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MINIMAL TRANSLATION SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

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Dedicated to Professor Young Ho Kim on his sixtieth birthday

Abstract. In this paper, we define translation surfaces in the Riemannian product space $\mathbb{H}^2 \times \mathbb{R}$ and completely classify minimal translation surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

1. INTRODUCTION

A homogenous space is a Riemannian manifold M such that for every two points p and q in M, there exists an isometry of M mapping p into q. This means that the space looks the same at every point. Homogenous geometries have main roles in the modern theory of manifolds. Homogenous spaces are, in a sense, the nicest examples of Riemannian manifolds and have applications in physics [2]. To underline their importance from the mathematical point of view we roughly cite the famous Thurston conjecture. This conjecture asserts that every compact orientable 3-dimensional manifold has a canonical decomposition into pieces, each of which admits a canonical geometric structure from among the eight maximal simple connected homogenous Riemannian 3-dimensional geometries [10]. These eight spaces are: $\mathbb{E}^3, \mathbb{H}^3, \mathbb{S}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, Nil_3 and Sol_3.

Constant mean curvature and constant Gaussian curvature surfaces are one of main objects which have drawn geometers' interest for a very long time [1, 6, 7, 8, 9]. In particular, as the study of minimal surfaces, L. Euler found that the only minimal surfaces of revolution are the planes and the catenoids, and E. Catalan proved that the planes and the helicoids are the only minimal ruled surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. Also, H. F. Scherk in 1835 studied translation surfaces in $\mathbb{E}^3$ defined as graph of the function $z(x, y) = f(x) + g(y)$ and he proved that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z = \frac{1}{a} \log \left| \frac{\cos(ax)}{\cos(ay)} \right| = \frac{1}{a} \log |\cos(ax)| - \frac{1}{a} \log |\cos(ay)|,$$

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where $f(x)$ and $g(y)$ are smooth functions on some interval of $\mathbb{R}$ and a is a non-zero constant.

Recently, R. López [4] studied translation surfaces in the 3-dimensional hyperbolic space $\mathbb{H}^3$ and classified minimal translation surfaces, and J. Inoguchi, R. López and M. I. Munteanu [3] defined translation surfaces in the 3-dimensional Heisenberg group Nil_3 in terms of a pair of two planar curves lying in orthogonal planes. They classified minimal translation surfaces in Nil_3. Also, in [11] the present author and C. W. Lee considered translation surfaces in Nil_3 generated as product of two planar curves lying in planes, which are not orthogonal, and the authors classified such minimal translation surfaces. R. López and M. I. Munteanu [5] constructed translation surfaces in Sol_3 and investigated properties of minimal one.

The purpose of this paper is to study and classify minimal translation surfaces in the Riemannian product space $\mathbb{H}^2 \times \mathbb{R}$.

2. Preliminaries

Let $\mathbb{H}^2$ be represented by the upper half-plane model $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the metric $g_{\mathbb{H}} = (dx^2+dy^2)/y^2$. The space $\mathbb{H}^2$, with the group structure derived by the composition of proper affine maps, is a Lie group and the metric $g_{\mathbb{H}}$ is left invariant. Therefore the Riemannian product space $\mathbb{H}^2 \times \mathbb{R}$ is a Lie group with respect to the operation (cf. [9])

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = (\bar{x}y + x, y\bar{y}, z + \bar{z})$$

and the left invariant product metric

$$g = \frac{dx^2 + dy^2}{y^2} + dz^2.$$

With respect to the metric g an orthonormal basis of left invariant vector fields on $\mathbb{H}^2 \times \mathbb{R}$ is

$$e_1 = y\frac{\partial}{\partial x}, \quad e_2 = y\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

with the only nontrivial commutator relation $[e_1, e_2] = -e_1$. It follows that the Levi-Civita connection $\tilde{\nabla}$ of $\mathbb{H}^2 \times \mathbb{R}$ is expressed as

$$\begin{aligned}
&\tilde{\nabla}_{e_1}e_1 = e_2, && \tilde{\nabla}_{e_1}e_2 = -e_1, && \tilde{\nabla}_{e_1}e_3 = 0,\\
&\tilde{\nabla}_{e_2}e_1 = 0, && \tilde{\nabla}_{e_2}e_2 = 0, && \tilde{\nabla}_{e_2}e_3 = 0,\\
&\tilde{\nabla}_{e_3}e_1 = 0, && \tilde{\nabla}_{e_3}e_2 = 0, && \tilde{\nabla}_{e_3}e_3 = 0.
\end{aligned}$$

