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# THE EXISTENCE OF HETEROCLINIC ORBITS FOR A SECOND ORDER HAMILTONIAN SYSTEM

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**Abstract.** In this paper, via variational methods and critical point theory, we study the existence of heteroclinic orbits for the following second order nonautonomous Hamiltonian system

$$\ddot{u} - \nabla F(t, u) = 0,$$

where $u \in R^n$ and $F \in C^1(R \times R^n, R), F \geq 0$. $\mathcal{M} \subset R^n$ be set of isolated points and $\sharp\mathcal{M} \geq 2$. For each $\xi \in \mathcal{M}$, there exists a positive number $\rho_0$ such that if $y \in B_{\rho_0}(\xi)$, then $F(t, y) \geq F(t, \xi)$for all $t \in R$, where $B_{\rho_0}(\xi) = \{y \in R^n \mid\mid y - \xi \mid< \rho_0\}$.Under some more assumptions on $F(t, x)$ and $\mathcal{M}$, we prove that each point in $\mathcal{M}$ is joined to another point in $\mathcal{M}$ by a solution of our system.

## 1. Introduction and Main Results

In this section, we introduce some fundamental knowledge concerned our topic and give out the main results (i.e. Theore 1.1 and Theore 1.2). Consider the following second order Hamiltonian system

$$\ddot{u} - \nabla F(t, u) = 0, \tag{1.1}$$

where $u \in R^n$ and $F \in C^1(R \times R^n, R), F \geq 0$. $\mathcal{M} \subset R^n$ be set of isolated points. We will suppose that $F$ and $\mathcal{M}$ satisfy the following assumptions:

$(F1)$ $F \in C^1(R \times R^n, R), F \geq 0, \sup_{\xi \in \mathcal{M}} \int_{-\infty}^{\infty} F(t, \xi)dt < \infty$.

$(F2)$ $\sharp\mathcal{M} \geq 2$ and $\gamma = \frac{1}{3}\inf\{\mid \xi - \eta \mid: \xi \neq \eta; \xi, \eta \in \mathcal{M}\} > 0$, if $\xi \in \mathcal{M}$, then $\nabla F(t, \xi) = 0$ for all $t \in R$.

$(F3)$ There exists a positive constant $\rho_0 < \gamma$ such that if $y \in B_{\rho_0}(\xi)$ for some $\xi \in \mathcal{M}$, then $F(t, y) \geq F(t, \xi)$ for all $t \in R$.

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$(F4)$ There exist positive numbers $\mu_1, \mu_2$ and $r_1 < \gamma$ such that if $\mid y - \xi \mid \leq r_1$ for some $\xi \in \mathcal{M}$, then $\mu_2 \mid y - \xi \mid^2 \geq F(t, y) - F(t, \xi) \geq \mu_1 \mid y - \xi \mid^2$ for all $t \in R$.

$(F5)$ There exists a $\mu_0 > 0$ such that if $F(t, \xi) \leq F(t, y) \leq F(t, \xi) + \mu_0$ for some $t \in R$ and some $\xi \in \mathcal{M}$, then $\mid y - \xi \mid \leq \rho_0$.

$(F6)$ There exists a positive constant $r_0$ such that $\sup_{x \neq y, x, y \in R^n} \frac{|\bigtriangledown F(t,x) - \bigtriangledown F(t,y)|}{|x-y|} \leq r_0$.

Here and subsequently, $\bigtriangledown F(t, x)$ denotes the gradient of $F(t, x)$ in $x$.

We say that a solution $u(t)$ of (1.1) is a *heteroclinic orbit* (i.e. heteroclinic solution) if there exisit $\xi, \eta \in R^n, \xi \neq \eta$, such that $u$ joins $\xi$ to $\beta$ , i.e.

(1.2) $$u(-\infty) \doteq \lim_{t \to -\infty} u(t) = \xi,$$

and

(1.3) $$u(+\infty) \doteq \lim_{t \to \infty} u(t) = \eta.$$

In the last years, the existence of connecting (i.e. homoclinic and heteroclinic) orbits of (1.1) have been intensively studied by many authors with the aid of critical point theory and variational methods. Among the previous studies of homoclinic orbits are those of [4-9] and heteroclinic orbits are studied for example, in [10-13].

We are motivated by [3] written by C. N. Chen. He studied the following nonautonomous second order Hamiltonian system:

(HS) $$\ddot{q} - V'(t, q) = 0,$$

where $q : R \to R^n, V \in C^2(R \times R^n, R)$ and $V'(t, y) = D_y V(t, y)$. The basic assumptions for the function $V(t, y)$ are the following:

(V1) There is a set $\mathcal{K}_1 \subset R^n$ such that if $\eta \in \mathcal{K}_1$ then $V(t, \eta) = \inf_{y \in R^n} V(t, y) = V_0 = 0$ for all $t \in R^n$.

(V2) There are positive numbers $\mu_1, \mu_2$ and $\rho_0$ such that if $\mid y - \eta \mid \leq \rho_0$ for some $\eta \in \mathcal{K}_1$ then $\mu_2 \mid y - \eta \mid^2 \geq V(t, y) - V_0 \geq \mu_1 \mid y - \eta \mid^2$ for all $t \in R$.Moreover, if $\eta_i, \eta_j \in \mathcal{K}_1$ and $i \neq j$, then $\mid \eta_i - \eta_j \mid > 8\rho_0$.

