exp\Big\{(\log t)^{\frac{1}{3}} \cos\,(\log t)^{\frac{1}{3}}\Big\}\,.$$

On the other hand its order of growth or decay is severely limited as is shown in the following Proposition.

Proposition 1. *If* $L(t) \in \mathrm{SV}$*, then for any* $\varepsilon > 0$*,*

$$\lim_{t\to\infty} t^{\varepsilon} L(t) = \infty, \qquad \lim_{t\to\infty} t^{-\varepsilon} L(t) = 0.$$

For brevity of exposition the properties of regularly varying functions needed later are not reproduced here, but will be cited where necessary, from the books of Bingham et al. [1] and Marić [15]. However, we state the following result, termed *Karamata's integration theorem*, since it will play a central role in establishing our main results.

Proposition 2. *Let* $L(t) \in \mathrm{SV}$*. Then,*

(i) *if* $\alpha > -1$*,*

$$\int_a^t s^{\alpha} L(s)ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) *if* $\alpha < -1$*,*

$$\int_t^{\infty} s^{\alpha} L(s)ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) *if* $\alpha = -1$*,*

$$l(t) = \int_a^t \frac{L(s)}{s} ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,$$

and

$$m(t) = \int_t^{\infty} \frac{L(s)}{s} ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0\,.$$

3. Existence and Asymptotic Behavior of Regularly Varying Solutions of Type (I_2)

This section is devoted to the precise asymptotic analysis of intermediate regularly varying solutions $x(t)$ of (A) of type (I_2) under the assumption that (A) is sub-half-linear equation with regularly varying coefficient $q(t)$ of index σ expressed as

$$q(t) = t^{\sigma} l(t), \quad l(t) \in \mathrm{SV}\,. \tag{8}$$

If $x(t) \in \mathrm{RV}(\rho)$, i.e. $x(t) = t^{\rho}\xi(t)$ with $\xi(t) \in \mathrm{SV}$, then it is clear that $2 \le \rho \le 2 + \frac{1}{\alpha}$. In the case $\rho = 2$, $\xi(t) \to \infty$ as $t \to \infty$, while in the case $\rho = 2 + 1/\alpha$, $\xi(t) \to 0$ as $t \to \infty$, which means that $x(t)$ belongs to one of the following three classes

$$\text{(9)} \qquad \text{ntr-RV}\left(2 + \frac{1}{\alpha}\right), \quad \mathrm{RV}(\rho) \ \text{ for some } \ \rho \in \left(2, 2 + \frac{1}{\alpha}\right), \quad \text{ntr-RV}(2).$$

Each of these three subclasses can be completely analyzed by means of regular variation combined using fixed point techniques and the following lemma (see [18]).

Lemma 1. *Let $f, g \in C^1[a, \infty)$ and*

$$\lim_{t\to\infty} g(t) = \infty \quad \textit{and} \quad g'(t) > 0 \quad \textit{for all large } t$$

or

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \quad \textit{and} \quad g'(t) < 0 \quad \textit{for all large } t\,.$$

Then

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}\,.$$

3.1. Main results

The existence and asymptotic behavior of all members of classes listed in (9) are fully understood as the following theorems assert.

Theorem 6. *Let $q(t) \in \mathrm{RV}(\sigma)$. Equation* (A) *possesses nontrivial regularly varying solutions of index $2 + \frac{1}{\alpha}$ if and only if*

$$\text{(10)} \qquad \sigma = -\left(2 + \frac{1}{\alpha}\right)\beta - 1 \quad \textit{and} \quad \int_a^{\infty} t^{(2+\frac{1}{\alpha})\beta} q(t)\,dt < \infty,$$

in which case any such solution $x(t)$ has the asymptotic behavior

$$\text{(11)} \qquad x(t) \sim t^{2+\frac{1}{\alpha}} \left[\frac{\alpha - \beta}{\alpha[(2+\frac{1}{\alpha})(1+\frac{1}{\alpha})]^{\alpha}} \int_t^{\infty} s^{(2+\frac{1}{\alpha})\beta} q(s)\,ds\right]^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Theorem 7. *Let $q(t) \in \mathrm{RV}(\sigma)$. Equation* (A) *possesses regularly varying solutions of index $\rho \in \left(2, 2 + \frac{1}{\alpha}\right)$ if and only if*

$$\text{(12)} \qquad \sigma \in \left(-2\beta - 2, -\left(2 + \frac{1}{\alpha}\right)\beta - 1\right),$$

in which case ρ is given by

$$\rho = \frac{\sigma + 2\alpha + 2}{\alpha - \beta}, \tag{13}$$

and the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula

$$x(t) \sim \left[\frac{t^{2\alpha+2}q(t)}{\lambda^{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \quad t \to \infty, \tag{14}$$

where

$$\lambda = [(1 - (\rho - 2)\alpha)(\rho - 2)\alpha]^{\frac{1}{\alpha}}\, \rho(\rho - 1). \tag{15}$$

Theorem 8. *Let $q(t) \in \mathrm{RV}(\sigma)$. Equation* (A) *possesses nontrivial regularly varying solutions of index* 2 *if and only if*

$$\sigma = -2\beta - 2 \quad \text{and} \quad \int_a^{\infty} t^{2\beta+1} q(t)dt = \infty, \tag{16}$$

in which case any such solution $x(t)$ has the asymptotic behavior

$$x(t) \sim t^2 \left[\frac{\alpha - \beta}{\alpha\, 2^{\alpha}} \int_a^t s^{2\beta+1} q(s)ds\right]^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{17}$$

3.2. Preliminaries

If $x(t)$ is a solution of (A) on $[t_0, \infty)$ such that $t^2 \prec x(t) \prec t^{2+\frac{1}{\alpha}}$ as $t \to \infty$, since $D_3x(\infty) = 0$, $x''(\infty) = x'(\infty) = x(\infty) = \infty$, integration of equation (A) first from t to ∞ and then three times on $[t_0, t]$ gives

$$\begin{aligned} x(t) &= x(t_0) + x'(t_0)(t - t_0) \\ &\quad + \int_{t_0}^t (t - s)\left(x''(t_0)^{\alpha} + \int_{t_0}^s \int_r^{\infty} q(u)x(u)^{\beta}dudr\right)^{\frac{1}{\alpha}} ds, \end{aligned} \tag{18}$$

implying the integral asymptotic relation

$$x(t) \sim \int_a^t (t - s)\left(\int_a^s \int_r^{\infty} q(u)x(u)^{\beta}dudr\right)^{\frac{1}{\alpha}} ds \quad \text{as} \quad t \to \infty\,. \tag{19}$$

Common way of determining the desired intermediate solution of (A) would be solving the integral equation (18) with the help of fixed point technique. This is, however, an extremely difficult task in general case of continuous coefficient $q(t)$, but in the framework of regular variation, the closed convex subset $\mathcal{X}$ of $C[T_0,\infty)$, which should be chosen in a such way that appropriate integral operator $\mathcal{F}$ is a continuous self-map on $\mathcal{X}$ and send $\mathcal{X}$ into a relatively compact subset of $C[T_0,\infty)$, can be found by means of regularly varying functions satisfying the integral asymptotic relation. In fact, regularly varying functions

$$X_1(t)=t^{2+\frac{1}{\alpha}}\left[\frac{\alpha-\beta}{\alpha\left((2+\frac{1}{\alpha})(1+\frac{1}{\alpha})\right)^{\alpha}}\int_t^{\infty}s^{(2+\frac{1}{\alpha})\beta}q(s)ds\right]^{\frac{1}{\alpha-\beta}}\in \mathrm{RV}\left(2+\frac{1}{\alpha}\right), \tag{20}$$

$$X_2(t)=\left[\frac{t^{2\alpha+2}q(t)}{\lambda^{\alpha}}\right]^{\frac{1}{\alpha-\beta}}\in \mathrm{RV}\left(\frac{\sigma+2\alpha+2}{\alpha-\beta}\right), \tag{21}$$

$$X_3(t)=t^2\left[\frac{\alpha-\beta}{\alpha\,2^{\alpha}}\int_a^t s^{2\beta+1}q(s)ds\right]^{\frac{1}{\alpha-\beta}}\in \mathrm{RV}(2), \tag{22}$$

appearing, respectively, in the asymptotic formulas (11), (14), (17), satisfy the integral asymptotic relation (19), which will be verified in the following three Lemmas.

Lemma 2. *If* $q(t)\in\mathrm{RV}(\sigma)$ *satisfies* (10)*, then the function* $X_1(t)$ *defined by* (20) *satisfies the integral asymptotic relation* (19).

Proof. Let (10) holds. Then, $q(t)$ and $X_1(t)\in\mathrm{RV}\left(2+\frac{1}{\alpha}\right)$ are expressed as

$$q(t)=t^{-(2+\frac{1}{\alpha})\beta-1}l(t),\qquad X_1(t)=t^{2+\frac{1}{\alpha}}L_1(t),\quad l(t),L_1(t)\in\mathrm{SV}. \tag{23}$$

Using (23) we have

$$\begin{aligned}\int_t^{\infty}q(s)X_1(s)^{\beta}ds &= \int_t^{\infty}s^{-1}l(s)\left[\frac{\alpha-\beta}{\alpha\left((2+\frac{1}{\alpha})(1+\frac{1}{\alpha})\right)^{\alpha}}\int_s^{\infty}r^{-1}l(r)dr\right]^{\frac{\beta}{\alpha-\beta}}ds\\ &= \left(\left(2+\frac{1}{\alpha}\right)\left(1+\frac{1}{\alpha}\right)\right)^{\alpha}L_1(t)^{\alpha}.\end{aligned} \tag{24}$$

Integrating the above from a to t and using Karamata's integration theorem (Proposition 2-(i)), we get

$$\int_a^t\int_s^{\infty}q(r)X_1(r)^{\beta}drds\sim\left(\left(2+\frac{1}{\alpha}\right)\left(1+\frac{1}{\alpha}\right)\right)^{\alpha}t\,L_1(t)^{\alpha},\qquad t\to\infty. \tag{25}$$

From (25), applying Karamata's integration theorem twice, we obtain

$$\begin{aligned}\int_a^t(t-s)\left(\int_a^s\int_r^{\infty}q(u)X_1(u)^{\beta}dudr\right)^{\frac{1}{\alpha}}ds &\sim \left(2+\frac{1}{\alpha}\right)\left(1+\frac{1}{\alpha}\right)\int_a^t\int_a^s r^{\frac{1}{\alpha}}L_1(r)drds\\ &\sim t^{2+\frac{1}{\alpha}}L_1(t)=X_1(t),\qquad t\to\infty,\end{aligned}$$

proving that $X_1(t)$ satisfies the integral asymptotic relation (19). ■

Lemma 3. *If $q(t) \in \mathrm{RV}(\sigma)$ satisfies* (12)*, then the function $X_2(t)$ defined by* (21) *satisfies the integral asymptotic relation* (19).