On the other hand, for any vectors $X = x_1e_1+y_1e_2+z_1e_3$ and $Y = x_2e_1+y_2e_2+z_2e_3$ in $\mathbb{H}^2 \times \mathbb{R}$ the cross product $\times$ is defined by

$$X \times Y = (y_1z_2 - y_2z_1)e_1 + (x_2z_1 - x_1z_2)e_2 + (x_1y_2 - x_2y_1)e_3.$$

A translation surface in the 3-dimensional Lie group equipped with a left invariant metric is a surface in the group parametrized as a product of two curves (cf. [3]). Therefore, a translation surfaces $\Sigma(\alpha, \beta)$ in $\mathbb{H}^2 \times \mathbb{R}$ is a surface parametrized by

$$x : \Sigma \to \mathbb{H}^2 \times \mathbb{R}, \quad x(s,t) = \alpha(s) * \beta(t),$$

where α and β are any generating planar curves lying in orthogonal planes of $\mathbb{R}^3$. Since the group operation $*$ is not commutative, we have two translation surfaces, namely $\Sigma(\alpha, \beta)$ and $\Sigma(\beta, \alpha)$, which are different. According to planar curves α and β, we distinguish two types as follows:

We assume that $\alpha(s)$ and $\beta(t)$ lie in the yz-plane and xy-plane of $\mathbb{R}^3$, respectively. That is,

$$\alpha(s) = (0, s, f(s)),$$
$$\beta(t) = (g(t), t, 0),$$

where $f(s)$ and $g(t)$ are smooth functions and $s, t > 0$.

In this case, we have two translation surfaces $\Sigma_1(\alpha, \beta)$ and $\Sigma_2(\beta, \alpha)$ parametrized by

$$\begin{aligned} x(s,t) &= \alpha(s) * \beta(t) \\ &= (sg(t), st, f(s)) \end{aligned}$$

and

$$\begin{aligned} x(s,t) &= \beta(t) * \alpha(s) \\ &= (g(t), st, f(s)) \end{aligned}$$

which are called the translation surfaces of type 1 and 2, respectively.

Remarks. 1. If one curve lies in the xz-plane, then the translation surface is a part of xz-plane.

2. The translation surfaces generated by $\alpha(s) = (0, c_1, s)$ and $\beta(t) = (t, c_2, 0)$ $(c_1, c_2 \in \mathbb{R}^+)$ are planes. So, translation surfaces except for Remarks 1 and 2 are meaningful for our study, because planes are trivial minimal surfaces.

3. Minimal Translation Surfaces of Type 1

Let Σ_1 be a translation surface of type 1 in Riemannian product space $\mathbb{H}^2 \times \mathbb{R}$. Then, Σ_1 is parametrized by

$$x(s,t) = (sg(t), st, f(s)) \tag{3.1}$$

for all $s > 0$ and $t > 0$. We have the natural frame $\{x_s, x_t\}$ given by

$$\begin{aligned} \frac{\partial x}{\partial s} &:= x_s = \frac{g(t)}{st} e_1 + \frac{1}{s} e_2 + f'(s) e_3, \\ \frac{\partial x}{\partial t} &:= x_t = \frac{g'(t)}{t} e_1 + \frac{1}{t} e_2. \end{aligned}$$

From this, the unit normal vector field U of Σ_1 is given by

$$U = \frac{x_s \times x_t}{||x_s \times x_t||} = -\frac{f'(s)}{wt}e_1 + \frac{f'(s)g'(t)}{wt}e_2 + \left(\frac{g(t) - tg'(t)}{wst^2}\right)e_3,$$

where $w = ||x_s \times x_t||$.