(V3) There is a $\mu_0 > 0$ such that if $V(t, y) \leq V_0 + \mu_0$ for some $t \in R$ then $\mid y - \eta \mid \leq \rho_0$ for some $\eta \in \mathcal{K}_1$.

(V4) For any $r_0 > 0$ there is an $M > 0$ such that $\sup_{t \in R} \parallel D_y^2 V(t, y) \parallel_\infty \leq M$ if $\mid y \mid \leq r_0$.

**Remark 1.1.** (i) In [3], $V \in C^2(R \times R^n, R)$, but here we only assume that $F \in C^1(R \times R^n, R)$; (ii) (V1) implies that for every $\eta \in \mathcal{K}_1$, $V(t, \eta) \equiv 0$ for all $t \in R$, but from (F2) we know that for every $\xi \in \mathcal{M}$, $F(t, \xi)$needn't equal to a constant in this paper. For the case where $n = 1$, assume $F(t, x) = (1 + \cos x) + f(t)$, where $f \in L^1(R, R^+)$, $\int_{\mathbb{R}} f(t)dt > 0$ and $\mathcal{M} = \{2k\pi + \pi \mid k \in \mathbb{Z}\}$, then it is easy to show that $F(t, x)$ satisfies (F1)-(F6), but $F(t, x)$ doesn't satisfy assumption (V1)in [3], because for every $\xi \in \mathcal{M}$, $F(t, \xi) = f(t)$ is not a constant.

In order to demonstrate a simple description of the main idea of our method. We consider the case where $\mathcal{M} = \{\xi_1, \xi_2\}$ at first. Let $U \in C^2(R, R^n)$ be a fixed function which satisfies

$$U(t) = \begin{cases} \xi_1 & if\ t \leq -1 \\ \xi_2 & if\ t \geq 1 \end{cases} \tag{1.4}$$

Let $E = W^{1,2}(R, R^n)$ with norm

$$\|u\| = \Big(\int_{-\infty}^{+\infty} [|\ \dot{u}\ |^2 + |\ u\ |^2]dt\Big)^{\frac{1}{2}}, \tag{1.5}$$

it is obvious that $u(t) \to 0$ as $|t| \to \infty$ for every $u \in E$. Define

$$\varphi_U(u) = \int_{-\infty}^{+\infty} [\frac{1}{2} \mid \dot{u} + \dot{U} \mid^2 + F(t, u + U)]dt. \tag{1.6}$$

We will prove that $\varphi_U \in C^1(E, R^+)$. Moreover, if $\varphi_U'(u) = 0$ for some $u \in E$, then the function $v(t) = U(t) + u(t)$ is a heteroclinic orbit of (1.1). Let

$$\alpha = \inf_{u \in E} \varphi_U(u) \tag{1.7}$$

It is not difficult to check that $\alpha$ is independent to the choice of $U$. A sequence $\{u_m\} \in E$ is called a *minimizing sequence* of $\varphi_U$ if $\varphi_U(u_m) \to \alpha$ as $m \to \infty$. It is well known that one of the most difficulties arised in the study of variational problem on unbounded domain is that the compact condition (i.e. Palais-Smale) may not be satisfied. Our method in this article is to search the critical point of $\varphi_U$ by investigating the convergence of the minimizing sequence.

For $k \in \mathbb{N}$, let

$$E_k = \{u \in E \mid u(t) + U(t) = \xi_1, if\ t \leq k\ \},$$

and

$$E_{-k} = \{u \in E \mid u(t) + U(t) = \xi_2, if\ t \geq -k\ \}.$$

Define

$$\alpha_k = \inf_{u \in E_k} \varphi_U(u) \tag{1.8}$$

and

$$\alpha_{-k} = \inf_{u \in E_{-k}} \varphi_U(u). \tag{1.9}$$

It is obvious that

$$\alpha_k \leq \alpha_{k+1}$$

and

$$\alpha_{-k} \leq \alpha_{-k-1}$$

for all $k \in \mathbb{N}$.

For this case, (i.e. $\sharp\mathcal{M} = 2$), in [3], the author assert that:

**Theorem A.** ([3]). *Under assumptions*$(V1) - (V4)$*, if there exists an* $k \in \mathbb{N}$ *such that*

$$\alpha < \min\{\alpha_k, \alpha_{-k}\},$$

*then there is a solution* $q(t)$ *of* $(HS)$ *which satisfies*

$$\lim_{t \to -\infty} q(t) = \xi_1,$$

*and*

$$\lim_{t \to +\infty} q(t) = \xi_2.$$

Our main result for our case is the following

**Theorem 1.1.** *Under assumptions*$(F1) - (F6)$*, if there exists an* $k \in \mathbb{N}$ *such that*

$$\alpha < \min\{\alpha_k, \alpha_{-k}\}, \tag{1.10}$$

*then there is a solution* $v(t)$ *of* $(1.1)$ *which satisfies*

$$\lim_{t \to -\infty} v(t) = \xi_1, \tag{1.11}$$

*and*

$$\lim_{t \to +\infty} v(t) = \xi_2. \tag{1.12}$$

For the case where $\sharp\mathcal{M} \geq 2$, we extend the notation as follows. Let $U_{i,j} \in C^2(R, R^n)$ be a fixed function which satisfies the follwing condition

$$U_{i,j}(t) = \begin{cases} \xi_i & if\ t \leq -1 \\ \xi_j & if\ t \geq 1. \end{cases} \tag{1.13}$$