Proof. Let $q(t)$ be regularly varying of index σ satisfying (12). Note that $X_2(t) \in \mathrm{RV}(\rho)$ with ρ satisfying

$$\rho \in \left(2, 2 + \frac{1}{\alpha}\right),$$

due to (12). We may express the regularly varying function $X_2(t)$ by

$$X_2(t) = t^{\rho} \lambda^{-\frac{\alpha}{\alpha-\beta}} l(t)^{\frac{1}{\alpha-\beta}} \tag{26}$$

so that using (8)

$$q(t) X_2(t)^{\beta} = t^{\sigma+\rho\beta} \lambda^{-\frac{\alpha\beta}{\alpha-\beta}} l(t)^{\frac{\alpha}{\alpha-\beta}}. \tag{27}$$

Noting that $\sigma+\rho\beta = \rho\alpha - 2\alpha - 2 < -1$, integration of (27) from t to ∞ by Proposition 2-(ii) implies

$$\int_t^{\infty} q(s) X_2(s)^{\beta} ds \sim \frac{t^{(\rho-2)\alpha-1}}{-((\rho-2)\alpha-1)} \lambda^{-\frac{\alpha\beta}{\alpha-\beta}} l(t)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty\,.$$

Then, since $(\rho-2)\alpha - 1 > -1$, application of Proposition 2-(i) yields

$$\int_a^t \int_s^{\infty} q(r) X_2(r)^{\beta} dr ds \sim \frac{t^{(\rho-2)\alpha}}{-((\rho-2)\alpha-1)(\rho-2)\alpha} \lambda^{-\frac{\alpha\beta}{\alpha-\beta}} l(t)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty\,. \tag{28}$$

Now, using that $\rho - 2 > 0$, from (28), applying Karamata's integration theorem twice, we obtain

$$\begin{aligned}
&\int_a^t (t-s) \left(\int_a^s \int_r^{\infty} q(u) X_2(u)^{\beta} du dr \right)^{\frac{1}{\alpha}} ds \\
&\quad \sim \frac{\lambda^{-\frac{\beta}{\alpha-\beta}}}{[(1-(\rho-2)\alpha)(\rho-2)\alpha]^{\frac{1}{\alpha}}} \int_a^t \int_a^s r^{\rho-2} l(r)^{\frac{1}{\alpha-\beta}} dr ds \\
&\quad \sim \frac{\lambda^{-\frac{\beta}{\alpha-\beta}}}{[(1-(\rho-2)\alpha)(\rho-2)\alpha]^{\frac{1}{\alpha}}(\rho-1)\rho} t^{\rho} l(t)^{\frac{1}{\alpha-\beta}} = \frac{t^{\rho} l(t)^{\frac{1}{\alpha-\beta}}}{\lambda^{\frac{\alpha}{\alpha-\beta}}}, \quad t \to \infty,
\end{aligned}$$

which due to (26) proves that $X_2(t)$ satisfies the integral asymptotic relation (19). ■

Lemma 4. *Suppose that $q(t) \in \mathrm{RV}(\sigma)$ satisfies* (16)*. Then, the function $X_3(t)$ defines by* (22) *satisfies the integral asymptotic relation* (19).

Proof. Suppose that (16) holds. Using the expressions

$$q(t) = t^{-2\beta-2} l(t), \qquad X_3(t) = t^2 L_3(t), \qquad l(t), L_3(t) \in \mathrm{SV}\,,$$

and Proposition 2-(ii) we get

$$\begin{aligned}
\int_t^\infty q(s) X_3(s)^\beta ds &= \int_t^\infty s^{-2} l(s) \left[\frac{\alpha-\beta}{\alpha\, 2^\alpha} \int_a^s r^{2\beta+1} q(r) dr\right]^{\frac{\beta}{\alpha-\beta}} ds \\
&\sim t^{-1} l(t) \left[\frac{\alpha-\beta}{\alpha\, 2^\alpha} \int_a^t s^{2\beta+1} q(s) ds\right]^{\frac{\beta}{\alpha-\beta}} \qquad (29)\\
&= t^{2\beta+1} q(t) \left[\frac{\alpha-\beta}{\alpha\, 2^\alpha} \int_a^t s^{2\beta+1} q(s) ds\right]^{\frac{\beta}{\alpha-\beta}}, \quad t \to \infty.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_a^t \int_s^\infty q(r) X_3(r)^\beta dr ds &\sim 2^\alpha \left[\frac{\alpha-\beta}{\alpha\, 2^\alpha} \int_a^t s^{2\beta+1} q(s) ds\right]^{\frac{\alpha}{\alpha-\beta}} \qquad (30)\\
&= 2^\alpha L_3(t)^\alpha, \quad t \to \infty
\end{aligned}$$

which implies

$$\int_a^t (t-s) \left(\int_a^s \int_r^\infty q(u) X_3(u)^\beta du dr\right)^{\frac{1}{\alpha}} ds \sim 2 \int_a^t \int_a^s L_3(r) dr ds \sim t^2 L_3(t) = X_3(t),$$

as $t \to \infty$, proving that $X_3(t)$ satisfies the integral asymptotic relation (19). ■

To simplify the proof of the "only if"parts of proof of our main results we first prove the following lemma.

Lemma 5. *If* $q(t) \in \mathrm{RV}(\sigma)$ *is expressed with* (8) *and* $x(t) = t^\rho \xi(t)$, $\xi(t) \in \mathrm{SV}$ *is a solution of* (A) *of type* (I_2), *then one of the following three statements holds:*
(i) $\sigma + \rho\beta = -1$ *and*

$$(31) \quad x(t) \sim \frac{t^{2+\frac{1}{\alpha}}}{\left(2+\frac{1}{\alpha}\right)\left(1+\frac{1}{\alpha}\right)} \left(\int_t^\infty s^{-1} l(s) \xi(s)^\beta ds\right)^{\frac{1}{\alpha}} \in \mathrm{RV}\Big(2+\frac{1}{\alpha}\Big), \quad t \to \infty.$$

(ii) $-2 < \sigma + \rho\beta < -1$ *and*

$$(32) \qquad x(t) \sim \frac{t^{\frac{\sigma+\rho\beta+2}{\alpha}+2} l(t)^{\frac{1}{\alpha}} \xi(t)^{\frac{\beta}{\alpha}}}{L} \in \mathrm{RV}\Big(\frac{\sigma+\rho\beta+2}{\alpha}+2\Big), \; t \to \infty\,,$$

where

$$L=\Big[-(\sigma+\rho\beta+1)(\sigma+\rho\beta+2)\Big]^{\frac{1}{\alpha}}\left(\frac{\sigma+\rho\beta+2}{\alpha}+2\right)\left(\frac{\sigma+\rho\beta+2}{\alpha}+1\right).$$

(iii) $\sigma+\rho\beta=-2$ *and*

$$x(t)\sim\frac{t^2}{2}\left(\int_{t_0}^{t}s^{-1}l(s)\xi(s)^{\beta}ds\right)^{\frac{1}{\alpha}}\in \mathrm{RV}(2),\quad t\to\infty. \tag{33}$$

Proof. Let $x(t)=t^{\rho}\xi(t)$, $\xi(t)\in\mathrm{SV}$ be a solution of (A) on $[t_0,\infty)$ satisfying (I_2). Then, using expression (8) we obtain from (A)

$$D_3x(t)=\int_t^{\infty}q(s)x(s)^{\beta}ds=\int_t^{\infty}s^{\sigma+\rho\beta}l(s)\xi(s)^{\beta}ds,\quad t\geq t_0. \tag{34}$$

The convergence of the last integral in (34) means that $\sigma+\rho\beta\leq-1$. We distinguish the two cases $\sigma+\rho\beta=-1$ and $\sigma+\rho\beta<-1$.

(a) Let $\sigma+\rho\beta=-1$. Then, since

$$\int_t^{\infty}q(s)x(s)^{\beta}ds=\int_t^{\infty}s^{-1}l(s)\xi(s)^{\beta}ds\in\mathrm{SV}$$

(cf. (iii) of Proposition 2), integrating (34) on $[t_0,t]$ and using Karamata's integration theorem ((i) of Proposition 2) we get

$$\left(\int_{t_0}^{t}\int_s^{\infty}q(r)x(r)^{\beta}drds\right)^{\frac{1}{\alpha}}\sim t^{\frac{1}{\alpha}}\left(\int_t^{\infty}s^{-1}l(s)\xi(s)^{\beta}ds\right)^{\frac{1}{\alpha}},\qquad t\to\infty\,,$$

which integrated twice over $[t_0,t]$ gives (31).

(b) Let $\sigma+\rho\beta<-1$. Applying Karamata's integration theorem ((ii) of Proposition 2) to the last integral in (34), we have

$$D_3x(t)=\int_t^{\infty}q(s)x(s)^{\beta}ds\sim\frac{t^{\sigma+\rho\beta+1}l(t)\xi(t)^{\beta}}{[-(\sigma+\rho\beta+1)]},\quad t\to\infty. \tag{35}$$

Let us distinguish the following three cases:

(b-1) $\sigma+\rho\beta<-2$, (b-2) $\sigma+\rho\beta=-2$, (b-3) $-2<\sigma+\rho\beta<-1$.