On the other hand, the coefficients of the first fundamental form of Σ_1 are

$$\begin{aligned} E &= \langle x_s, x_s\rangle = \left(\frac{g(t)}{st}\right)^2 + \frac{1}{s^2} + (f'(s))^2, \\ F &= \langle x_s, x_t\rangle = \frac{g(t)g'(t)}{st^2} + \frac{1}{st}, \\ G &= \langle x_t, x_t\rangle = \left(\frac{g'(t)}{t}\right)^2 + \frac{1}{t^2}. \end{aligned}$$

To compute the second fundamental form of Σ_1, we have to calculate the following:

$$\begin{aligned} \tilde{\nabla}_{x_s}x_s &= -\frac{2g(t)}{s^2t}e_1 + \left(\frac{g(t)^2}{s^2t^2} - \frac{1}{s^2}\right)e_2 + f''(s)e_3, \\ \tilde{\nabla}_{x_s}x_t &= -\frac{g(t)}{st^2}e_1 + \frac{g(t)g'(t)}{st^2}e_2, \\ \tilde{\nabla}_{x_t}x_t &= \left(\frac{tg''(t) - 2g'(t)}{t^2}\right)e_1 + \left(\frac{g'(t)^2 - 1}{t^2}\right)e_2, \end{aligned}$$

which imply the coefficients of the second fundamental form of Σ_1 are given by

$$\begin{aligned} L = \langle\tilde{\nabla}_{x_s}x_s, U\rangle &= \frac{1}{ws^2t^3}\big(2tf'(s)g(t) + f'(s)g(t)^2g'(t) - t^2f'(s)g'(t) \\ &\quad + stf''(s)g(t) - st^2f''(s)g'(t)\big), \\ M = \langle\tilde{\nabla}_{x_s}x_t, U\rangle &= \frac{1}{wst^3}\left(f'(s)g(t) + f'(s)g(t)g'(t)^2\right), \\ N = \langle\tilde{\nabla}_{x_t}x_t, U\rangle &= \frac{1}{wt^3}\left(-tf'(s)g''(t) + f'(s)g'(t) + f'(s)g'(t)^3\right). \end{aligned}$$

We suppose that the translation surface Σ_1 of type 1 is minimal. Then we obtain

$$\begin{aligned} &s^2f'(s)^3[t^2g''(t) - tg'(t) - tg'(t)^3] + sf''(s)[tg'(t)^3 \\ &+tg'(t) - g(t)g'(t)^2 - g(t)] + f'(s)[g(t)^2g''(t) + t^2g''(t)] = 0. \end{aligned} \tag{3.2}$$

If $f'(s) = 0$, that is, $f(s) = k$ ($k \in \mathbb{R}$), the surface Σ_1 is parametrized by

$$x(s,t) = (sg(t), st, k),$$

where $g(t)$ is an arbitrary function.

Now, we assume that $f'(s) \neq 0$ on an open interval. Since $s > 0$, divide (3.2) by $s^2 f'(s)^3$ and take the derivative with respect to s. Then we have

$$\begin{aligned} &\frac{d}{ds}\left(\frac{f''(s)}{sf'(s)^3}\right)[tg'(t)^3 + tg'(t) - g(t)g'(t)^2 - g(t)] \\ &+\frac{d}{ds}\left(\frac{1}{s^2 f'(s)^2}\right)[g(t)^2 g''(t) + t^2 g''(t)] = 0. \end{aligned}$$

Hence, we deduce the existence of a real number $a \in \mathbb{R}$ such that

$$\begin{aligned} &\frac{d}{ds}\left(\frac{f''(s)}{sf'(s)^3}\right) = -a\frac{d}{ds}\left(\frac{1}{s^2 f'(s)^2}\right), \\ &g(t)^2 g''(t) + t^2 g''(t) = a[tg'(t)^3 + tg'(t) - g(t)g'(t)^2 - g(t)]. \end{aligned} \tag{3.3}$$

Let us distinguish the following cases:

1 If $a = 0$, then $\frac{f''(s)}{sf'(s)^3} = b$ and $g''(t) = 0$, that is, $g(t) = c_1 t + c_2$ $(b, c_1, c_2 \in \mathbb{R})$.