Define

$$\alpha_{i,j} = \inf_{u \in E} \varphi_{U_{i,j}}(u). \tag{1.14}$$

For $k \in \mathbb{N}$, let

$$E_k(j,l) = \{u + \xi_j | u \in W^{1,2}([k,\infty), R^n) and\ u(k) = \xi_l - \xi_j\ \}.$$

Define

$$\alpha_k(j,l) = \inf_{u\in E_k(j,l)} \int_k^{\infty} [\frac{1}{2} \mid \dot{u} \mid^2 + F(t,u)]dt. \tag{1.15}$$

Similarly, we define

$$E_{-k}(i,l) = \{u+\xi_i | u \in W^{1,2}((-\infty,-k],R^n) and\ u(k) = \xi_l - \xi_i\ \},$$

and

$$\alpha_{-k}(i,l) = \inf_{u\in E_{-k}(i,l)} \int_{-\infty}^{-k} [\frac{1}{2} \mid \dot{u} \mid^2 + F(t,u)]dt. \tag{1.16}$$

Let

$$\bar{\alpha}_k(j) = \inf_{\xi_l\in\mathcal{M}\backslash\{\xi_j\}} \alpha_k(j,l) \qquad \bar{\alpha}_{-k}(i) = \inf_{\xi_l\in\mathcal{M}\backslash\{\xi_i\}} \alpha_k(i,l).$$

For this case, (i.e. $\sharp\mathcal{M} > 2$), in [3], the author assert that:

**Theorem B.** ([3]). *Under assumptions* $(V1)-(V4)$. *if there exists a* $k \in \mathbb{N}$*, such that*

$$\alpha_{i,j} < \min\{\bar{\alpha}_{-k}(i), \bar{\alpha}_k(j)\},$$

*then* $(HS)$ *possess a solution* $q(t)$ *which satisfies* (1.2) *and* (1.3).

Our result for this case is the following:

**Theorem 1.2.** *Under assumptions* $(F1)-(F6)$. *if there exists a* $k \in \mathbb{N}$*, such that*

$$\alpha_{i,j} < \min\{\bar{\alpha}_{-k}(i), \bar{\alpha}_k(j)\}, \tag{1.17}$$

*then* (1.1) *possess a solution* $v(t)$ *which satisfies* (1.2) *and* (1.3).

## 2. Proof of Theorem 1.1 and Theorem 1.2

Our proof is divided into a sequence of lemmas.

**Lemma 2.1.** $\varphi_U \in C^1(E,R^+)$*, and if* $u$ *is a critical point of* $\varphi_U$ *, then* $U+u$ *is a classical solution of* (1.1).

*Proof.* $(F1)$ and (1.6) imply that $\varphi_U(u) \geq 0$ for all $u \in E$. By $F \in C^1(R \times R^n, R)$, for $x,y \in R^n$,

$$F(t,x+y) = F(t,x) + \int_0^1 \langle \nabla F(t,x+sy), y\rangle ds.$$

This together with (F2) , (F6) and mean value theorem, for every $u \in E$,

$$\varphi_U(u) = \int_{-\infty}^{+\infty} [\frac{1}{2} \mid \dot{U} + \dot{u} \mid^2 + F(t,U+u)]dt$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} [\frac{1}{2} \mid \dot{u} \mid^2 + \int_0^1 \langle \bigtriangledown F(t, U+su), u\rangle ds + F(t,U)]dt \\
&\quad + \int_{-1}^{1} [\frac{1}{2} \mid \dot{U} \mid^2 + \langle \dot{U}, \dot{u}\rangle ]dt \\
&= \int_{-\infty}^{+\infty} [\frac{1}{2} \mid \dot{u} \mid^2 + \langle \bigtriangledown F(t, U+\tau u), u\rangle ]dt \\
&+ \int_{-\infty}^{+\infty} F(t,U)dt + \int_{-1}^{1} [\frac{1}{2} \mid \dot{U} \mid^2 + \langle \dot{U}, \dot{u}\rangle ]dt \\
&= \int_{-\infty}^{+\infty} [\frac{1}{2} \mid \dot{u} \mid^2 + \langle \bigtriangledown F(t, U+\tau u), u\rangle - \langle \bigtriangledown F(t,U), u\rangle + \langle \bigtriangledown F(t,U), u\rangle ]dt \\
&\quad + \int_{-1}^{1} [\frac{1}{2} \mid \dot{U} \mid^2 + \langle \dot{U}, \dot{u}\rangle ]dt + \int_{-\infty}^{\infty} F(t,U)dt \\
&\leq \int_{-\infty}^{+\infty} [\frac{1}{2} \mid \dot{u} \mid^2 + r_0\tau \mid u \mid^2]dt + \int_{-\infty}^{+\infty} \langle \bigtriangledown F(t,U), u\rangle + \int_{-1}^{1} [\frac{1}{2} \mid \dot{U} \mid^2 + \langle \dot{U}, \dot{u}\rangle ]dt \\
&\quad + \int_{-\infty}^{\infty} F(t,U)dt \\
&= \int_{-\infty}^{+\infty} [\frac{1}{2} \mid \dot{u} \mid^2 + r_0\tau \mid u \mid^2]dt + \int_{-1}^{1} \langle \bigtriangledown F(t,U), u\rangle + \int_{-1}^{1} [\frac{1}{2} \mid \dot{U} \mid^2 + \langle \dot{U}, \dot{u}\rangle ]dt \\
&\quad + \int_{-\infty}^{\infty} F(t,U)dt \\
&\leq \int_{-\infty}^{+\infty} [\frac{1}{2} \mid \dot{u} \mid^2 + r_0\tau \mid u \mid^2]dt + D
\end{aligned}
$$