The case (b-1) is impossible, because in this case the right-hand side of (35) is integrable on $[t_0,\infty)$, which means that

$$\int_t^{\infty}\int_s^{\infty}q(r)x(r)^{\beta}drds\sim\frac{t^{\sigma+\rho\beta+2}l(t)\xi(t)^{\beta}}{[-(\sigma+\rho\beta+1)][-(\sigma+\rho\beta+2)]}\to 0,\quad t\to\infty.$$

contradicting the fact that $x''(\infty)=\infty$.

If (b-2) holds, from (35) we have

$$\int_{t_0}^{t}\int_{s}^{\infty} q(r)x(r)^{\beta}drds \sim \int_{t_0}^{t} s^{-1}l(s)\xi(s)^{\beta}ds \in \mathrm{SV},$$

so that

$$\int_{t_0}^{t}(t-s)\left(\int_{t_0}^{s}\int_{r}^{\infty} q(u)x(u)^{\beta}dudr\right)^{\frac{1}{\alpha}}ds \sim \frac{t^2}{2}\left(\int_{t_0}^{t} s^{-1}l(s)\xi(s)^{\beta}ds\right)^{\frac{1}{\alpha}}, \quad t\to\infty,$$

which due to (19) implies (33).

Finally, if (b-3) holds, Karamata's integration theorem applied to (35) gives

$$\left(\int_{t_0}^{t}\int_{s}^{\infty} q(r)x(r)^{\beta}drds\right)^{\frac{1}{\alpha}} \sim \frac{t^{\frac{\sigma+\rho\beta+2}{\alpha}}l(t)^{\frac{1}{\alpha}}\xi(t)^{\frac{\beta}{\alpha}}}{[-(\sigma+\rho\beta+1)(\sigma+\rho\beta+2)]^{\frac{1}{\alpha}}}, \qquad t\to\infty\,,$$

which integrated twice over $[t_0, t]$ yields (32). ■

3.3. Proof of main results

Proof of the "only if" part of Theorem 6. Suppose that $x(t) \in \mathrm{ntr}-\mathrm{RV}\left(2+\frac{1}{\alpha}\right)$ is a solution of (A) on $[t_0,\infty)$ satisfying (I_2). Let $x(t) = t^{2+\frac{1}{\alpha}}\xi(t)$, $\xi(t)\in\mathrm{SV}$. For such $x(t)$ only statement (i) in Lemma 5 is possible with $\rho = 2+\frac{1}{\alpha}$. This means that $\sigma = -\left(2+\frac{1}{\alpha}\right)\beta - 1$ i.e.

$$q(t) = t^{-(2+\frac{1}{\alpha})\beta-1}l(t), \qquad l(t)\in\mathrm{SV}$$

and $x(t)$ must satisfy the asymptotic relation (31), which is equivalent to

$$\xi(t)^{\alpha} \sim \frac{1}{[(2+\frac{1}{\alpha})(1+\frac{1}{\alpha})]^{\alpha}}\int_{t}^{\infty} s^{-1}l(s)\xi(s)^{\beta}ds, \quad t\to\infty. \tag{36}$$

Let $\eta(t)$ denote the right-hand side of (36). Then, (36) is transformed into the differential asymptotic relation

$$-\eta(t)^{-\frac{\beta}{\alpha}}\eta'(t) \sim \frac{t^{-1}l(t)}{[(2+\frac{1}{\alpha})(1+\frac{1}{\alpha})]^{\alpha}} = \frac{t^{(2+\frac{1}{\alpha})\beta}q(t)}{[(2+\frac{1}{\alpha})(1+\frac{1}{\alpha})]^{\alpha}}, \quad t\to\infty. \tag{37}$$

Noting that $\alpha > \beta$ and

$$\eta(t) \sim \xi(t)^{\alpha} = \left(\frac{x(t)}{t^{(2+\frac{1}{\alpha})\beta}}\right)^{\alpha} \to 0, \quad t\to\infty,$$

integrating (37) on $[t,\infty)$, we see that the second condition in (10) must be satisfied and

$$\eta(t) \sim \left[\frac{\alpha-\beta}{\alpha\big[(2+\frac{1}{\alpha})(1+\frac{1}{\alpha})\big]^{\alpha}}\int_t^{\infty} s^{(2+\frac{1}{\alpha})\beta} q(s)ds\right]^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty,$$

which since $x(t) \sim t^{(2+\frac{1}{\alpha})\beta}\eta(t)^{\frac{1}{\alpha}}$, $t\to\infty$, implies the validity of (11). ■

Proof of the "only if"part of Theorem 7. Suppose that $x(t) \in \mathrm{RV}(\rho)$ for some $\rho \in \left(2, 2+\frac{1}{\alpha}\right)$ is the solution of (A). Then, clearly only the statement (ii) of Lemma 5 could hold. Thus, $x(t)$ satisfy (32), so it must be

$$\rho = \frac{\sigma+\rho\beta+2}{\alpha} + 2, \tag{38}$$

which implies that ρ is given by (13). This combined with $\rho \in \left(2, 2+\frac{1}{\alpha}\right)$ determines the range of σ with (12). Using (38) we have in (32) that $L=\lambda$, where λ is given by (15), so (32) can be rewritten as

$$x(t) \sim \frac{t^{\frac{2\alpha+2}{\alpha}} q(t)^{\frac{1}{\alpha}} x(t)^{\frac{\beta}{\alpha}}}{\lambda}, \quad t\to\infty,$$

verifying that asymptotic formula for $x(t)$ is be given by (14). ■

Proof of the "only if"part of Theorem 8. Suppose that $x(t) \in$ ntr-RV(2) is a solution of (A) on $[t_0,\infty)$. Let $x(t) = t^2\xi(t)$, $\xi(t) \in \mathrm{SV}$. It is clear that only statement (iii) of Lemma 5 is admissible for $x(t)$. Therefore, $\sigma = -2\beta-2$ and $x(t)$ satisfies (33) which is equivalent to

$$\xi(t)^{\alpha} \sim \frac{1}{2^{\alpha}}\int_{t_0}^{t} s^{-1} l(s)\xi(s)^{\beta} ds, \quad t\to\infty. \tag{39}$$

The right-hand side of (39), denoted by $\eta(t)$, satisfies the differential asymptotic relation

$$\eta(t)^{-\frac{\beta}{\alpha}}\eta'(t) \sim \frac{t^{-1}l(t)}{2^{\alpha}} = \frac{t^{2\beta+1}q(t)}{2^{\alpha}}, \quad t\to\infty\,. \tag{40}$$

Noting that $\eta(t)^{1-\frac{\beta}{\alpha}} \to \infty$ as $t\to\infty$, it follows from (40) that $\int_a^{\infty} t^{2\beta+1}q(t)dt = \infty$ and integration of (40) on $[t_0,t]$ gives

$$\eta(t) \sim \left[\frac{\alpha-\beta}{2^{\alpha}\alpha}\int_{t_0}^{t} s^{2\beta+1}q(s)ds\right]^{\frac{\alpha}{\alpha-\beta}} \sim \left[\frac{\alpha-\beta}{2^{\alpha}\alpha}\int_{a}^{t} s^{2\beta+1}q(s)ds\right]^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty,$$

which since $x(t) = t^2\xi(t) \sim t^2\eta(t)^{1/\alpha}$ as $t\to\infty$, verify the truth of the asymptotic formula (17). ■

Proof of the "if"part of Theorems 6, 7 *and* 8. Suppose that $q(t) \in \mathrm{RV}(\sigma)$ with σ satisfying either (10) or (12) or (16). By Lemma 2, 3 and 4 the functions $X_1(t), X_2(t), X_3(t)$ defined, respectively, by (20), (21), (22) satisfy the asymptotic relation (19). We perform the simultaneous proof for $X_i(t)$, $i = 1, 2, 3$ so the subscripts $i = 1, 2, 3$ will be deleted in the rest of the proof. By (19), there exists $T_0 > a$ such that

$$(41) \qquad \int_{T_0}^{t} (t-s) \left(\int_{T_0}^{s} \int_{r}^{\infty} q(u) X(u)^{\beta} du dr \right)^{\frac{1}{\alpha}} ds \le 2X(t), \quad t \ge T_0 \,.$$

Let such a T_0 be fixed. We may assume that $X(t)$ is increasing on $[T_0, \infty)$. Since from (19) we have

$$\int_{T_0}^{t} (t-s) \left(\int_{T_0}^{s} \int_{r}^{\infty} q(u) X(u)^{\beta} du dr \right)^{\frac{1}{\alpha}} ds \sim X(t), \quad t \to \infty,$$

there exists $T_1 > T_0$ such that

$$(42) \qquad \int_{T_0}^{t} (t-s) \left(\int_{T_0}^{s} \int_{r}^{\infty} q(u) X(u)^{\beta} du dr \right)^{\frac{1}{\alpha}} ds \ge \frac{X(t)}{2}, \quad t \ge T_1.$$

Let such a T_1 be fixed. Let $m \in (0, 1)$ be a fixed positive constant such that

$$(43) \qquad m^{1-\frac{\beta}{\alpha}} \le \frac{1}{2} \,,$$

and choose a constant $M > 1$ such that

$$(44) \qquad M^{1-\frac{\beta}{\alpha}} \ge 4 \quad \text{and} \quad M \ge 2m \frac{X(T_1)}{X(T_0)} \,.$$

Let us define the set $\mathcal{X}$ to be the set of continuous functions $x(t)$ on $[T_0, \infty)$ satisfying

$$(45) \qquad \begin{cases} mX(T_1) \le x(t) \le MX(t), & \text{for } T_0 \le t \le T_1 \,, \\ mX(t) \le x(t) \le MX(t), & \text{for } t \ge T_1 \,. \end{cases}$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[T_0, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0, \infty)$. We now define the integral operator

$$(46) \qquad \mathcal{F}x(t) = mX(T_1) + \int_{T_0}^{t} (t-s) \left(\int_{T_0}^{s} \int_{r}^{\infty} q(u) x(u)^{\beta} du dr \right)^{\frac{1}{\alpha}} ds, \quad t \ge T_0$$

and let it act on the set $\mathcal{X}$ defined above. It can be shown that $\mathcal{F}$ is a self-map on $\mathcal{X}$ and sends $\mathcal{X}$ continuously on a relatively compact subset of $C[T_0, \infty)$.

