ON SHARP INEQUALITIES OF HOMOGENEOUS EXPANSIONS FOR STARLIKE MAPPINGS OF ORDER α IN SEVERAL COMPLEX VARIABLES

Xiaosong Liu*, Taishun Liu and Qinghua Xu

Abstract. In this paper, we establish sharp inequalities of homogeneous expansions for starlike mappings and starlike mappings of order α defined on the unit ball of Banach complex spaces. As corollaries, we also obtain the sharp estimates of the third homogeneous expansions for the above mappings defined on the unit polydisk in $\mathbb{C}^n$ with some restricted conditions.

1. INTRODUCTION

In one complex variable, there are the following well-known and classical theorems.

Theorem A. [1]. *If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ is a univalent function on the unit disk U, then*

$$|a_3 - a_2^2| \leq 1.$$

Theorem B. [6]. *If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ is a univalent convex function on the unit disk U in $\mathbb{C}$, then*

$$|a_3 - a_2^2| \leq \frac{1 - |a_2|^2}{3}.$$

From Theorems B and Alexander's Theorem, we can easily obtain the following corollaries.

Corollary A. *If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ is a univalent convex function on the unit disc U in $\mathbb{C}$, then*

$$|3a_3 - 2a_2^2| \leq 1.$$

Corollary B.   *If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ is a univalent starlike function on the unit disc U in $\mathbb{C}$, then*

$$|2a_3 - a_2^2| \leq 2.$$

It is easy to see that we derive the sharp estimates of the third coefficients for univalent convex functions and univalent starlike functions from the above inequalities of coefficients. Therefore, it is very necessary to investigate these inequalities of coefficients.

However, there are almost no result concerning the inequalities of homogeneous expansions for subclasses of biholomorphic mappings in several complex variables up to now. In 2009, we establish the sharp inequality of homogeneous expansions for quasi-convex mappings (include quasi-convex mappings of type $\mathbb{A}$ and quasi-convex mappings of type $\mathbb{B}$) on the unit polydisc in $\mathbb{C}^n$ in [10]. Moreover, we obtain the estimate of the third homogeneous expansion for quasi-convex mappings from the sharp inequality of homogeneous expansions. At present, only a few results about the estimates of mth homogeneous expansions ($m \geq 3$) for subclasses of biholomorphic mappings are obtained. Especially, it is rare for biholomorphic starlike mappings and starlike mappings of order α. The relevant references may consult [8, 9, 11, 12, 13, 14] and [15] . These results seem to be very significant in several complex variables due to the following conjecture proposed by S. Gong.

Conjecture A. [2].   *If $f : U^n \longrightarrow \mathbb{C}^n$ is a normalized biholomorphic starlike mapping, where U^n is the open unit polydisc in $\mathbb{C}^n$, then*

$$\frac{\|D^m f(0)(z^m)\|}{m!} \leq m \, \|z\|^m, z \in U^n, m = 2, 3, \cdots.$$

In fact, the above conjecture is the Bieberbach conjecture in several complex variables. Two important reasons are that the Bieberbach conjecture for biholomorphic mappings in several complex variables does not hold and the properties of biholomorphic starlike mappings are the most analogous to biholomorphic mappings among the subclasses of biholomorphic mappings.

Let X denote a complex Banach space with the norm $\| \, . \, \|$, X^* be the dual space of X, B be the open unit ball in X, and U be the Euclidean open unit disc in $\mathbb{C}$. Also, we denote by U^n the open unit polydisc in $\mathbb{C}^n$, and $\mathbb{N}$ the set of all positive integers. Let ∂U^n denote the boundary of U^n, and $\partial_0 U^n$ mean the distinguished boundary of U^n. Let the symbol $'$ stands for the transpose of vectors and matrices. For each $x \in X \backslash \{0\}$, we define

$$T(x) = \{T_x \in X^* : \|T_x\| = 1, T_x(x) = \|x\|\}.$$

By the Hahn-Banach's theorem, $T(x)$ is nonempty.