where $\tau \in (0,1)$, D is a finite positive number. That is, $\varphi_U(u) < \infty$ for each $u \in E$.

Now, we are going to prove that $\varphi_U$ is differetiable for any given $u \in E$ and

$$
\langle \varphi_U'(u), \phi\rangle = \int_{-\infty}^{\infty} [\langle \dot{U} + \dot{u}, \dot{\phi}\rangle + \langle \nabla F(t, U+u), \phi\rangle ]dt
$$

for every $\phi \in E$. By (F6) and mean value theorem, we compute

$$
\begin{aligned}
&\varphi_U(u+\phi) - \varphi_U(u) - \int_{-\infty}^{\infty} [\langle \dot{U} + \dot{u}, \dot{\phi}\rangle + \langle \nabla F(t, U+u), \phi\rangle ]dt \\
&= \int_{-\infty}^{\infty} [\frac{1}{2} \mid \dot{\phi} \mid^2 + F(t, U+u+\phi) - F(t, U+u) - \langle \nabla F(t, U+u), \phi\rangle ]dt \\
&= \int_{-\infty}^{\infty} [\frac{1}{2} \mid \dot{\phi} \mid^2 + \int_0^1 \langle \nabla F(t, U+u+s\phi), \phi\rangle ds - \langle \nabla F(t, U+u), \phi\rangle ]dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} [\frac{1}{2} \mid \dot{\phi} \mid^2 + \langle \nabla F(t, U+u+\tau\phi), \phi\rangle - \langle \nabla F(t, U+u), \phi\rangle] dt \\
&= \int_{-\infty}^{\infty} [\frac{1}{2} \mid \dot{\phi} \mid^2 + \langle \nabla F(t, U+u+\tau\phi) - \nabla F(t, U+u), \phi\rangle] dt \\
&\leq \int_{-\infty}^{\infty} [\frac{1}{2} \mid \dot{\phi} \mid^2 + r_0\tau \mid \phi \mid^2 \\
&\leq C \parallel \phi \parallel^2,
\end{aligned}
$$

where $\tau \in (0,1)$ and $C = \max\{\frac{1}{2}, r_0\tau\}$. Thus

$\varphi_U(u+\phi) - \varphi_U(u) - \langle \varphi'_U(u), \phi\rangle \to 0(\parallel \phi \parallel)$, as $\parallel \varphi \parallel \to 0$.

Furthermore, for $u_1, u_2, \phi \in E$, where $\parallel \phi \parallel = 1$ , by $(F6)$, then

$$
\begin{aligned}
&\langle \phi'_U(u_1), \phi\rangle - \langle \phi'_U(u_2), \phi\rangle \\
&= \int_{-\infty}^{\infty} [\langle \dot{u_1} - \dot{u_2}, \dot{\phi}\rangle + \langle \nabla F(t, U+u_1) - \nabla F(t, U+u_2), \phi\rangle] dt \\
&\leq \int_{-\infty}^{\infty} [\mid \langle \dot{u_1} - \dot{u_2}, \dot{\phi}\rangle \mid + \mid \nabla F(t, U+u_1) - \nabla F(t, U+u_2) \mid\mid \phi \mid] dt \\
&\leq \parallel \dot{u_1} - \dot{u_2} \parallel_{L^2} \cdot \parallel \dot{\phi} \parallel_{L^2} + r_0 \parallel u_1 - u_2 \parallel_{L^2} \cdot \parallel \phi \parallel_{L^2} \\
&\leq (1+r_0) \parallel u_1 - u_2 \parallel \parallel \phi \parallel .
\end{aligned}
$$

This implies that $\varphi'_U$ is continuous.

Since $\phi \in C_0^\infty$ implies $\phi \in E$ , if $u \in E$ is a critical point of the functional $\varphi_U$, then $\langle \phi'_U(u), \phi\rangle = 0$ for any $\varphi \in C_0^\infty$, that is $U + u$ is a weak solution of (1.1). By standard regularity argument we know that $U + u$ is a classical solution of (1.1).

**Remark 2.1.** (i) By (1.6) it is easy to show that $\varphi_U : E \to R$ is weakly lower semi-continuous; (ii) Lemma 1 shows that $\varphi_U : E \to R$ bounded from below and differentiable on $E$. Thus by Corollary 4.1 in [1], there exists a minimizing sequence $(u_k) \subset E$ of $\varphi_U$ such that $\varphi'_U(u_k) \to 0$ and $\varphi_U(u_k) \to \alpha$ as $k \to \infty$.