(i) Let $b = 0$. Then $f(s) = d_1 s + d_2$ $(d_1, d_2 \in \mathbb{R}, d_1 \neq 0)$. In this case, equation (3.2) becomes $c_1(1 + c_1^2)d_1^3 s^2 t = 0$, it follows that $c_1 = 0$. Thus, the surface can be parametrize as

$$x(s,t) = (c_2 s, st, d_1 s + d_2).$$

(ii) If $b = -k^2 \neq 0$, then $f''(s) = -k^2 s f'(s)^3$ and the general solution of the ODE is given by

$$f(s) = \frac{1}{k}\ln\left(s + \sqrt{s^2 + \frac{2d_1}{k^2}}\right) + d_2, \tag{3.4}$$

Substituting (3.4) into (3.2), we easily obtain $c_1 = c_2 = 0$. Thus, $g(t) = 0$. where d_1 and d_2 are constants of integration.

(iii) If $b = k^2 \neq 0$, then the general solution of the ODE $f''(s) = k^2 s f'(s)^3$ is given by

$$f(s) = \frac{1}{k}\sin^{-1}\frac{ks}{\sqrt{2}d_1} + d_2, d_1 \neq 0$$

which implies from (3.2) we can also obtain $c_1 = c_2 = 0$, that is, $g(t) = 0$.

2 Suppose now $a \neq 0$. From the first equation in (3.3), we obtain

$$f''(s) + \frac{a}{s}f'(s) = c_1 s f'(s)^3, \tag{3.5}$$

where c_1 is a constant of integration. We put $f'(s) = p(s)$. Then we find the Bernoulli's equation as follows:

$$\frac{dp}{ds} + \frac{a}{s}p = c_1 s p^3.$$

From this, the function $p(s)$ is given by

$$p^{-2} = s^{2a}\left(\int -2c_1 s^{-2a+1} ds + c_2\right), \tag{3.6}$$

where c_2 is a constant of integration.

(i) Let $a = 1$. Then from (3.6) we have

$$f(s) = -\frac{1}{c_1}\sqrt{c_2 - 2c_1 \ln s} + c_3, \tag{3.7}$$

where $c_3 \in \mathbb{R}$ and $c_1 \neq 0$.

Substituting (3.7) into (3.2) and using the second equation in (3.3), we get

$$[(1 + c_1)t^2 + c_1 g(t)^2]g''(t) = tg'(t)[1 + g'(t)^2]. \tag{3.8}$$

If $c_1 = 0$, then equation (3.8) becomes $g''(t) - \frac{1}{t}g'(t) - \frac{1}{t}g'(t)^3 = 0$, it follows that the general solution is given by $g(t) = -\sqrt{d_1 - t^2}$. And from (3.5) $f(s) = d_2 \ln s + d_3$ $(d_1, d_2, d_3 \in \mathbb{R})$.

(ii) Let $a \neq 1$. In this case, the function $f(s)$ satisfying equation (3.5) appears in the form

$$f(s) = \frac{1}{\sqrt{|c_2|}}\int \frac{1}{s\sqrt{s^{2(a-1)} + \frac{c_1}{c_2(a-1)}}} ds. \tag{3.9}$$

Substituting (3.9) into (3.2) and using the second equation in (3.3), we get

$$s^2[a(t^2 g''(t) - tg'(t) - tg'(t)^3) + c_1(g(t)^2 g''(t) + t^2 g''(t))] = 0,$$

which implies

$$[(a + c_1)t^2 + c_1 g(t)^2]g''(t) = atg'(t)[1 + g'(t)^2]. \tag{3.10}$$

If $c_1 = 0$, then the general solution of (3.10) is given by $g(t) = -\sqrt{d_1 - t^2}$. And from (3.5) $f(s) = \frac{d_2}{1-a}s^{1-a} + d_3$ $(d_1, d_2, d_3 \in \mathbb{R})$ (see Fig. 1).