Let $H(B)$ be the set of all holomorphic mappings from B into X. We know that if $f \in H(B)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)((y-x)^n),$$

for all y in some neighborhood of $x \in B$, where $D^n f(x)$ is the nth Fréchet derivative of f at x, and for $n \geq 1$,

$$D^n f(x)((y-x)^n) = D^n f(x)(\underbrace{y-x, \cdots, y-x}_{n}).$$

Furthermore, $D^n f(x)$ is a bounded symmetric n-linear mapping from $\prod_{j=1}^{n} X$ into X.

We say that a holomorphic mapping $f : B \to X$ is biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B$. If $f : B \to X$ is a holomorphic mapping, then we say that f is normalized if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator from X into X.

We say that a normalized biholomorphic mapping $f : B \to X$ is a starlike mapping if $f(B)$ is a starlike domain with respect to the origin.

Suppose that $\Omega \in \mathbb{C}^n$ is a bounded circular domain. The first Fréchet derivative and the mth Fréchet derivative $(m \geqslant 2)$ of a mapping $f \in H(\Omega)$ at point $z \in \Omega$ are written by $Df(z), D^m f(z)(a^{m-1}, .)$ respectively. Their matrix representations are

$$Df(z) = \left(\frac{\partial f_p(z)}{\partial z_k} \right)_{1 \leqslant p, k \leqslant n},$$

$$D^m f(z)(a^{m-1}, .) = \left(\sum_{l_1, l_2, \cdots, l_{m-1}=1}^{n} \frac{\partial^m f_p(z)}{\partial z_k \partial z_{l_1} \cdots \partial z_{l_{m-1}}} a_{l_1} \cdots a_{l_{m-1}} \right)_{1 \leqslant p, k \leqslant n},$$

where $f(z) = (f_1(z), f_2(z), \cdots, f_n(z))', a = (a_1, a_2, \cdots, a_n)' \in \mathbb{C}^n$.

Now we recall some definitions.

Definition 1.1. [5].   Suppose that $f : B \to X$ is a normalized locally biholomorphic mapping. If $\alpha \in (0, 1)$ and

$$\left| \frac{1}{\|x\|} T_x[(Df(x))^{-1} f(x)] - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad x \in B \setminus \{0\},$$

then we say that f is a starlike mapping of order α.

Let $S^*(B)$ be the set of all normalized biholomorphic starlike mappings on B, and $S_{\alpha}^*(B)$ be the set of all starlike mappings of order α on B.

In this paper, we will establish the sharp inequalities of homogeneous expansions for starlike mappings and starlike mappings of order α defined on the unit ball in

Banach complex spaces. Moreover, we will also establish the sharp estimates of the third homogeneous expansions for the above mappings defined on the unit polydisc in $\mathbb{C}^n$ with some restricted conditions from the previous sharp inequalities. Our results generalize Theorem 1.2 in [13] and our proofs are very concise.

2. Two Inequalities of Homogeneous Expansions for Starlike Mappings of Order α and Starlike Mappings

To prove the desired theorem in this section, we need the following lemmas.

Lemma 2.1. [3]. *If* $f(z) = a_0 + \sum\limits_{n=1}^{\infty} a_n z^n \in H(U)$, *and* $f(U) \subset U$, *then*

$$|a_n| \leq 1 - |a_0|^2, n = 1, 2, \cdots.$$

Lemma 2.2. [3]. *Let* $p(z) = 1 + \sum\limits_{n=1}^{\infty} b_n z^n \in H(U)$, *and* $\Re\, ep(z) > 0, z \in U.$ *Then*

$$\left| b_2 - \frac{1}{2} b_1^2 \right| \leq 2 - \frac{1}{2}|b_1|^2.$$

Furthermore,

$$(2.1) \qquad\qquad\qquad |b_2 - b_1^2| \leq 2.$$

Lemma 2.3. *Let* $p(z) = 1 + \sum\limits_{n=1}^{\infty} b_n z^n \in H(U)$, *and* $\left| p(z) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} (0 < \alpha < 1), z \in U \setminus \{0\}.$ *Then*

$$\left| b_2 - \frac{1 - 2\alpha}{2(1 - \alpha)} b_1^2 \right| \leq 2(1 - \alpha) - \frac{|b_1|^2}{2(1 - \alpha)}.$$