(i) $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$. Let $x(t) \in \mathcal{X}$. Using (41), (44), (45) and (46) we get

$$\begin{aligned}\mathcal{F}x(t) &\leq mX(T_1) + M^{\frac{\beta}{\alpha}} \int_{T_0}^{t} (t-s) \left(\int_{T_0}^{s} \int_{r}^{\infty} q(u) X(u)^{\beta} du dr \right)^{\frac{1}{\alpha}} ds \\ &\leq \frac{M}{2} X(T_0) + \frac{M}{2} X(t) \leq \frac{M}{2} X(t) + \frac{M}{2} X(t) = MX(t), \quad t \geq T_0 .\end{aligned}$$

On the other hand, using (42), (43), (45) and (46) we have

$$\mathcal{F}x(t) \geq mX(T_1) \quad \text{for} \quad T_0 \leq t \leq T_1 ,$$

and

$$\begin{aligned}\mathcal{F}x(t) &\geq m^{\frac{\beta}{\alpha}} \int_{T_0}^{t} (t-s) \left(\int_{T_0}^{s} \int_{r}^{\infty} q(u) X(u)^{\beta} du dr \right)^{\frac{1}{\alpha}} ds \\ &\geq m^{\frac{\beta}{\alpha}} \frac{X(t)}{2} \geq mX(t), \quad t \geq T_1 .\end{aligned}$$

This shows that $\mathcal{F}x(t) \in \mathcal{X}$, that is, $\mathcal{F}$ maps $\mathcal{X}$ into itself.

(ii) $\mathcal{F}(\mathcal{X})$ *is relatively compact.* The inclusion $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$ ensures that $\mathcal{F}(\mathcal{X})$ is locally uniformly bounded on $[T_0, \infty)$. From the inequality

$$0 \leq (\mathcal{F}x)'(t) \leq M^{\frac{\beta}{\alpha}} \int_{T_0}^{t} \left(\int_{T_0}^{s} \int_{r}^{\infty} q(u) X(u)^{\beta} du dr \right)^{\frac{1}{\alpha}} ds, \quad t \geq T_0,$$

holding for all $x(t) \in \mathcal{X}$ it follows that $\mathcal{F}(\mathcal{X})$ is locally equicontinuous on $[T_0, \infty)$. The relative compactness of $\mathcal{F}(\mathcal{X})$ then follows from the Arzela-Ascoli.

(iii) $\mathcal{F}$ *is continuous.* Let $\{x_n(t)\}$ be a sequence in $\mathcal{X}$ converging to $x(t) \in \mathcal{X}$ uniformly on compact subintervals of $[T_0, \infty)$. Then, by (46) we have

$$|\mathcal{F}x_n(t) - \mathcal{F}x(t)| \leq \int_{T_0}^{t} (t-s) F_n(s) ds, \quad t \geq T_0, \tag{47}$$

where

$$F_n(t) = \left| \left(\int_{T_0}^{t} \int_{s}^{\infty} q(r) x_n(r)^{\beta} dr ds \right)^{\frac{1}{\alpha}} - \left(\int_{T_0}^{t} \int_{s}^{\infty} q(r) x(r)^{\beta} dr ds \right)^{\frac{1}{\alpha}} \right| .$$

Using the inequality $|x^{\lambda} - y^{\lambda}| \leq |x - y|^{\lambda}$, $x, y \in \mathbb{R}^{+}$ holding for $\lambda \in (0,1)$, we see that

$$F_n(t) \leq \left(\int_{T_0}^{t} \int_{s}^{\infty} q(r) |x_n(r)^{\beta} - x(r)^{\beta}| dr ds \right)^{\frac{1}{\alpha}}, \quad t \geq T_0, \quad \text{if} \quad \alpha \geq 1. \tag{48}$$

On the other hand, using the mean value theorem we get

$$F_n(t) \le \theta \int_{T_0}^{t} \int_{s}^{\infty} q(r)|x_n(r)^\beta - x(r)^\beta| dr ds, \quad t \ge T_0, \quad \text{if} \quad \alpha < 1\,, \tag{49}$$

where

$$\theta = \frac{1}{\alpha}\left(M^\beta \int_{T_0}^{t} \int_{s}^{\infty} q(r) X(r)^\beta dr ds \right)^{\frac{1-\alpha}{\alpha}}.$$

Thus, using that $q(t)\big|x_n(t)^\beta - x(t)^\beta\big| \to 0$ as $n \to \infty$ at each point $t \in [T_0, \infty)$ and $q(t)\big|x_n(t)^\beta - x(t)^\beta\big| \le M^\beta q(t)X(t)^\beta$ for $t \ge T_0$, while $q(t)X(t)^\beta$ is integrable on $[T_0, \infty)$, the uniform convergence $F_n(t) \to 0$ on compact subinterval of $[T_0, \infty)$ follows by the application of the Lebesgue dominated convergence theorem. We conclude that $\mathcal{F}x_n(t) \to \mathcal{F}x(t)$ uniformly on any compact subinterval of $[T_0, \infty)$ as $n \to \infty$, which proves the continuity of $\mathcal{F}$.

Thus all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled, and there exists a fixed point $x(t) \in \mathcal{X}$ of $\mathcal{F}$, which satisfies the integral equation

$$x(t) = mX(T_1) + \int_{T_0}^{t} (t-s) \left(\int_{T_0}^{s} \int_{r}^{\infty} q(u) x(u)^\beta du dr \right)^{\frac{1}{\alpha}} ds, \quad t \ge T_0\,. \tag{50}$$

Differentiating the above four times we show that $x(t)$ is a solution of (A) on $[T_0, \infty)$, which due to (45) satisfies

$$m\,X(t) \le x(t) \le M\,X(t), \qquad \text{for} \ \ t \ge T_1\,. \tag{51}$$

Therefore,

$$0 < \liminf_{t\to\infty} \frac{x(t)}{X(t)} \le \limsup_{t\to\infty} \frac{x(t)}{X(t)} < \infty\,,$$

or in view of the fact that

$$X(t) \sim \chi_1(t) = \int_{a}^{t} (t-s) \left(\int_{a}^{s} \int_{r}^{\infty} q(u) X(u)^\beta du dr \right)^{\frac{1}{\alpha}} ds, \quad t \to \infty, \tag{52}$$

we have

$$0 < \liminf_{t\to\infty} \frac{x(t)}{\chi_1(t)} \le \limsup_{t\to\infty} \frac{x(t)}{\chi_1(t)} < \infty\,.$$

Application of Lemma 1 four times, using (50) and (52), yields

$$\begin{aligned}
K &= \limsup_{t\to\infty} \frac{x(t)}{\chi_1(t)} \le \limsup_{t\to\infty} \frac{x'(t)}{\chi_1'(t)} \le \limsup_{t\to\infty} \frac{x''(t)}{\chi_1''(t)} \\
&= \limsup_{t\to\infty} \frac{\left(\int_{T_0}^t \int_s^\infty q(r)x(r)^\beta dr ds\right)^{\frac{1}{\alpha}}}{\left(\int_{T_0}^t \int_s^\infty q(r)X(r)^\beta dr ds\right)^{\frac{1}{\alpha}}} \\
&= \left(\limsup_{t\to\infty} \frac{\int_{T_0}^t \int_s^\infty q(r)x(r)^\beta dr ds}{\int_{T_0}^t \int_s^\infty q(r)X(r)^\beta dr ds}\right)^{\frac{1}{\alpha}} \\
&\le \left(\limsup_{t\to\infty} \frac{-q(t)x(t)^\beta}{-q(t)X(t)^\beta}\right)^{\frac{1}{\alpha}} \\
&= \left(\limsup_{t\to\infty} \frac{x(t)}{X(t)}\right)^{\frac{\beta}{\alpha}} = \left(\limsup_{t\to\infty} \frac{x(t)}{\chi_1(t)}\right)^{\frac{\beta}{\alpha}} = K^{\frac{\beta}{\alpha}} \,.
\end{aligned}$$

Since $\beta/\alpha < 1$, from above we conclude that

$$0 < K \le 1 \,. \tag{53}$$

Similary, we can see that

$$k = \liminf_{t\to\infty} \frac{x(t)}{\chi_1(t)}$$

satisfies

$$1 \le k < \infty \,. \tag{54}$$

From (53) and (54) we obtain that $k = K = 1$, which means that

$$x(t) \sim \chi_1(t) \sim X(t), \quad t \to \infty \,,$$

and ensures that $x(t)$ is a regularly varying solution of (A) with requested regularity index and the asymptotic behavior (11), (14), (17) depending on if $q(t) \in \mathrm{RV}(\sigma)$ satisfies, respectively, (10) or (12) or (16). ■

4. Existence and Asymptotic Behavior of Regularly Varying Solutions of (A) of Type (I_1)

It remains to study intermediate regularly varying solutions $x(t)$ of (A) satisfying (I_1). Let $x(t)$ be one such solution and write it as $x(t) = t^\rho \xi(t)$ with $\xi(t) \in$ SV. Thus, the regularity index ρ must lie in the interval $[0, 1]$, and if $\rho = 0$ (or $\rho = 1$), then the slowly varying part $\xi(t)$ of $x(t)$ must satisfy $\lim_{t\to\infty} \xi(t) = \infty$ (or $\lim_{t\to\infty} \xi(t) = 0$).

It follows that the class of regularly varying solutions of type (I_1) consists of three different types of regularly varying solutions:

(55) $x(t) \in \mathrm{ntr} - \mathrm{RV}(1), \quad x(t) \in \mathrm{RV}(\rho)$ for some $\rho \in (0,1), \quad x(t) \in \mathrm{ntr} - \mathrm{SV}.$

We establish necessary and sufficient for the existence of these three types of regularly varying solutions with precise asymptotic behavior at infinity and show that the regularity index ρ of such solution is uniquely determined by α, β and the regularity index σ of $q(t)$.