We conclude with the following:

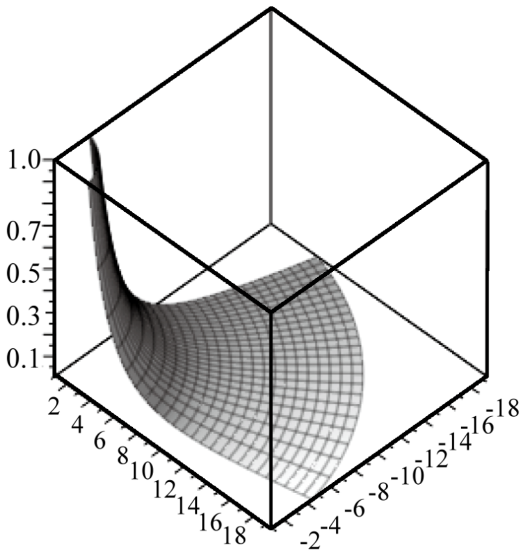

Fig. 1.

Theorem 3.1. *Let Σ_1 be a translation surface of type 1 in $\mathbb{H}^2 \times \mathbb{R}$. If Σ_1 is minimal surface, then Σ_1 is a plane or parametrized as*

$$x(s,t) = (sg(t), st, f(s)),$$

where

(1) *either $f(s) = c_1 s + c_2$ and $g(t) = c_3$ or*
(2) *$f(s) = c_1 \ln s + c_2$ and $g(t) = -\sqrt{c_3 - t^2}$ or*
(3) *$f(s) = \frac{c_1}{1-a} s^{1-a} + c_2$ and $g(t) = -\sqrt{c_3 - t^2}$, or*
(4) *$f(s) = -\frac{1}{c_1}\sqrt{c_2 - 2c_1 \ln s} + c_3$ and $g(t)$ is the function satisfying equation (3.8)*
(5) *or $f(s) = \frac{1}{\sqrt{|c_2|}} \int \frac{1}{s\sqrt{s^{2(a-1)} + \frac{c_1}{c_2(a-1)}}} ds$ and $g(t)$ is the function satisfying equation (3.10).*

4. Minimal Translation Surfaces of Type 2

Let Σ_2 be a translation surface of type 2 in Riemannian product space $\mathbb{H}^2 \times \mathbb{R}$. Then, Σ_2 is parametrized by

$$x(s,t) = (g(t), st, f(s)) \tag{4.1}$$

for all $s > 0$ and $t > 0$. It follows that we have

$$x_s = \frac{1}{s}e_2 + f'(s)e_3, \quad x_t = \frac{g'(t)}{st}e_1 + \frac{1}{t}e_2,$$

the unit normal vector U of Σ_2 is

$$U = -\frac{f'(s)}{wt}e_1 + \frac{f'(s)g'(t)}{wst}e_2 - \frac{g'(t)}{ws^2t}e_3,$$

where $w = ||x_s \times x_t||$.

On the other hand, the coefficients of the first fundamental form of Σ_2 are given by

$$E = \frac{1}{s^2} + f'(s)^2, \quad F = \frac{1}{st}, \quad G = \frac{g'(t)^2}{s^2t^2} + \frac{1}{t^2}.$$

By a straightforward computation, we get

$$\begin{aligned}
\tilde{\nabla}_{x_s}x_s &= -\frac{1}{s^2}e_2 + f''(s)e_3,\\
\tilde{\nabla}_{x_s}x_t &= -\frac{g'(t)}{s^2t}e_1,\\
\tilde{\nabla}_{x_t}x_t &= \left(\frac{tg''(t) - 2g'(t)}{st^2}\right)e_1 + \left(\frac{g'(t)^2 - s^2}{s^2t^2}\right)e_2,
\end{aligned}$$

which imply the coefficients of the second fundamental form of Σ_2 are given by

$$\begin{aligned}
L &= -\frac{g'(t)}{ws^3t}(f'(s) + sf''(s)),\\
M &= \frac{1}{ws^2t^2}f'(s)g'(t),\\
N &= \frac{1}{ws^3t^3}[f'(s)g'(t)(g'(t)^2 - s^2) - s^2f'(s)(tg''(t) - 2g'(t))].
\end{aligned}$$