Furthermore,

$$(2.2) \qquad\qquad\qquad |b_2 - b_1^2| \leq 2(1 - \alpha).$$

Proof. Define

$$q(z) = \begin{cases} \frac{2\alpha p(z) - 1 - (2\alpha - 1)}{z(1 - (2\alpha - 1)(2\alpha p(z) - 1))}, & z \in U \setminus \{0\}, \\ \frac{b_1}{2(1 - \alpha)}, & z = 0. \end{cases}$$

Then $q(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots \in H(U)$, and $|q(z)| < 1.$ When $z \in U \setminus \{0\}$, we drive

$$(2.3) \qquad\qquad 2\alpha(p(z) - 1) = 2\alpha z q(z)(1 - (2\alpha - 1)p(z))$$

from the definition of $q(z)$. Considering the Taylor series expansions of the two sides of (2.3), we have

$$(2.4) \quad b_1 z + b_2 z^2 + \cdots = z(c_0 + c_1 z + \cdots)(1 - (2\alpha - 1)(1 + b_1 z + b_2 z^2 + \cdots)).$$

Comparing with the coefficients of the two sides of (2.4), we obtain

$$b_1 = c_0(2 - 2\alpha), \, b_2 = c_1(2 - 2\alpha) + c_0 b_1(1 - 2\alpha) = c_1(2 - 2\alpha) + \frac{b_1(1 - 2\alpha)}{2(1 - \alpha)}.$$

A simple computation shows that

$$c_1 = \frac{1}{2(1 - \alpha)}\left(b_2 - \frac{b_1^2(1 - 2\alpha)}{2(1 - \alpha)}\right).$$

This yields that

$$\left|b_2 - \frac{1 - 2\alpha}{2(1 - \alpha)}b_1^2\right| \le 2(1 - \alpha) - \frac{|b_1|^2}{2(1 - \alpha)}$$

from Lemma 2.1. Obviously, we see that

$$|b_2 - b_1^2| \le 2(1 - \alpha).$$

Lemma 2.4. *If f is a normalized locally biholomorphic mapping on B, and $g(x) = (Df(x))^{-1}f(x) \in H(B)$, then*

$$-\frac{D^2 f(0)(x^2)}{2!} = \frac{D^2 g(0)(x^2)}{2!},$$

$$-2 \cdot \frac{D^3 f(0)(x^3)}{3!} = \frac{D^3 g(0)(x^3)}{3!} - 2 \cdot \frac{D^2 f(0)}{2!}\left(x, \frac{D^2 f(0)(x^2)}{2!}\right).$$

Proof. Let $g(x) = (Df(x))^{-1}f(x), \, x \in B$. Then $f(x) = Df(x)g(x)$. Hence,

$$x + \frac{1}{2!}D^2 f(0)(x^2) + \frac{1}{3!}D^3 f(0)(x^3) + \cdots$$

$$= (I + \frac{2}{2!}D^2 f(0)(x, .) + \frac{3}{3!}D^3 f(0)(x^2, .) + \cdots)$$

$$(Dg(0)x + \frac{D^2 g(0)(x^2)}{2!} + \frac{D^3 g(0)(x^3)}{3!} + \cdots).$$

Comparing with the homogeneous expansions of two sides of the above equality, we obtain

$$Dg(0)x = x, \frac{D^2 f(0)(x^2)}{2!} = 2 \cdot \frac{D^2 f(0)(x^2)}{2!} + \frac{D^2 g(0)(x^2)}{2!},$$

(2.5)

$$\frac{D^3 f(0)(x^3)}{3!} = 3 \cdot \frac{D^3 f(0)(x^3)}{3!} + 2 \cdot \frac{D^2 f(0)}{2!}\left(x, \frac{D^2 g(0)(x^2)}{2!}\right) + \frac{D^3 g(0)(x^3)}{3!}.$$

We derive

$$-\frac{D^2 f(0)(x^2)}{2!} = \frac{D^2 g(0)(x^2)}{2!},$$

$$-2 \cdot \frac{D^3 f(0)(x^3)}{3!} = \frac{D^3 g(0)(x^3)}{3!} - 2 \cdot \frac{D^2 f(0)}{2!} \left(x, \frac{D^2 f(0)(x^2)}{2!} \right)$$

from (2.5). This completes the proof.