**Lemma 2.2.** (see [3]). *For any* $t_1, t_2 \in R, u \in W^{1,2}([t_1, t_2], R^n)$ *and* $\rho \in (0, \rho_0]$, *if* $\inf_{t\in[t_1,t_2],\xi\in\mathcal{M}} \mid u(t) - \xi \mid \geq \rho$, *then*

$$
\int_{t_1}^{t_2} F(t,u)dt \geq (t_2 - t_1)\theta(\rho), \tag{2.1}
$$

*where* $\theta(\rho) = \min\{\mu_1\rho^2, \mu_0\}$.

*Proof.* $(F4)$ and $(F5)$ imply (2.1) .

**Lemma 2.3.** (see [3]). *Let $\rho \in (0, \rho_0]$ and $\theta(\rho)$ is the same as in Lemma 2.2, suppose that $u(t_1) \in \partial B_\rho(\xi_i)$, $u(t_2) \in \partial B_\rho(\xi_j)$ for some $\xi_i, \xi_j \in \mathcal{M}$and $u(t)\overline{\in}\bigcup_{\xi\in\mathcal{M}} B_\rho(\xi)$ for $t \in (t_1, t_2)$. If $i \neq j$, then*

$$\int_{t_1}^{t_2}[\frac{1}{2} \mid \dot{u}(t) \mid^2 + F(t,u)]dt \geq \frac{1}{2(t_2-t_1)}(\mid \xi_i - \xi_j \mid -2\rho)^2 + \theta(\rho)(t_2-t_1). \tag{2.2}$$

*For the convenience of readers, we give out the detail of the proof as follow.*

*Proof.* $\| \xi_i - \xi_j \mid -2\rho \mid \leq \mid u(t_1) - u(t_2) \mid = \mid \int_{t_1}^{t_2} \dot{u}(t)dt \mid \leq \sqrt{t_2 - t_1}(\int_{t_1}^{t_2} \mid \dot{u} \mid^2 dt)^{1/2}$. Thus

$$\int_{t_1}^{t_2} \mid \dot{u}(t) \mid^2 dt \geq \frac{1}{t_2 - t_1}(\mid \xi_i - \xi_j \mid -2\rho)^2,$$

this together with lemma 2.2 yields (2.2).

**Lemma 2.4.** *Let $\{u_m\} \subset E$ be a sequence such that $\varphi_U(u_m) \to \alpha$ and $\varphi_U'(u_m) \to 0$, as $m \to \infty$. Then there exists a positive constant $C_0$ such that*

$$\sup_m \| \dot{u}_m \|_{L^2(R)} \leq C_0 \tag{2.3}$$

*Furthermore, $\{u_m\}$ is bounded in $W^{1,2}_{loc}(R, R^n)$.*

*Proof.* By Remark 2.1 (ii), there indeed exists $\{u_m\} \subset E$ such that $\varphi_U(u_m) \to \alpha$ and $\varphi_U'(u_m) \to 0$, as $m \to \infty$. Since $\{\varphi_U(u_m)\}$ is a bounded sequence in $\mathbb{R}$, without loss of generality, we assume that

$$\varphi_U(u_m) \leq \alpha + 1$$

for all $m \in \mathbb{N}$. By (1.4), (1.6) and $(F1)$

$$\begin{aligned}
\varphi_U(u_m) &= \int_{-\infty}^{-1}[\frac{1}{2} \mid \dot{u}_m \mid^2 + F(t, \xi_1 + u_m)]dt + \int_{-1}^{1}[\frac{1}{2} \mid \dot{U} + \dot{u}_m \mid^2 + F(t, U + u_m)]dt \\
&\quad + \int_{1}^{\infty}[\frac{1}{2} \mid \dot{u}_m \mid^2 + F(t, \xi_2 + u_m)]dt \\
&\geq \int_{-\infty}^{-1} \frac{1}{2} \mid \dot{u}_m \mid^2 dt + \int_{-1}^{1} \frac{1}{2} \mid \dot{U} + \dot{u}_m \mid^2 dt + \int_{1}^{\infty} \frac{1}{2} \mid \dot{u}_m \mid^2 dt.
\end{aligned}$$

Thus

$$\int_{-\infty}^{-1} \frac{1}{2} \mid \dot{u}_m \mid^2 dt + \int_{-1}^{1} \frac{1}{2} \mid \dot{U} + \dot{u}_m \mid^2 dt + \int_{1}^{\infty} \frac{1}{2} \mid \dot{u}_m \mid^2 dt \leq \alpha + 1$$

for all $m \in \mathbb{N}$, that is

$$\begin{aligned}\alpha+1 &\geq \int_{-\infty}^{\infty} \frac{1}{2} \mid \dot{u}_m \mid^2 dt + \int_{-1}^{1} [\langle \dot{U}, \dot{u}_m \rangle + \frac{1}{2} \mid \dot{U} \mid^2] dt \\ &\geq \int_{-\infty}^{\infty} \frac{1}{2} \mid \dot{u}_m \mid^2 dt - \int_{-1}^{1} \mid \dot{U} \mid \cdot \mid \dot{u}_m \mid dt + \frac{1}{2} \int_{-1}^{1} \mid \dot{U} \mid^2 dt \\ &\geq \int_{-\infty}^{\infty} \frac{1}{2} \mid \dot{u}_m \mid^2 dt - \int_{-1}^{1} (\mid \dot{U} \mid^2 + \frac{1}{4} \mid \dot{u}_m \mid^2) dt + \frac{1}{2} \int_{-1}^{1} \mid \dot{U} \mid^2 dt \\ &\geq \frac{1}{4} \int_{-\infty}^{\infty} \mid \dot{u}_m \mid^2 dt - \frac{1}{2} \int_{-1}^{1} \mid \dot{U} \mid^2 dt.\end{aligned}$$