Suppose that the translation surface Σ_2 is minimal. Then we have

$$\begin{aligned}
&tg''(t)[sf'(s) + s^3f'(s)^3] + g'(t)[2sf'(s) - s^3f'(s)^3 + s^2f''(s)]\\
&+ g'(t)^3[f''(s) - sf'(s)^3] = 0.
\end{aligned} \tag{4.2}$$

If $g'(t) = 0$, that is, $g(t) = c$ ($c \in \mathbb{R}$), the surface Σ_2 is parametrized by

$$x(s,t) = (c, st, f(s)),$$

where $f(s)$ is an arbitrary function.

Now, we assume that $g'(t) \neq 0$ on an open interval. Dividing (4.2) by $g'(t)$ and taking the derivative with respect to t we have

$$\frac{d}{dt}\left(\frac{tg''(t)}{g'(t)}\right)(sf'(s)+s^3f'(s)^3)+\frac{d}{dt}\left(g'(t)^2\right))(f''(s)-sf'(s)^3)=0.$$

Therefore, there exists a real number $a \in \mathbb{R}$ such that

$$\frac{d}{dt}\left(\frac{tg''(t)}{g'(t)}\right)=-a\frac{d}{dt}\left(g'(t)^2\right),$$
$$f''(s)-sf'(s)^3=a(sf'(s)+s^3f'(s)^3). \tag{4.3}$$

Let us distinguish the following cases:

1. Suppose that $a=0$. Then the first equation of (4.3) leads to $tg''(t)=bg'(t)$ ($b \in \mathbb{R}$). It follows that $g'(t)=c_1t^b$, where c_1 is a constant of integration. If $b \neq -1$, then $g(t)=\frac{c_1}{b+1}t^{b+1}+c_2$ ($c_1, c_2 \in \mathbb{R}$) and if $b=-1$, $g(t)=c_1 \ln t + c_2$ (see Fig. 2). From the second equation of (4.3), we have the ordinary differential equation $f''(s)=sf'(s)^3$, and the general solution is given by $f(s)=$ constant or $f(s)=\sin^{-1}\frac{s}{c_3}+c_4$ $(c_3 \neq 0, c_4 \in \mathbb{R})$.
2. If $a \neq 0$, then the first equation of (4.3) writes as

$$g''(t)-\frac{b}{t}g'(t)=-\frac{a}{t}g'(t)^3, \tag{4.4}$$

where b is a constant of integration. We put $g'(t)=q(t)$. Then we can obtain the Bernoulli's equation as follows:

$$\frac{dq}{dt}-\frac{b}{t}q=-\frac{a}{t}q^3$$

and its solution is given by

$$q^{-2}=\frac{1}{t^{2b}}\int 2at^{2b-1}dt. \tag{4.5}$$

(i) If $b=0$, then the general solution of (4.4) appears in the form

$$g(t)=\int \frac{1}{\sqrt{2a\ln t-d_1}}dt. \tag{4.6}$$

Substituting (4.6) into (4.2) and using the second equation in (4.3), we get

$$(2a\ln t-2d_1)[2sf'(s)-s^3f'(s)^3+s^2f''(s)]=0. \tag{4.7}$$

From this, we obtain $2sf'(s)-s^3f'(s)^3+s^2f''(s)=0$, and it's solution is $f(s)=\pm\ln\left(\frac{1+\sqrt{1+d_2s^2}}{s}\right)+d_3$ $(d_2, d_3 \in \mathbb{R})$.

(ii) If $b = 1$, then from (4.5) the function $g(t)$ is given by $g(t) = \frac{1}{a}\sqrt{c_1 + at^2} + c_2$ ($c_2 \in \mathbb{R}$). In this case, the left hand side of equation (4.2) is polynomial in t with functions of s as the coefficients. Therefore, the leading coefficient must vanish. Thus $s^2 f''(s) + 3sf'(s) = 0$ and so, $f(s) = -\frac{d_1}{2s^2} + d_2$ $(d_1, d_2 \in \mathbb{R})$ (see Fig. 3).