We now establish the sharp inequalities of homogeneous expansions for starlike mappings of order α and biholomorphic starlike mappings.

Theorem 2.1.   *Suppose that $\alpha \in (0, 1)$. If $f \in S_\alpha^*(B)$, then*

$$\left| 2T_x \left(\frac{D^3 f(0)(x^3)}{3!} \right) \|x\| - 2T_x \left(\frac{D^2 f(0)}{2!} \left(x, \frac{D^2 f(0)(x^2)}{2!} \right) \right) \|x\| \right.$$

$$\left. + \left(T_x \left(\frac{D^2 f(0)(x^2)}{2!} \right) \right)^2 \right| \le 2(1 - \alpha)\|x\|^4.$$

The above inequality is sharp.

Proof.   Fix $x \in B \setminus \{0\}$, and let $x_0 = \frac{x}{\|x\|}$. Define

$$p(\xi) = \begin{cases} \dfrac{T_{x_0}(g(\xi x_0))}{\xi}, & \xi \in U \setminus \{0\}, \\ 1, & \xi = 0, \end{cases}$$

where $g(x) = (Df(x))^{-1} f(x)$. We have $p \in H(U)$ and

$$(2.6) \qquad\qquad \left| p(\xi) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}$$

by the hypothesis of Theorem 2.1. Note that

$$p(\xi) = 1 + \frac{T_{x_0}(D^2 g(0)(x_0^2))}{2!} \xi + \cdots + \frac{T_{x_0}(D^m g(0)(x_0^m))}{m!} \xi^{m-1} + \cdots.$$

We conclude that

$$(2.7) \qquad \left| T_x \left(\frac{D^3 g(0)(x_0^3)}{3!} \right) - \left(T_x \left(\frac{D^2 g(0)(x_0^2)}{2!} \right) \right)^2 \right| \le 2(1 - \alpha)$$

by (2.2) and (2.6).

Also, by using Lemma 2.4, we have

$$\frac{D^3 g(0)(x_0^3)}{3!} = -2 \cdot \frac{D^3 f(0)(x_0^3)}{3!} + 2 \cdot \left(\frac{D^2 f(0)}{2!} \left(x_0, \frac{D^2 f(0)(x_0^2)}{2!} \right) \right).$$

This yields that

$$T_x\left(\frac{D^3 g(0)(x_0^3)}{3!}\right) - \left(T_x\left(\frac{D^2 g(0)(x_0^2)}{2!}\right)\right)^2$$

$$= -2T_x\left(\frac{D^3 f(0)(x_0^3)}{3!}\right) + 2T_x\left(\frac{D^2 f(0)}{2!}\left(x_0, \frac{D^2 f(0)(x_0^2)}{2!}\right)\right)$$

$$- \left(T_x\left(\frac{D^2 f(0)(x_0^2)}{2!}\right)\right)^2$$

from Lemma 2.4. Therefore,

$$\left|2T_x\left(\frac{D^3 f(0)(x_0^3)}{3!}\right) - 2T_x\left(\frac{D^2 f(0)}{2!}\left(x_0, \frac{D^2 f(0)(x_0^2)}{2!}\right)\right)\right.$$

$$\left. + \left(T_x\left(\frac{D^2 f(0)(x_0^2)}{2!}\right)\right)^2\right| \le 2(1-\alpha).$$

That is,

$$\left|2T_x\left(\frac{D^3 f(0)(x^3)}{3!}\right)\|x\| - 2T_x\left(\frac{D^2 f(0)}{2!}\left(x, \frac{D^2 f(0)(x^2)}{2!}\right)\right)\|x\|\right.$$

$$\left. + \left(T_x\left(\frac{D^2 f(0)(x^2)}{2!}\right)\right)^2\right|$$

$$\le 2(1-\alpha)\|x\|^4.$$

It is not difficult to check that

$$f(x) = \frac{T_u(x)}{(1-T_u(x))^{2(1-\alpha)}}x,\ x\in B$$

satisfies the condition of Theorem 2.1. We set $x = ru(0 \le r < 1)$. By a direct computation, we obtain that

$$\left|2T_x\left(\frac{D^3 f(0)(x^3)}{3!}\right)\|x\| - 2T_x\left(\frac{D^2 f(0)}{2!}\left(x, \frac{D^2 f(0)(x^2)}{2!}\right)\right)\|x\|\right.$$

$$\left. + \left(T_x\left(\frac{D^2 f(0)(x^2)}{2!}\right)\right)^2\right|$$

$$= 2(1-\alpha)r^4,$$

where $\|u\| = 1$. Hence, the inequality of Theorem 2.1 is sharp. This completes the proof.