4.1. Main results

The main result of this section asserts that the membership of three classes of regularly varying solutions of (A) listed in (55) can be characterized completely.

Theorem 9. *Let $q(t) \in \mathrm{RV}(\sigma)$. Equation* (A) *possesses nontrivial regularly varying solutions of index* 1 *if and only if*

(56) $$\sigma = -\alpha - \beta - 2 \quad \textit{and} \quad \int_a^\infty \big(t^{\beta+2} q(t)\big)^{\frac{1}{\alpha}} dt < \infty,$$

in which case any such solution $x(t)$ has the asymptotic behavior

(57) $$x(t) \sim t \left[\frac{\alpha - \beta}{\alpha\big(\alpha(\alpha+1)\big)^{\frac{1}{\alpha}}} \int_t^\infty \big(s^{\beta+2} q(s)\big)^{\frac{1}{\alpha}} ds \right]^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty.$$

Theorem 10. *Let $q(t) \in \mathrm{RV}(\sigma)$. Equation* (A) *possesses regularly varying solutions of index $\rho \in (0,1)$ if and only if*

(58) $$\sigma \in (-2\alpha - 2, -\alpha - \beta - 2),$$

in which case ρ is given by (13) *and the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula*

(59) $$x(t) \sim \left[\frac{t^{2\alpha+2} q(t)}{\omega^\alpha} \right]^{\frac{1}{\alpha-\beta}}, \quad t \to \infty,$$

where

(60) $$\omega = [(1 - (\rho - 2)\alpha)(2 - \rho)\alpha]^{\frac{1}{\alpha}} \rho(1 - \rho).$$

Theorem 11. *Let* $q(t) \in \mathrm{RV}(\sigma)$*. Equation* (A) *possesses nontrivial slowly varying solutions if and only if*

$$\sigma = -2\alpha - 2 \quad \textit{and} \quad \int_a^\infty \big(t^{\alpha+2} q(t)\big)^{\frac{1}{\alpha}} dt = \infty, \tag{61}$$

in which case any such solution $x(t)$ *has the asymptotic behavior*

$$x(t) \sim \left[\frac{\alpha-\beta}{\alpha[2\alpha(2\alpha+1)]^{\frac{1}{\alpha}}} \int_a^t \big(s^{\alpha+2} q(s)\big)^{\frac{1}{\alpha}} ds\right]^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty. \tag{62}$$

4.2. Preliminaries

Let $x(t)$ be a solution of (A) on $[t_0, \infty)$ such that $1 \prec x(t) \prec t$ as $t \to \infty$. Since $D_3 x(\infty) = x''(\infty) = x'(\infty) = 0$ and $x(\infty) = \infty$, integration of equation (A) three times from t to ∞ and then on $[t_0, t]$ implies the integral asymptotic relation

$$x(t) \sim \int_a^t \int_s^\infty \left(\int_r^\infty (u-r) q(u) x(u)^\beta du\right)^{\frac{1}{\alpha}} dr ds \quad \text{as} \quad t \to \infty\,. \tag{63}$$

We first prove that regularly varying functions appearing on the righ-hand side of asymptotic formula (57), (59) and (62), satisfy the integral asymptotic relation (63).

Lemma 6. *Suppose that* $q(t) \in \mathrm{RV}(\sigma)$ *satisfies* (56)*. Then, the function*

$$X_4(t) = t\left[\frac{\alpha-\beta}{\alpha\big(\alpha(\alpha+1)\big)^{\frac{1}{\alpha}}} \int_t^\infty \big(s^{\beta+2} q(s)\big)^{\frac{1}{\alpha}} ds\right]^{\frac{\alpha}{\alpha-\beta}}, \quad t \geq a \tag{64}$$

satisfies the integral asymptotic relation (63)*.*

Proof. Suppose that (56) holds. Using that $q(t) = t^{-\alpha-\beta-2} l(t)$, $l(t) \in \mathrm{SV}$ and $X_4(t) = t L_4(t) \in \mathrm{RV}(1)$, $L_4(t) \in \mathrm{SV}$, we get $q(t) X_4(t)^\beta = t^{-\alpha-2} l(t) L_4(t)^\beta$, so integration on $[t, \infty)$ two times gives

$$\int_t^\infty (s-t) q(s) X_4(s)^\beta ds \sim \frac{t^{-\alpha} l(t) L_4(t)^\beta}{(\alpha+1)\alpha}, \qquad t \to \infty\,. \tag{65}$$

Then,

$$\begin{aligned} \int_t^\infty \left[\int_s^\infty (r-s) q(r) X_4(r)^\beta dr\right]^{\frac{1}{\alpha}} ds &\sim \frac{1}{[(\alpha+1)\alpha]^{\frac{1}{\alpha}}} \int_t^\infty s^{-1} l(s)^{\frac{1}{\alpha}} L_4(s)^{\frac{\beta}{\alpha}} ds \\ &= \frac{1}{[(\alpha+1)\alpha]^{\frac{1}{\alpha}}} \int_t^\infty \big(s^{\beta+2} q(s)\big)^{\frac{1}{\alpha}} \left[\frac{\alpha-\beta}{\alpha\big(\alpha(\alpha+1)\big)^{\frac{1}{\alpha}}} \int_s^\infty \big(r^{\beta+2} q(r)\big)^{\frac{1}{\alpha}} dr\right]^{\frac{\beta}{\alpha-\beta}} ds \\ &= L_4(t),\ t \to \infty. \end{aligned} \tag{66}$$

Therefore, integration of above on $[a, t]$ and application of Proposition 2-(i) yields the desired conclusion that $X_4(t)$ satisfies (63). ■

Lemma 7. *Let* ρ, ω *be defined by* (13) *and* (60), *respectively. Suppose that* $q(t) \in \mathrm{RV}(\sigma)$ *satisfies* (58). *Then, the function*

$$X_5(t) = \left[\frac{t^{2\alpha+2}q(t)}{\omega^{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \qquad t \geq a\,, \tag{67}$$

satisfies the integral asymptotic relation (63).

Proof. Function $X_5(t)$ is regularly varying of index ρ and using (8) it may be expressed as

$$X_5(t) = t^{\rho}\omega^{-\frac{\alpha}{\alpha-\beta}}l(t)^{\frac{1}{\alpha-\beta}}. \tag{68}$$

Then, $q(t)X_5(t)^{\beta} = t^{\rho\alpha-2\alpha-2}\omega^{-\frac{\alpha\beta}{\alpha-\beta}}l(t)^{\frac{\alpha}{\alpha-\beta}}$, so that integration two times on $[t, \infty)$ gives

$$\int_t^{\infty}(s-t)q(s)X_5(s)^{\beta}ds \sim \frac{t^{(\rho-2)\alpha}\omega^{-\frac{\alpha\beta}{\alpha-\beta}}l(t)^{\frac{\alpha}{\alpha-\beta}}}{[-(\rho\alpha-2\alpha-1)]\alpha(2-\rho)}, \quad t \to \infty\,,$$

implying that

$$\begin{aligned}
&\int_t^{\infty}\left[\int_s^{\infty}(r-s)q(r)X_5(r)^{\beta}dr\right]^{\frac{1}{\alpha}}ds \\
&\quad\sim \frac{\omega^{-\frac{\beta}{\alpha-\beta}}}{[-(\rho\alpha-2\alpha-1)\alpha(2-\rho)]^{\frac{1}{\alpha}}}\int_t^{\infty}s^{\rho-2}l(s)^{\frac{1}{\alpha-\beta}}ds \\
&\quad\sim \frac{\omega^{-\frac{\beta}{\alpha-\beta}}}{[-(\rho\alpha-2\alpha-1)\alpha(2-\rho)]^{\frac{1}{\alpha}}}\cdot\frac{t^{\rho-1}}{1-\rho}l(t)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.
\end{aligned}$$

Therefore, integrating above on $[a, t]$ it follows that

$$\int_a^t\int_s^{\infty}\left[\int_r^{\infty}(u-r)q(u)X_5(u)^{\beta}du\right]^{\frac{1}{\alpha}}drds \sim \frac{t^{\rho}\omega^{-\frac{\beta}{\alpha-\beta}}l(t)^{\frac{1}{\alpha-\beta}}}{\omega} = X_5(t), \quad t \to \infty\,,$$

verifying that $X_5(t)$ satisfies the integral asymptotic relation (63). ■

Lemma 8. *Suppose that* $q(t) \in \mathrm{RV}(\sigma)$ *satisfies* (61). *Then, the function*

$$X_6(t) = \left[\frac{\alpha-\beta}{\alpha[2\alpha(2\alpha+1)]^{\frac{1}{\alpha}}}\int_a^t\left(s^{\alpha+2}q(s)\right)^{\frac{1}{\alpha}}ds\right]^{\frac{\alpha}{\alpha-\beta}}, \quad t \geq a \tag{69}$$

satisfies the integral asymptotic relation (63).