This implies that (2.3) holds for some $C_0$(for example, under the assumption $\varphi_U(u_m) \leq \alpha + 1$ for all $m \in \mathbb{N}$, we can choose $C_0 = [4(\alpha+1) + 2\int_{-1}^{1} \mid \dot{U} \mid^2 dt]^{\frac{1}{2}}$).

Now, we are going to prove that $\{u_m\}$ is bounded in $W^{1,2}_{loc}(R, R^n)$. Once more we assume that $\varphi_U(u_m) \leq \alpha + 1$ for all $m \in \mathbb{N}$. Let

$$d_m(\tau) = \inf_{\xi \in \mathcal{M}} \{\mid U(\tau) + u_m(\tau) - \xi \mid\} \tag{2.4}$$

and

$$S_m = \{\tau \in \mathbb{R} \mid d_m(\tau) < \rho_0\}, \tag{2.5}$$

we claim that

$$S_m \cap [-\hat{n} + t, \hat{n} + t] \neq \emptyset \tag{2.6}$$

for any $t \in \mathbb{R}$, where $\hat{n} = \frac{\alpha+2}{2\theta(\rho_0)}$. If (2.6) is false, then there exists some $t_0 \in \mathbb{R}$, such that

$$S_m \cap [-\hat{n} + t_0, \hat{n} + t_0] = \emptyset$$

i.e. $v_m(t) = U(t) + u_m$ satisfies

$$\inf_{\xi \in \mathcal{M}} \{\mid v_m(t) - \xi \mid\} \geq \rho_0$$

for all $t \in [-\hat{n} + t_0, \hat{n} + t_0]$. By lemma 2.2,

$$\varphi_U(u_m) \geq \int_{-\hat{n}+t_0}^{\hat{n}+t_0} F(t, U + u_m) dt \geq \theta(\rho_0) \cdot 2\hat{n} = \alpha + 2 > \alpha + 1$$

This is obviously contrary to the hypothesis $\varphi_U(u_m) \leq \alpha + 1$ for all $m \in \mathbb{N}$. Choose a $t_m \in S_m \cap [-\hat{n} + t, \hat{n} + t]$, by (2.3) and (2.4)

$$\mid u_m(t) \mid \leq \mid u_m(t_m) \mid + \mid \int_{t_m}^{t} \dot{u}_m(s) ds \mid \leq \parallel U \parallel_{L^\infty(R)} + \sup_{\xi \in \mathcal{M}} \mid \xi \mid + \rho_0 + \sqrt{2\hat{n}} C_0. \tag{2.7}$$

We divide $\mathcal{M}$ into two cases:

Case one, $\mathcal{M}$ is a bounded subset of $R^n$, then there exists some positive constant $L$ such that $\sup_{\xi\in\mathcal{M}} \mid \xi \mid\leq L$, this together with (2.7) shows that for each given $s > 0$, there exists a positive number $R_s$ depend on $s$ but not on $m$, such that

$$\| u_m \|_{W^{1,2}([-s,s],R^n)}\leq R_s,$$

that is $\{u_m\}$ is bounded in $W^{1,2}_{loc}(R, R^n)$.

Case two, $\mathcal{M}$ is a unbounded subset of $R^n$, let

$$\mathcal{M}(m) = \{\xi \in \mathcal{M} \mid \text{there exists a } t \in \mathbb{R} \text{ such that } U(t) + u_m(t) \in B_{\rho_0}(\xi)\}.$$

By relabeling the elements of $\mathcal{M}$ if necessary, we assume that

$$\lim_{t\to-\infty}(U + u_m)(t) = \xi_1 \text{ and } \lim_{t\to\infty}(U + u_m)(t) = \xi_2.$$

Thus $\xi_1, \xi_2 \in \mathcal{M}(m)$ for every $m \in \mathbb{N}$, we are going to prove that

$$\sharp\mathcal{M}(m) < \infty. \tag{2.8}$$

For any fixed $m$, there exists $t_1, s_1 \in \mathbb{R}$ and $\bar{\xi} \in \mathcal{M}(m)\backslash\{\xi_1\}$, such that $(U+u_m)(t_1) \in B_{\rho_0}(\xi_1)$, $(U + u_m)(s_1) \in B_{\rho_0}(\bar{\xi})$ and $(U + u_m)(t)\bar{\in}(\cup_{\xi\in\mathcal{M}}B_{\rho_0}(\xi))$ for $t \in (t_1, s_1)$, this together with lemma 2.3 and $F(2), F(3)$ shows that

$$\begin{aligned} \alpha + 1 \geq \varphi_U(u_m) &\geq \int_{t_1}^{s_1}[\frac{1}{2} \mid \dot{U} + \dot{u}_m \mid^2 +F(t, U + u_m)]dt \\ &\geq \frac{1}{2(s_1 - t_1)}(\mid \xi_1 - \bar{\xi} \mid -2\rho_0)^2 + \theta(\rho_0)(s_1 - t_1) \\ &\geq \frac{\rho_0^2}{2(s_1 - t_1)} + \theta(\rho_0)(s_1 - t_1) \geq \rho_0\sqrt{\frac{\theta(\rho_0)}{2}}. \end{aligned} \tag{2.9}$$