When $X = \mathbb{C}$, $B = U$, we can easily see that Theorem 2.1 reduces to the following corollary.

Corollary 2.1. *Suppose that $\alpha \in (0, 1)$. If $f = z + \sum\limits_{n=2}^{\infty} a_n z^n \in S_\alpha^*(U)$, then*

$$|2a_3 - a_2^2| \le 2(1 - \alpha).$$

The above inequality is sharp.

Applying (2.1) and Lemma 2.4, we can readily deduce the following theorem, which we only state here without the proof.

Theorem 2.2. *If $f \in S^*(B)$, then*

$$\left| 2T_x \left(\frac{D^3 f(0)(x^3)}{3!} \right) \|x\| - 2T_x \left(\frac{D^2 f(0)}{2!} \left(x, \frac{D^2 f(0)(x^2)}{2!} \right) \right) \|x\| \right.$$
$$\left. + \left(T_x \left(\frac{D^2 f(0)(x^2)}{2!} \right) \right)^2 \right|$$
$$\le 2\|x\|^4.$$

The above inequality is sharp.

It is not difficult to verify that

$$f(x) = \frac{T_u(x)}{(1 - T_u(x))^2} x, x \in B$$

satisfies the condition of Theorem 2.2. With the same argument of Theorem 2.1, we can show that the inequality of Theorem 2.2 is sharp.

When $X = \mathbb{C}$, $B = U$, Theorem 2.2 is equivalent to the following corollary.

Corollary 2.2. *If $f = z + \sum\limits_{n=2}^{\infty} a_n z^n \in S^*(U)$, then*

$$|2a_3 - a_2^2| \le 2.$$

The above inequality is sharp.

3. APPLICATIONS OF INEQUALITIES

It is necessary to provide the following lemmas in order to establish the desired theorems in this section.

We can readily prove the following lemma (The proof is omitted here).

Lemma 3.1. *Suppose that f is a normalized locally biholomorphic mapping on U^n. Then $f \in S_\alpha^*(U^n)$ if and only if*

$$\left| \frac{g_j(z)}{z_j} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \, z \in U^n \setminus \{0\} \,,$$

where $g(z) = (g_1(z), g_2(z), \cdots, g_n(z))' = (Df(z))^{-1}(f(z))$ is a column vector in $\mathbb{C}^n$, and j satisfies $|z_j| = \|z\| = \max\limits_{1 \leqslant k \leqslant n} \{|z_k|\}$.

Lemma 3.2. [16]. *Suppose that f is a normalized locally biholomorphic mapping on U^n. Then $f \in S^*(U^n)$ if and only if*

$$\Re e \frac{g_j(z)}{z_j} \geqslant 0 \,, z \in U^n \,,$$

where $g(z) = (g_1(z), g_2(z), \cdots, g_n(z))' = (Df(z))^{-1}(f(z))$, and $|z_j| = \|z\| = \max\limits_{1 \leqslant k \leqslant n} \{|z_k|\}$.

Example 3.1. *Suppose that $\alpha \in (0,1)$. If $\sum\limits_{s=2}^{n} |a_s| < \frac{1-|2\alpha-1|}{2\alpha}$, then $f(z) = (z_1 + \sum\limits_{s=2}^{n} a_s z_s^2, z_2, \cdots, z_n)' \in S_\alpha^*(U^n)$.*

Proof. A straightforward calculation shows that

$$(Df(z))^{-1}f(z) = \left(z_1 - \sum_{s=2}^{n} a_s z_s^2, z_2, \cdots, z_n \right)'.$$