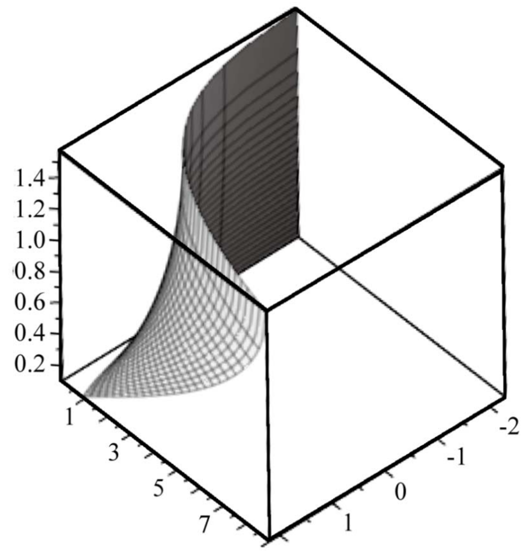

Fig. 2.

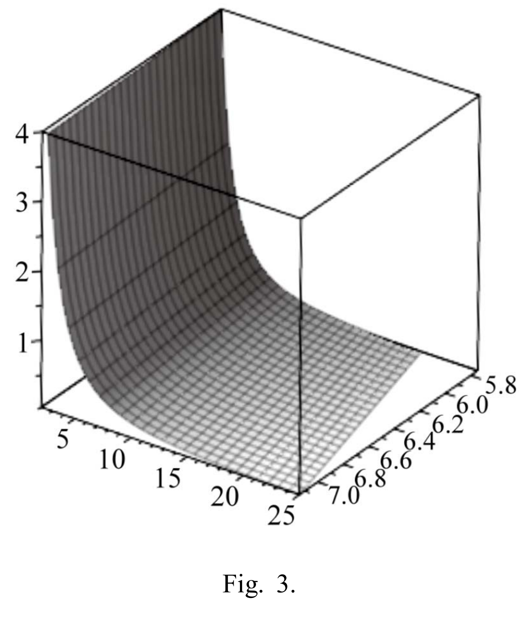

Fig. 3.

(iii) If $b \notin \mathbb{R}-\{0,1\}$, then the general solution of (4.4) is $g(t) = \sqrt{|b|}\int \frac{t^b}{\sqrt{at^{2b}+bc_1}}dt$, it follows that equation (4.2) is polynomial equation on t with functions of s as the coefficients. So, the leading coefficient must vanish, that is,

$$f''(s) + (b+2)\frac{1}{s}f'(s) = s(1-b)f'(s)^3.$$

This yields $f(s) = \int \frac{1}{s\sqrt{d_1 s^{2(b+1)} - \frac{b-1}{b+1}}}ds$ $(d_1 \in \mathbb{R})$.

Thus, we have the following:

Theorem 4.1. *Let Σ_2 be a translation surface of type 2 in $\mathbb{H}^2 \times \mathbb{R}$. If Σ_2 is minimal surface, then Σ_2 is a plane or parametrized as*

$$x(s,t) = (g(t), st, f(s)),$$

where

(1) *either* $f(s) = \sin^{-1}\frac{s}{c_3} + c_4$ *and* $g(t) = c_1 \ln t + c_2$ *or*

(2) $f(s) = \sin^{-1}\frac{s}{c_3} + c_4$ *and* $g(t) = \frac{c_1}{b+1}t^{b+1} + c_2$ *or*

(3) $f(s) = \pm\ln\left(\frac{1+\sqrt{1+d_2 s^2}}{s}\right) + d_3$ *and* $g(t) = \int \frac{1}{\sqrt{2a\ln t - d_1}}dt$ *or*

(4) $f(s) = -\frac{d_1}{2s^2} + d_2$ *and* $g(t) = \frac{1}{a}\sqrt{c_1 + at^2} + c_2$

(5) *or* $f(s) = \int \frac{1}{s\sqrt{d_1 s^{2(b+1)} - \frac{b-1}{b+1}}}ds$ *and* $g(t) = \sqrt{|b|}\int \frac{t^b}{\sqrt{at^{2b}+bc_1}}dt.$

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