Furthermore, (2.9) implies that

$$\theta(\rho_0)(s_1 - t_1) \leq \alpha + 1, \tag{2.10}$$

and

$$(\mid \xi_1 - \bar{\xi} \mid -2\rho_0)^2 \leq 2(s_1 - t_1)(\alpha + 1). \tag{2.11}$$

Thus

$$\mid \xi_1 - \bar{\xi} \mid\leq (\alpha + 1)\sqrt{\frac{2}{\theta(\rho_0)}} + 2\rho_0$$

For any $\xi_i, \xi_j \in \mathcal{M}, \xi_i \neq \xi_j$, if $(U + u_m)(t_i) \in B_{\rho_0}$, $(U + u_m)(s_i) \in B_{\rho_0}$ and $(U + u_m)(t)\bar{\in}(\cup_{\xi\in\mathcal{M}}B_{\rho_0}(\xi))$ for $t \in (t_i, s_i)$, then for the same reasoning as above shows that

$$\int_{t_i}^{s_i}[\frac{1}{2} \mid \dot{U} + \dot{u}_m \mid^2 +F(t, U + u_m)]dt \geq \rho_0\sqrt{\frac{\theta(\rho_0)}{2}}$$

and

$$|\xi_i - \xi_j| \leq (\alpha+1)\sqrt{\frac{2}{\theta(\rho_0)}} + 2\rho_0. \tag{2.12}$$

For

$$u_m \in E, \alpha+1 \geq \varphi_U(u_m) \geq \sum_{i=1}^{\sharp\mathcal{M}(m)} \int_{t_i}^{s_i} [\frac{1}{2} | \dot{U} + \dot{u}_m |^2 + F(t, U+u_m)]dt$$
$$\geq \sharp\mathcal{M}(m)\rho_0\sqrt{\frac{\theta(\rho_0)}{2}}.$$

Thus

$$Card(\mathcal{M}(m)) \leq \frac{\alpha+1}{\rho_0}\sqrt{\frac{2}{\theta(\rho_0)}},$$

i.e. (2.8) is true. By (2.12), for any $\xi \in \mathcal{M}(m)$

$$|\xi - \xi_1| \leq \sharp\mathcal{M}(m) \cdot ((\alpha+1)\sqrt{\frac{2}{\theta(\rho_0)}} + 2\rho_0) \tag{2.13}$$

Replace $\sup_{\xi\in\mathcal{M}} | \xi |$ in (2.7) by

$$|\xi_1| + (\frac{\alpha+1}{\rho_0}\sqrt{\frac{2}{\theta(\rho_0)}})((\alpha+1)\sqrt{\frac{2}{\theta(\rho_0)}} + 2\rho_0),$$

thus $u_m(t) \in W^{1,2}_{loc}(R, R^n)$.

**Corollary 2.1.** *If* $\{u_m\} \subset E$ *is a sequence such that* $\varphi_U(u_m) \to \alpha$ *and* $\varphi'_U(u_m) \to 0$*, then* $u_m \in L^\infty(R, R^n)$.

*Proof.* It is directly from lemma 2.4.

*Proof of Theorem 1.1.* Let $\{u_m\} \subset E$ be a sequence such that $\varphi_U(u_m) \to \alpha$ and $\varphi'_U(u_m) \to 0$ as $m \to \infty$, by lemma 2.4, $\{u_m\}$ is bounded in $W^{1,2}_{loc}(R, R^n)$. By the reflexivity of $W^{1,2}_{loc}(R, R^n)$, there exists a subsequence of $\{u_m\}$, for convenience, also denoted by $\{u_m\}$ and a $u \in W^{1,2}_{loc}(R, R^n)$ such that $u_m \rightharpoonup u$ in $W^{1,2}_{loc}(R, R^n)$ and $u_m \to u$ in $L^\infty_{loc}$. Furthermore, it follows from Corollary 2.1 that $u \in L^\infty(R, R^n)$. For each $l \in \mathbb{N}$, let

$$a_l(u) = \int_{-l}^{l} [\frac{1}{2} | \dot{U} + \dot{u} |^2 + F(t, U+u)]dt,$$

then $a_l(u)$ is weakly lower semi-continuous on $W^{1,2}([-l, l], R^n)$

$$a_l(u) \leq \liminf_{m\to\infty} a_l(u_m).$$

Thus

$$\lim_{l\to\infty} a_l(u) \le \lim_{l\to\infty} \liminf_{m\to\infty} a_l(u_m) \le \alpha,$$

i.e.