Therefore, by $\sum\limits_{s=2}^{n} |a_s| < \frac{1-|2\alpha-1|}{2\alpha}$, when $|z_s| < |z_1|$, $s = 2, \cdots, n$, we have

$$\begin{aligned}
\left| \frac{p_1(z)}{z_1} - \frac{1}{2\alpha} \right| &= \left| 1 - \frac{1}{2\alpha} - \sum_{s=2}^{n} a_s \frac{z_s^2}{z_1} \right| \\
&\leqslant \left| 1 - \frac{1}{2\alpha} \right| + \sum_{s=2}^{n} |a_s| \\
&< \left| 1 - \frac{1}{2\alpha} \right| + \frac{1 - |2\alpha - 1|}{2\alpha} = \frac{1}{2\alpha} \,;
\end{aligned}$$

when there exists $j (2 \leqslant j \leqslant n)$ satisfying $|z_j| \geqslant |z_1|$, then

$$\left| \frac{p_j(z)}{z_j} - \frac{1}{2\alpha} \right| = \left| 1 - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \,,$$

where $p(z) = (p_1(z), \cdots, p_n(z))' = (Dg(z))^{-1}Df(z)z$, and j satisfies $|z_j| = \|z\| = \max\limits_{1 \leqslant k \leqslant n} \{|z_k|\}$. According to Lemma 2.3, $f \in S_\alpha^*(U^n)$. This completes the proof.

Remark 3.1. With the same argument of the proof in Example 3.1, we have $f(z) = (z_1 + \sum\limits_{s=2}^{n} a_s z_s^2, z_2, \cdots, z_n)' \in S^*(U^n)$ if $\sum\limits_{s=2}^{n} |a_s| \leqslant 1$.

Remark 3.2. Example 3.1 and Remark 3.1 show that $f(z) = (z_1g_1(z), z_2g_2(z), \cdots, z_ng_n(z))'$ fails if $f(z) \in S_\alpha^*(U^n)(S^*(U^n))$, where $g_k(z) : U^n \to \mathbb{C} \in H(U^n), k = 1, 2, \cdots, n$.

Now we begin to present the desired Theorems in this section.

Theorem 3.1. *If* $f \in S_\alpha^*(U^n)(0 < \alpha < 1)$, *and*

$$2z_k\frac{D^2f_k(0)}{2!}\left(z_0, \frac{D^2f(0)(z_0^2)}{2!}\right)\|z\| - \left(\frac{D^2f_k(0)(z_0^2)}{2!}\right)^2\|z\|^2$$
$$= z_k^2(\sum_{k=1}^n a_{km}\frac{z_m}{\|z\|}(\sum_{l=1}^n a_{ml}\frac{z_l}{\|z\|}))$$

for $z \in U^n \setminus \{0\}$, *where* $k = 1, 2, \cdots, n, z_0 = \frac{z}{\|z\|}, a_{ml} = \frac{1}{2!}\frac{\partial^2 f_m(0)}{\partial z_m \partial z_l}$, *and* $m, l = 1, 2, \cdots, n$, *then*

$$\frac{\|D^3f(0)(z^3)\|}{3!} \leqslant \left(1 - \alpha + \frac{M^2}{2}\right)\|z\|^3 \leqslant \frac{(2 - 2\alpha)(3 - 2\alpha)}{2!}\|z\|^3, z \in U^n,$$

where $M = \max\limits_{1 \leqslant k \leqslant n}\{\sum\limits_{l=1}^n |a_{kl}|\}$. *The above estimate is sharp.*

Proof. For $\forall z \in U^n \setminus \{0\}$, let $z_0 = \frac{z}{\|z\|}$. Note that $T_z = \left(0, \cdots, 0, \frac{|z_j|}{z_j}, 0, \cdots, 0\right)$, where j satisfies $|z_j| = \|z\| = \max\limits_{1 \leq k \leq n}\{|z_k|\}$. According to Theorem 2.1, we have

$$\left|2\frac{D^3f_j(0)(z_0^3)}{3!}\frac{\|z\|}{z_j} - 2 \cdot \frac{D^2f_j(0)}{2!}\left(z_0, \frac{D^2f(0)(z_0^2)}{2!}\right)\frac{\|z\|}{z_j}\right.$$
$$\left. + \left(\frac{D^2f_j(0)(z_0^2)}{2!}\right)^2\frac{\|z\|^2}{z_j^2}\right| \leq 2(1 - \alpha).$$