Proof. Suppose that (61) holds. The function $X_6(t)$ is a slowly varying, so since

$$q(t)X_6(t)^\beta = t^{-2\alpha-2}l(t)X_6(t)^\beta, \quad l(t), X_6(t) \in \mathrm{SV}, \tag{70}$$

application of Proposition 2-(ii) gives

$$\int_t^\infty (s-t)q(s)X_6(s)^\beta ds \sim \frac{t^{-2\alpha}l(t)X_6(t)^\beta}{(2\alpha+1)2\alpha}, \quad t \to \infty,$$

which implies

$$\begin{aligned}\int_t^\infty \left[\int_s^\infty (r-s)q(r)X_6(r)^\beta dr\right]^{\frac{1}{\alpha}} ds &\sim \int_t^\infty \frac{s^{-2}l(s)^{\frac{1}{\alpha}}X_6(s)^{\frac{\beta}{\alpha}}}{[(2\alpha+1)2\alpha]^{\frac{1}{\alpha}}}ds \\ &\sim \frac{t^{-1}l(t)^{\frac{1}{\alpha}}X_6(t)^{\frac{\beta}{\alpha}}}{[(2\alpha+1)2\alpha]^{\frac{1}{\alpha}}}, \ t \to \infty.\end{aligned} \tag{71}$$

Integration on $[a,t]$ then implies

$$\begin{aligned}&\int_a^t \int_s^\infty \left[\int_r^\infty (u-r)q(u)X_6(u)^\beta du\right]^{\frac{1}{\alpha}} drds \\ &\sim \frac{1}{[(2\alpha+1)2\alpha]^{\frac{1}{\alpha}}}\int_a^t s^{-1}l(s)^{\frac{1}{\alpha}}X_6(s)^{\frac{\beta}{\alpha}}ds \\ &= \frac{1}{[(2\alpha+1)2\alpha]^{\frac{1}{\alpha}}}\int_a^t \left(s^{\alpha+2}q(s)\right)^{\frac{1}{\alpha}}\left[\frac{\alpha-\beta}{\alpha[2\alpha(2\alpha+1)]^{\frac{1}{\alpha}}}\int_a^s \left(r^{\alpha+2}q(r)\right)^{\frac{1}{\alpha}}dr\right]^{\frac{\beta}{\alpha-\beta}} ds \\ &\sim X_6(t), \quad t \to \infty,\end{aligned}$$

and completes the proof of Lemma 8. ■

The next Lemma will be used in the proof of the "only if"part of Theorems 9, 10 and 11.

Lemma 9. *If* $q(t) \in \mathrm{RV}(\sigma)$ *is expressed with* (8) *and* $x(t) = t^\rho \xi(t)$*,* $\xi(t) \in \mathrm{SV}$ *is a solution of* (A) *satisfying* (I_1)*, then only one of the following three statements can be valid:*

(i) $\sigma + \rho\beta = -2\alpha - 2$ *and*

$$x(t) \sim \frac{1}{\left(2\alpha(2\alpha+1)\right)^{\frac{1}{\alpha}}}\int_{t_0}^t s^{-1}l(s)^{\frac{1}{\alpha}}\xi(s)^{\frac{\beta}{\alpha}}ds \in \mathrm{SV}, \quad t \to \infty. \tag{72}$$

(ii) $-2\alpha - 2 < \sigma + \rho\beta < -\alpha - 2$ *and*

$$x(t) \sim \frac{t^{\frac{\sigma+\rho\beta+2}{\alpha}+2}l(t)^{\frac{1}{\alpha}}\xi(t)^{\frac{\beta}{\alpha}}}{W} \in \mathrm{RV}\left(\frac{\sigma+\rho\beta+2}{\alpha}+2\right), \ t \to \infty, \tag{73}$$

where

$$W=\Big[\big(-(\sigma+\rho\beta+2)\big)\big(-(\sigma+\rho\beta+1)\big)\Big]^{\frac{1}{\alpha}}\left(-\left(\frac{\sigma+\rho\beta+2}{\alpha}+1\right)\right)\left(\frac{\sigma+\rho\beta+2}{\alpha}+2\right).$$

(iii) $\sigma+\rho\beta=-\alpha-2$ *and*

$$x(t)\sim\frac{t}{[\alpha(\alpha+1)]^{\frac{1}{\alpha}}}\int_t^\infty s^{-1}l(s)^{\frac{1}{\alpha}}\xi(s)^{\frac{\beta}{\alpha}}ds\in \mathrm{RV}(1). \tag{74}$$

Proof. Let $x(t)=t^\rho\xi(t)$, $\xi(t)\in\mathrm{SV}$ be a solution of (A) on $[t_0,\infty)$ such that $1\prec x(t)\prec t$, $t\to\infty$. Then, $x(t)$ satisfies (63) and integration of (A) two times from t to ∞ implies

$$x''(t)^\alpha=\int_t^\infty(s-t)q(s)x(s)^\beta ds=\int_t^\infty(s-t)s^{\sigma+\rho\beta}l(s)\xi(s)^\beta ds,$$

implying that $\sigma+\rho\beta<-2$. Via Karamata's integration theorem we obtain

$$\left(\int_t^\infty(s-t)q(s)x(s)^\beta ds\right)^{\frac{1}{\alpha}}\sim\frac{t^{\frac{\sigma+\rho\beta+2}{\alpha}}l(t)^{\frac{1}{\alpha}}\xi(t)^{\frac{\beta}{\alpha}}}{\big[(-(\sigma+\rho\beta+2))(-(\sigma+\rho\beta+1))\big]^{\frac{1}{\alpha}}},\quad t\to\infty. \tag{75}$$

The integrability of the right-hand side of (75) on $[t_0,\infty)$ implies that $\frac{\sigma+\rho\beta+2}{\alpha}\le -1$.

(a) If $\frac{\sigma+\rho\beta+2}{\alpha}=-1$, then we integrate (75) first on $[t,\infty)$ and then on $[t_0,t]$ to obtain

$$\int_{t_0}^t\int_s^\infty\left(\int_r^\infty(u-r)q(u)x(u)^\beta du\right)^{\frac{1}{\alpha}}drds$$
$$\sim\frac{t}{[\alpha(\alpha+1)]^{\frac{1}{\alpha}}}\int_t^\infty s^{-1}l(s)^{\frac{1}{\alpha}}\xi(s)^{\frac{\beta}{\alpha}}ds,\quad t\to\infty,$$

which combined with (63) gives (74).

(b) If $\frac{\sigma+\rho\beta+2}{\alpha}<-1$, integration of (75) on $[t,\infty)$ yields

$$\int_t^\infty\left(\int_s^\infty(r-s)q(r)x(r)^\beta dr\right)^{\frac{1}{\alpha}}ds$$
$$\sim\frac{t^{\frac{\sigma+\rho\beta+2}{\alpha}+1}l(t)^{\frac{1}{\alpha}}\xi(t)^{\frac{\beta}{\alpha}}}{\big[(-(\sigma+\rho\beta+2))(-(\sigma+\rho\beta+1))\big]^{\frac{1}{\alpha}}\big(-\big(\frac{\sigma+\rho\beta+2}{\alpha}+1\big)\big)},\quad t\to\infty. \tag{76}$$

We distinguish the three cases

(b-1) $\frac{\sigma+\rho\beta+2}{\alpha}<-2$, (b-2) $-2<\frac{\sigma+\rho\beta+2}{\alpha}<-1$, (b-3) $\frac{\sigma+\rho\beta+2}{\alpha}=-2$.

The case (b-1) is impossible, since then the right-hand side of (76) is integrable on $[t_0, \infty)$, contradicting the fact that $x(\infty) = \infty$. If case (b-2) occurs, then integrating (76) on $[t_0, t]$, we obtain (73). If case (b-3) occurs, then (76) takes the form

$$\int_t^\infty \left(\int_s^\infty (r-s) q(r) x(r)^\beta dr \right)^{\frac{1}{\alpha}} ds \sim \frac{t^{-1} l(t)^{\frac{1}{\alpha}} \xi(t)^{\frac{\beta}{\alpha}}}{\big(2\alpha(2\alpha+1)\big)^{\frac{1}{\alpha}}}, \quad t \to \infty\,, \tag{77}$$

which integrated on $[t_0, t]$ gives (72). ■

4.3. Proof of main results

Proof of the "only if"part of Theorem 9. Suppose that (A) has a solution $x(t) \in \mathrm{ntr} - \mathrm{RV}(1)$ defined on $[t_0, \infty)$. Then, $x(t) = t\xi(t)$, $\xi(t) \in \mathrm{SV}$, $\xi(t) = x(t)/t \to 0$ as $t \to \infty$. It is clear that only the statement (iii) of Lemma 9 could hold. Therefore, $\sigma = -\alpha - \beta - 2$ and $x(t)$ satisfies (74), which is equivalent to

$$\xi(t) \sim \frac{1}{[\alpha(\alpha+1)]^{\frac{1}{\alpha}}} \int_t^\infty s^{-1} l(s)^{\frac{1}{\alpha}} \xi(s)^{\frac{\beta}{\alpha}} ds, \quad t \to \infty. \tag{78}$$

Let $\eta(t)$ denote the right-hand side of (78). Then, (78) can be transformed into the following differential asymptotic relation for $\eta(t)$:

$$-\eta(t)^{-\frac{\beta}{\alpha}} \eta'(t) \sim \frac{t^{-1} l(t)^{\frac{1}{\alpha}}}{[\alpha(\alpha+1)]^{\frac{1}{\alpha}}} = \frac{\big(t^{\beta+2} q(t)\big)^{\frac{1}{\alpha}}}{[\alpha(\alpha+1)]^{\frac{1}{\alpha}}}, \quad t \to \infty. \tag{79}$$