Hence,

$$\frac{|D^3f_j(0)(z_0^3)|}{3!} \leq 1 - \alpha + \frac{M^2}{2}$$

from the hypothesis of Theorem 3.1, where $|z_j| = \|z\| = \max\limits_{1 \leq k \leq n}\{|z_k|\}$. In view of the maximum modulus theorem of holomorphic functions on the unit polydisc, Lemma 3.1, Lemmas 2.7 and 2.8 in [13], we obtain

$$\frac{\|D^3f(0)(z^3)\|}{3!} \leqslant \left(1 - \alpha + \frac{M^2}{2}\right)\|z\|^3 \leqslant \frac{(2 - 2\alpha)(3 - 2\alpha)}{2!}\|z\|^3, z \in U^n,$$

where $M = \max\limits_{1 \leqslant k \leqslant n}\{\sum\limits_{l=1}^n |a_{kl}|\}$.

The example which shows that the estimates of Theorem 3.1 are sharp is the same as the example of Theorem 1.2 in [13]. This completes the proof.

Corollary 3.1.   If $f \in S_\alpha^*(U^n)(0 < \alpha < 1)$, and $\frac{D^2 f_k(0)(z^2)}{2!} = z_k(\sum\limits_{l=1}^{n} a_{kl} z_l)$, $k = 1, 2, \cdots, n$, where $a_{kl} = \frac{1}{2!}\frac{\partial^2 f_k(0)}{\partial z_k \partial z_l}$, $k, l = 1, 2, \cdots, n$, then

$$\frac{\|D^3 f(0)(z^3)\|}{3!} \leqslant \left(1 - \alpha + \frac{M^2}{2}\right)\|z\|^3 \leqslant \frac{(2 - 2\alpha)(3 - 2\alpha)}{2!}\|z\|^3, \ z \in U^n,$$

where $M = \max\limits_{1 \leqslant k \leqslant n}\{\sum\limits_{l=1}^{n}|a_{kl}|\}$. The above estimate is sharp.

Proof.   A straightforward computation shows that $\frac{D^2 f_k(0)(z^2)}{2!} = z_k(\sum\limits_{l=1}^{n} a_{kl} z_l)(k = 1, 2, \cdots, n)$ satisfy the hypothesis of Theorem 3.1, where $a_{kl} = \frac{1}{2!}\frac{\partial^2 f_k(0)}{\partial z_k \partial z_l}$, $k, l = 1, 2, \cdots, n$. The desired result follows. This completes the proof.

Remark 3.1. Corollary 3.1 is Theorem 1.2 in [13].

Applying Lemma 3.2, Lemmas 2.7 and 2.8 in [13], we obtain the following theorem and corollary with the same arguments.

Theorem 3.2.   If $f \in S^*(U^n)$, and

$$2z_k \frac{D^2 f_k(0)}{2!}\left(z_0, \frac{D^2 f(0)(z_0^2)}{2!}\right)\|z\| - \left(\frac{D^2 f_k(0)(z_0^2)}{2!}\right)^2 \|z\|^2$$

$$= z_k^2\left(\sum\limits_{k=1}^{n} a_{km}\frac{z_m}{\|z\|}(\sum\limits_{l=1}^{n} a_{ml}\frac{z_l}{\|z\|})\right)$$

for $z \in U^n \setminus \{0\}$, *where* $k = 1, 2, \cdots, n, z_0 = \frac{z}{\|z\|}, a_{ml} = \frac{1}{2!}\frac{\partial^2 f_m(0)}{\partial z_m \partial z_l}, m, l = 1, 2, \cdots, n$, *then*

$$\frac{\|D^3 f(0)(z^3)\|}{3!} \leqslant \left(1 + \frac{M^2}{2}\right)\|z\|^3 \leqslant 3\|z\|^3, \ z \in U^n,$$

where $M = \max\limits_{1 \leqslant k \leqslant n}\{\sum\limits_{l=1}^{n}|a_{kl}|\}$. The above estimate is sharp.