Since $\eta(t)^{1-\frac{\beta}{\alpha}} \to 0$ as $t \to \infty$, it is clear that the second condition in (56) holds and integrating (79) from t to ∞, we obtain

$$\eta(t) \sim \left[\frac{\alpha - \beta}{\alpha[\alpha(\alpha+1)]^{\frac{1}{\alpha}}} \int_t^\infty \big(s^{\beta+2} q(s)\big)^{\frac{1}{\alpha}} ds \right]^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty,$$

proving the truth of the asymptotic formula (57). ■

Proof of the "only if"part of Theorem 10. Suppose next that $x(t) \in \mathrm{RV}(\rho)$ for some $\rho \in (0,1)$. Then, we find that $x(t)$ must falls into the case (ii) of Lemma 9 satisfying (73), so that the regularity index ρ of $x(t)$ is given by

$$\rho = \frac{\sigma + \rho\beta + 2}{\alpha} + 2 \quad \text{or} \quad \rho = \frac{\sigma + 2\alpha + 2}{\alpha - \beta},$$

which yields that in (73) is $W = \omega$, with ω given by (60). Taking the assumption $\rho \in (0,1)$, the range of σ is determined with (58). It follows from (73) that

$$x(t) \sim \frac{t^{\frac{2\alpha+2}{\alpha}} q(t)^{\frac{1}{\alpha}} x(t)^{\frac{\beta}{\alpha}}}{\omega}, \quad t \to \infty,$$

which is equivalent to (59). ■

Proof of the "only if"part of Theorem 11. Suppose that (A) has a nontrivial slowly varying solution $x(t)$ defined on $[t_0, \infty)$. For this solution only the statement (i) of Lemma 9 could hold, so that $\sigma = -2\alpha - 2$ and $x(t)$ must satisfy (72). Denoting by $\eta(t)$ the right hand side of (72) we obtain the differential asymptotic relation

$$\eta'(t) = \frac{t^{-1} l(t)^{\frac{1}{\alpha}} \xi(t)^{\frac{\beta}{\alpha}}}{\big(2\alpha(2\alpha+1)\big)^{\frac{1}{\alpha}}} \sim \frac{t^{-1} l(t)^{\frac{1}{\alpha}} \eta(t)^{\frac{\beta}{\alpha}}}{\big(2\alpha(2\alpha+1)\big)^{\frac{1}{\alpha}}}, \quad t \to \infty\,,$$

implying the differential asymptotic relation

$$\eta(t)^{-\frac{\beta}{\alpha}} \eta'(t) \sim \frac{\big(t^{\alpha+2} q(t)\big)^{\frac{1}{\alpha}}}{\big(2\alpha(2\alpha+1)\big)^{\frac{1}{\alpha}}}, \quad t \to \infty\,. \tag{80}$$

From (80) one easily observes that the second condition in (61) must hold and integration of (80) on $[t_0, t]$ then shows that $x(t)$ satisfies (62). ■

Proof of the "if"part of Theorems 9, 10 and 11. Suppose that $q(t) \in \mathrm{RV}(\sigma)$ satisfies either (56) or (58) or (61). By Lemma 6, 7 and 8 functions $X_4(t)$, $X_5(t)$, $X_6(t)$ defined, respectively, by (64), (67), (69) satisfy the asymptotic relation (63). Therefore, omitting index $i \in \{4, 5, 6\}$ in the rest of the proof, there exists $t_1 > a$ so that

$$\int_{t_1}^{t} \int_{s}^{\infty} \left[\int_{r}^{\infty} (u-r) q(u) X(u)^{\beta} du \right]^{\frac{1}{\alpha}} dr ds \le 2X(t), \quad \text{for } t \ge t_1\,, \tag{81}$$

and that $X(t)$ is increasing on $[t_1, \infty)$. Let such a t_1 be fixed. Since from (63) we have

$$\int_{t_1}^{t} \int_{s}^{\infty} \left[\int_{r}^{\infty} (u-r) q(u) X(u)^{\beta} du \right]^{\frac{1}{\alpha}} dr ds \sim X(t), \quad t \to \infty,$$

there exists $t_2 > t_1$ such that

$$\int_{t_1}^{t} \int_{s}^{\infty} \left[\int_{r}^{\infty} (u-r) q(u) X(u)^{\beta} du \right]^{\frac{1}{\alpha}} dr ds \ge \frac{X(t)}{2}, \quad t \ge t_2.$$

Let such a t_2 be fixed and let $k \in (0, 1)$ be a fixed positive constant such that $k^{1-\frac{\beta}{\alpha}} \le 1/2$. Choose a constant $K > 1$ such that

$$K^{1-\frac{\beta}{\alpha}} \ge 4 \quad \text{and} \quad K \ge 2k \frac{X(t_2)}{X(t_1)}\,.$$

Considering the integral operator

$$\mathcal{G}x(t) = kX(t_2) + \int_{t_1}^{t}\int_{s}^{\infty}\left[\int_{r}^{\infty}(u-r)q(u)x(u)^{\beta}du\right]^{\frac{1}{\alpha}}drds, \quad t \geq t_1, \tag{82}$$

we may verify that $\mathcal{G}$ is a continuous self-map on the set $\mathcal{Y}$ consisting of continuous functions $x(t)$ on $[t_1, \infty)$ satisfying

$$\begin{cases} kX(t_2) \leq x(t) \leq KX(t), & \text{for } t_1 \leq t \leq t_2\,, \\ kX(t) \leq x(t) \leq KX(t), & \text{for } t \geq t_2\,, \end{cases} \tag{83}$$

and that $\mathcal{G}$ sends $\mathcal{Y}$ into relatively compact subset of $C[t_1, \infty)$. Thus, $\mathcal{G}$ has a fixed point $x(t) \in \mathcal{Y}$, which generates a solution of equation (A) of type (I_1) satisfying inequalities (83) and thus yields that

$$0 < \liminf_{t\to\infty}\frac{x(t)}{X(t)} \leq \limsup_{t\to\infty}\frac{x(t)}{X(t)} < \infty\,.$$

Denoting

$$\chi_2(t) = \int_{t_1}^{t}\int_{s}^{\infty}\left[\int_{r}^{\infty}(u-r)q(u)X(u)^{\beta}du\right]^{\frac{1}{\alpha}}drds$$

and using that $X(t) \sim \chi_2(t)$, $t \to \infty$, we get

$$0 < \liminf_{t\to\infty}\frac{x(t)}{\chi_2(t)} \leq \limsup_{t\to\infty}\frac{x(t)}{\chi_2(t)} < \infty\,.$$

Then, proceeding exactly as in the proof of the "if"part of Theorems 6-8, with application of Lemma 1, we conclude that $x(t) \sim \chi_2(t) \sim X(t)$, $t \to \infty$. Therefore, $x(t)$ is regularly varying solution of (A) with the asymptotic behavior given by (57), (59), (62), respectively, if $q(t)$ satisfies either (56) or (58) or (61). ■

5. Overall Structure of Regularly Varying Solutions of Equation (A)

In Sections 3 and 4 thorough and accurate analysis has been made of the existence and asymptotic behavior of regularly varying solutions of the fourth order nonlinear differential equation (A) under the hypotheses that $\alpha > \beta$ and $q(t)$ is a regularly varying function of index σ. We have started on the premise that condition (1) does not hold, which means by Theorem 2 that (A) always possesses trivial regularly varying solutions of index $2+1/\alpha$. From Theorems 2, 3, 4 and 5 we easily see that as the value of σ decreases the remaining classes $\mathrm{tr}-\mathrm{RV}(j), j = 2, 1, 0$, become non-void one by one until all the four classes of primitive positive solutions of (A) have members. On the other hand, the set of intermediate positive solutions of (A) is divided into the six subsets of regularly varying listed in (9) and (55), which have been the object of our accurate analysis carried out in Sections 3 and 4. It turns out that equation (A) does not always possess intermediate solutions, but that the totality of such solutions, if exist, is identical to only one of the six subsets. As the main results (Theorems 3.1 - 3.3 and Theorems 4.1 - 4.3) clarify, the membership of each of the six subsets can be completely characterized and all the members are shown to have one and the same precise asymptotic behavior at infinity.

Denote with $\mathcal{R}$ the set of all regularly varying solutions of (A), and define the subsets

$$\mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{RV}(\rho), \quad \mathrm{tr} - \mathcal{R}(\rho) = \mathcal{R} \cap \text{tr-RV}(\rho), \quad \mathrm{ntr} - \mathcal{R}(\rho) = \mathcal{R} \cap \text{ntr-RV}(\rho).$$

By combining our main theorems with Theorems 2-5 one can depict a simple and clear picture of the overall structure of regularly varying solutions of equation (A) as follows:

- If $\sigma = -\left(2 + \dfrac{1}{\alpha}\right)\beta - 1$ and $\displaystyle\int_a^\infty t^{(2+\frac{1}{\alpha})\beta} q(t)dt < \infty$, then

$$\mathcal{R} = \text{tr-}\mathcal{R}\left(2 + \frac{1}{\alpha}\right) \cup \text{ntr-}\mathcal{R}\left(2 + \frac{1}{\alpha}\right).$$

- If $\sigma \in \left(-2\beta - 2, -\left(2 + \frac{1}{\alpha}\right)\beta - 1\right)$, then

$$\mathcal{R} = \text{tr-}\mathcal{R}\left(2 + \frac{1}{\alpha}\right) \cup \mathcal{R}\left(\frac{\sigma + 2\alpha + 2}{\alpha - \beta}\right).$$

- If $\sigma = -2\beta - 2$ and $\displaystyle\int_a^\infty t^{2\beta+1} q(t)dt = \infty$, then

$$\mathcal{R} = \text{tr-}\mathcal{R}\left(2 + \frac{1}{\alpha}\right) \cup \text{ntr-}\mathcal{R}(2).$$

- If $\sigma = -2\beta - 2$ and $\displaystyle\int_a^\infty t^{2\beta+1} q(t)dt < \infty$, then

$$\mathcal{R} = \text{tr-}\mathcal{R}\left(2 + \frac{1}{\alpha}\right) \cup \text{tr-}\mathcal{R}(2).$$

- If $\sigma \in (-\alpha - \beta - 2, -2\beta - 2)$, then

$$\mathcal{R} = \text{tr-}\mathcal{R}\left(2 + \frac{1}{\alpha}\right) \cup \text{tr-}\mathcal{R}(2).$$

- If $\sigma = -\alpha - \beta - 2$ and $\displaystyle\int_a^\infty \left(t^{\beta+2} q(t)\right)^{\frac{1}{\alpha}} dt = \infty$, then

$$\mathcal{R} = \text{tr-}\mathcal{R}\left(2 + \frac{1}{\alpha}\right) \cup \text{tr-}\mathcal{R}(2).$$

• If $\sigma = -\alpha - \beta - 2$ and $\displaystyle\int_a^\infty \big(t^{\beta+2}q(t)\big)^{\frac{1}{\alpha}}dt < \infty$, then

$$\mathcal{R} = \text{tr-}\mathcal{R}\left(2+\frac{1}{\alpha}\right) \cup \text{tr-}\mathcal{R}(2) \cup \text{tr-}\mathcal{R}(1) \cup \text{ntr-}\mathcal{R}(1)\,.$$

• If $\sigma \in (-2\alpha - 2, -\alpha - \beta - 2)$, then

$$\mathcal{R} = \text{tr-}\mathcal{R}\left(2+\frac{1}{\alpha}\right) \cup \text{tr-}\mathcal{R}(2) \cup \text{tr-}\mathcal{R}(1) \cup \mathcal{R}\left(\frac{\sigma+2\alpha+2}{\alpha-\beta}\right).$$

• If $\sigma = -2\alpha - 2$ and $\displaystyle\int_a^\infty \big(t^{\alpha+2}q(t)\big)^{\frac{1}{\alpha}}dt = \infty$, then

$$\mathcal{R} = \text{tr-}\mathcal{R}\left(2+\frac{1}{\alpha}\right) \cup \text{tr-}\mathcal{R}(2) \cup \text{tr-}\mathcal{R}(1) \cup \text{ntr-}\mathcal{R}(0)\,.$$

• If $\sigma = -2\alpha - 2$ and $\displaystyle\int_a^\infty \big(t^{\alpha+2}q(t)\big)^{\frac{1}{\alpha}}dt < \infty$, then

$$\mathcal{R} = \text{tr-}\mathcal{R}\left(2+\frac{1}{\alpha}\right) \cup \text{tr-}\mathcal{R}(2) \cup \text{tr-}\mathcal{R}(1) \cup \text{tr-}\mathcal{R}(0)\,.$$

• If $\sigma < -2\alpha - 2$, then

$$\mathcal{R} = \text{tr-}\mathcal{R}\left(2+\frac{1}{\alpha}\right) \cup \text{tr-}\mathcal{R}(2) \cup \text{tr-}\mathcal{R}(1) \cup \text{tr-}\mathcal{R}(0)\,.$$

The following examples illustrate the main results proven in Sections 3 and 4.

Example 1. Consider the sub-half-linear differential equation

$$(\mathrm{A}_1) \qquad (|x''|^{\alpha-1}x'')'' + q_1(t)|x|^{\beta-1}x = 0, \quad q_1(t) = \frac{k\exp\big(\mu(\log t)^{\frac{1}{2}}\big)}{t^\gamma(\log t)^{\frac{1}{2}}} \in \mathrm{RV}(-\gamma), \quad t \ge e,$$

where $\gamma > 0$, $k > 0$ and μ are constants.

(i) Let $\gamma = (2+\frac{1}{\alpha})\beta+1$. If $\mu > 0$, then since $\int_e^\infty t^{(2+\frac{1}{\alpha})\beta}q_1(t)dt = \infty$, by Theorem 2 and Theorem 6 equation (A_1) has neither trivial nor nontrivial RV $\big(2+\frac{1}{\alpha}\big)$-solution. If, on the other hand, $\mu < 0$, then

$$\int_t^\infty s^{(2+\frac{1}{\alpha})\beta}q_1(s)ds = \frac{2k}{|\mu|}\exp\left(-|\mu|(\log t)^{\frac{1}{2}}\right),$$

and so from Theorem 2 equation (A_1) possesses trivial RV $\big(2+\frac{1}{\alpha}\big)$-solutions, but also by Theorem 6 nontrivial RV $\big(2+\frac{1}{\alpha}\big)$-solutions $x(t)$, all of which have the same asymptotic behavior

$$x(t) \sim A\,t^{2+\frac{1}{\alpha}}\exp\Big(-\frac{|\mu|}{\alpha-\beta}(\log t)^{\frac{1}{2}}\Big), \quad t \to \infty,$$

where

$$A = \left(\frac{2k(\alpha-\beta)}{\alpha|\mu|((2+\frac{1}{\alpha})(1+\frac{1}{\alpha}))^{\alpha}} \right)^{\frac{1}{\alpha-\beta}}.$$

(ii) Let $(2+\frac{1}{\alpha})\beta+1 < \gamma < 2\beta+2$. Then, from Theorem 7 it follows that (A_1) has RV(ρ)-solutions $x(t)$, with $\rho = (2\alpha+2-\gamma)/(\alpha-\beta) \in (2, 2+\frac{1}{\alpha})$, all of which behave exactly like

$$x(t) \sim (k\lambda^{-\alpha})^{\frac{1}{\alpha-\beta}} t^{\rho} (\log t)^{-\frac{1}{2(\alpha-\beta)}} \exp\Big(\frac{\mu}{\alpha-\beta}(\log t)^{\frac{1}{2}}\Big), \quad t \to \infty,$$

where λ is a constant given by (15).

(iii) Let $\gamma = 2\beta+2$. Theorem 8 implies that if $\mu < 0$, then (A_1) admits no nontrivial $\mathrm{RV}(2)$-solutions because $\int_e^{\infty} t^{2\beta+1} q_1(t)dt < \infty$, but by Theorem 3 (A_1) does possess trivial $\mathrm{RV}(2)$-solutions and that if $\mu > 0$, then since

$$\int_e^t s^{2\beta+1} q_1(s)ds \sim \frac{2k}{\mu} \exp\Big(\mu(\log t)^{\frac{1}{2}}\Big), \quad t \to \infty,$$

(A_1) possesses nontrivial $\mathrm{RV}(2)$-solutions and the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula

$$x(t) \sim \left(\frac{k(\alpha-\beta)}{2^{\alpha-1}\mu\alpha}\right)^{\frac{1}{\alpha-\beta}} t^2 \exp\Big(\frac{\mu}{\alpha-\beta}(\log t)^{\frac{1}{2}}\Big), \quad t \to \infty\,.$$

Example 2. Consider the sub-half-linear differential equation

(A_2)

$$(|x''|^{\alpha-1}x'')'' + q_2(t)|x|^{\beta-1}x = 0, \quad q_2(t) = \frac{k}{t^{\gamma}(\log t)^{\alpha}(\log\log t)^{\alpha\nu}}, \quad t \geq t_e = \exp(e),$$

where $\gamma > 0$, $k > 0$ and $\nu > 0$ are constants.

(i) Let $\gamma = \alpha+\beta+2$. By Theorem 4 and Theorem 9, if $\nu \leq 1$, then since $\int_{t_e}^{\infty} \big(t^{\beta+2} q_2(t)\big)^{\frac{1}{\alpha}} dt = \infty$, equation (A_2) has neither trivial nor nontrivial $\mathrm{RV}(1)$-solution, but if $\nu > 1$, then

$$\int_t^{\infty} \big(s^{\beta+2} q_2(s)\big)^{\frac{1}{\alpha}} ds = \frac{k^{\frac{1}{\alpha}}}{(\nu-1)(\log\log t)^{\nu-1}},$$

and hence together with trivial $\mathrm{RV}(1)$-solution, equation (A_2) possesses nontrivial $\mathrm{RV}(2)$-solutions $x(t)$ all of which exhibit the unique asymptotic behavior

$$x(t) \sim B\, t(\log\log t)^{-\frac{\alpha(\nu-1)}{\alpha-\beta}}, \quad t \to \infty,$$

where

$$B = \left(\frac{\alpha-\beta}{\alpha(\nu-1)}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{k}{\alpha(\alpha+1)}\right)^{\frac{1}{\alpha-\beta}}$$

(ii) Let $\alpha+\beta+2<\gamma<2\alpha+2$. Put $\rho=(2\alpha+2-\gamma)/(\alpha-\beta)\in(0,1)$. From Theorem 10 equation (A_2) possesses RV(ρ)-solutions $x(t)$ all of which behave exactly like

$$x(t)\sim w^{-\frac{\alpha}{\alpha-\beta}}t^{\rho}(\log t)^{-\frac{\alpha}{\alpha-\beta}}(\log\log t)^{-\frac{\alpha\nu}{\alpha-\beta}}, \quad t\to\infty,$$

where w is given by (60).

(iii) Let $\gamma=2\alpha+2$. If $\nu>1$, then $\int_{t_e}^{\infty}(t^{\alpha+2}q_2(t))^{\frac{1}{\alpha}}dt<\infty$, and so by Theorem 11 equation (A_2) has no nontrivial nontrivial SV-solution. If $\nu\le 1$, then since

$$\int_{t_e}^{t}(s^{\alpha+2}q_2(s))^{\frac{1}{\alpha}}ds\sim\begin{cases}\frac{k^{\frac{1}{\alpha}}(\log\log t)^{1-\nu}}{1-\nu} & \text{if } \nu<1;\\ k^{\frac{1}{\alpha}}\log\log\log t & \text{if } \nu=1,\end{cases}$$

we conclude from Theorem 11 that equation (A_2) possesses nontrivial SV-solutions and any such solution $x(t)$ enjoys the same asymptotic behavior

$$x(t)\sim\begin{cases}C_{\nu}(\log\log t)^{\frac{\alpha(1-\nu)}{\alpha-\beta}} & \text{if } \nu<1;\\ C_1(\log\log\log t)^{\frac{\alpha}{\alpha-\beta}} & \text{if } \nu=1,\end{cases} \quad \text{as} \quad t\to\infty,$$

where

$$C_{\nu}=\begin{cases}\left(\frac{k}{2\alpha(2\alpha+1)}\right)^{\frac{1}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha(1-\nu)}\right)^{\frac{\alpha}{\alpha-\beta}} & \text{if } \nu<1;\\ \left(\frac{k}{2\alpha(2\alpha+1)}\right)^{\frac{1}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} & \text{if } \nu=1.\end{cases}$$

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Kusano Takaŝi
Hiroshima University
Department of Mathematics
Faculty of Science
Higashi-Hiroshima 739-8526
Japan
E-mail: kusanot@zj8.so-net.ne jp

Jelena Manojlović
University of Nis,
Faculty of Science and Mathematics
Department of Mathematics
Visegradska 33, 18000 Nis, Serbia
E-mail: jelenam@pmf.ni.ac.rs

Tomoyuki Tanigawa
Kumamoto University
Department of Mathematics
Faculty of Education
2-40-1, Kurokami Kumamoto 860-8555
Japan
E-mail: tanigawa@educ.kumamoto-u.ac.jp