Corollary 3.2. Corollary 3.2.   If $f \in S^*(U^n)$, and $\frac{D^2 f_k(0)(z^2)}{2!} = z_k(\sum\limits_{l=1}^{n} a_{kl} z_l)$, $k = 1, 2, \cdots, n$, where $a_{kl} = \frac{1}{2!}\frac{\partial^2 f_k(0)}{\partial z_k \partial z_l}$, $k, l = 1, 2, \cdots, n$, then

$$\frac{\|D^3 f(0)(z^3)\|}{3!} \leqslant \left(1 + \frac{M^2}{2}\right)\|z\|^3 \leqslant 3\|z\|^3, \ z \in U^n,$$

where $M = \max\limits_{1 \leqslant k \leqslant n} \{\sum\limits_{l=1}^{n} |a_{kl}|\}$. *The above estimate is sharp.*

Remark 3.2. Corollary 3.2 is Theorem 1.3 in [13].

REFERENCES

1. L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S.-B. Preuss, Akad. Wiss., 1916.

2. S. Gong, *The Bieberbach Conjecture*, Amer. Math. Soc., International Press, Providence, RI, 1999.

3. I. Graham and G. Kohr, *Geometric Function Theory in One and Higher Dimensions*, New York, Marcel Dekker, 2003.

4. H. Hamada and T. Honda, Sharp growth theorems and coefficient bounds for starlike mappings in several complex variables, *Chin. Ann. Math.*, **29(B)** (2008), 353-368.

5. H. Hamada, G. Kohr and P. Liczberski, Starlike mappings of order α on the unit ball in complex Banach spaces, *Glas Mat Ser III*, **36** (2001), 39-48.

6. J. A. Hummel, The coefficient regions of starlike functions, *Pacif. J. Math.,* **7** (1957), 1381-1389.

7. G. Kohr, On some best bounds for coefficients of several subclasses of biholomorphic mappings in $\mathbb{C}^n$, *Complex Var. Theory Appl.*, **36** (1998), 261-284.

8. X. S. Liu, On the quasi-convex mappings on the unit polydisc in $\mathbb{C}^n$, *J. Math. Anal. Appl.*, **335** (2007), 43-55.

9. X. S. Liu and M. S. Liu, Quasi-convex mappings of order α on the unit polydisc in $\mathbb{C}^n$, *Rocky Mountain J. of Math.*, **40** (2010), 1619-1644.

10. X. S. Liu and T. S. Liu, An inequality of homogeneous expansion for biholomorphic quasi-convex mappings on the unit polydisc and its application, *Acta Math. Sci.*, **19B** (2009), 201-209.

11. X. S. Liu and T. S. Liu, The sharp estimates of all homogeneous expansions for a class of quasi-convex mappings on the unit polydisc in $\mathbb{C}^n$, *Chin. Ann. Math.*, **32B** (2011), 241-252.

12. X. S. Liu, T. S. Liu, The sharp estimates for each item in the homogeneous polynomial expansions of a subclass of close-to-convex mappings, *Sci. China Math.*, **40** (2010), 1079-1090 (in Chinese).

13. X. S. Liu and T. S. Liu, The sharp estimate of the third homogeneous expansion for a class of starlike mappings of order α on the unit polydisc in $\mathbb{C}^n$, *Acta Math. Sci.*, **32B** (2012), 752-764.

14. T. S. Liu and X. S. Liu, A refinement about estimation of expansion coefficients for normalized biholomorphic mappings, *Sci. China Ser. A-Math.*, **48** (2005), 865-879.

15. Q. H. Xu and T. S. Liu, On coefficient estimates for a class of holomorphic mappings, *Sci. China Ser. A-Math.*, **52** (2009), 677-686.

16. T. J. Suffridge, Starlike and convex maps in Banach spaces, *Pacif. J. of Math.*, **46** (1973), 575-589.

Xiaosong Liu
School of Mathematics and Computation Science
Zhanjiang Normal University
Zhanjiang, Guangdong 524048
P. R. China
E-mail: liuxsgd@yahoo.com.cn

Taishun Liu
Department of Mathematics
Huzhou Teachers College
Huzhou, Zhejiang 313000
P. R. China
E-mail: lts@ustc.edu.cn

Qinghua Xu
College of Mathematics and Information Science
Jiangxi Normal University
Nanchang, Jiangxi 330027
P. R. China
E-mail: xuqhster@gmail.com