$$\varphi_U(u) = \lim_{l\to\infty} a_l(u) \le \alpha. \tag{2.14}$$

Let $v = U + u$, we are going to prove that

$$v(-\infty) \doteq \lim_{t\to-\infty} v(t) = \xi_1 and v(\infty) \doteq \lim_{t\to\infty} v(t) = \xi_2. \tag{2.15}$$

It is obviously that (2.15) will be satisfied if $u \in E$. Let

$$d(t) = \inf_{\xi\in\mathcal{M}} \{\mid v(t) - \xi \mid\}, S_\rho = \{t \in \mathbb{R} \mid d(t) < \rho\}, \hat{S} = \mathbb{R} \setminus S_\rho$$

It follows from lemma 2.2 that $meas\hat{S} < l(\rho)$, where $l(\rho) = \frac{\alpha+2}{\theta(\rho)}$. We claim that

$(*)$ there exists a $\rho_1 \in (0, r_1]$ such that $v(t)\overline{\in} B_{\rho_1}(\xi_1)$ if $t > k + 1$.

If not, then for any $\rho \in (0, r_1]$, there exists a $t_\rho \in (k+1, \infty)$, such that $v(t_\rho) \in B_\rho(\xi_1)$. For $m \in \mathbb{N}$ and $\rho \in (0, r_1]$, define

$$U_{m,\rho}(t) = \begin{cases} \xi_1 - U(t) & if t \le t_\rho - \rho \\ \frac{t_\rho - t}{\rho}(\xi_1 - U(t_\rho - \rho)) + \frac{t-(t_\rho-\rho)}{\rho} u_m(t_\rho) & if t \in (t_\rho - \rho, t_\rho) \\ u_m & if t \ge t_\rho \end{cases}$$

Then $U_{m,\rho} \in E_k$ if $\rho < 1$. Moreover

$$\begin{aligned} \varphi_U(U_{m,\rho}) &= \int_{t_\rho-\rho}^{t_\rho} [\frac{1}{2} \mid \dot{U} + \dot{U}_{m,\rho} \mid^2 + F(t, U_{m,\rho} + U)]dt \\ &\quad + \int_{t_\rho}^{\infty} [\frac{1}{2} \mid \dot{u}_m + \dot{U} \mid^2 + F(t, u_m + U)]dt \\ &\le \int_{t_\rho-\rho}^{t_\rho} \frac{1}{\rho^2} \mid U(t_\rho - \rho) - \xi_1 + u_m(t_\rho) \mid^2 dt \\ &\quad + \int_{t_\rho-\rho}^{t_\rho} F(t, U_{m,\rho} + U)dt + \varphi_U(u_m). \end{aligned}$$

It is directly from $F \in C^1(R \times R^n, R)$ and the definition of $U_{m,\rho}$ that

$$\int_{t_\rho-\rho}^{t_\rho} F(t, U_{m,\rho} + U)dt \le b\rho$$

where $b$ is a constant independent of $m$ and $\rho$. Moreover,

$$\begin{aligned}&| U(t_\rho-\rho)-\xi_1+u_m(t_\rho) |\\ \le&| U(t_\rho-\rho)-U(t_\rho) | + | u_m(t_\rho)+U(t_\rho)-v(t) | + | v(t)-\xi_1 |\\ \le&\rho | \dot{U} |_{L^\infty} + | u_m(t_\rho)-u(t_\rho) | +\rho.\end{aligned}$$

Choose $\overline{m}=\overline{m}(\rho)$ large enough, such that $| u_m(t_\rho)-u(t_\rho) |<\rho$ and $\varphi_U(u_m)<\alpha+\rho$ for all $m\ge\overline{m}(\rho)$, then

$$\varphi_U(U_{m,\rho})\le(\| \dot{U} \|_{L^\infty}+2)^2\rho+b\rho+\rho+\alpha, \tag{2.16}$$

this together with (1.10) implies that

$$\varphi_U(U_{m,\rho})<\alpha_k=\inf_{u\in E_k}\varphi_U(u)$$

for $\rho$ small enough, but this is impossible for $U_{m,\rho}\in E_k$, so $(*)$ must be true. By the same method, there exists a $\rho_2\in(0,r_1]$ such that $v(t)\bar{\in}B_{\rho_2}(\xi_2)$ if $t<-k-1$. Thus $v(t)\in B_{\rho_1}(\xi_2)$ if $t\in S_{\rho_1}\cap(k+1,\infty)$ and $v(t)\in B_{\rho_2}(\xi_1)$ if $t\in S_{\rho_2}\cap(-\infty,-k-1)$. Let $\bar{\rho}=\min\{\rho_1,\rho_2\}$, $A_1=S_{\bar{\rho}}\cap(-\infty,-k-1)$, $A_2=S_{\bar{\rho}}\cap(k+1,\infty)$ and $A_3=\mathbb{R}\backslash(A_1\cup A_2)$, By (2.14) and $(F4)$

$$\alpha\ge\varphi_U(u)\ge\int_{-\infty}^{\infty}\frac{1}{2} | \dot{U}+\dot{u} |^2 dt+\int_{A_1\cup A_2}\mu_1 | u |^2+\int_{A_3}F(t,U+u)dt$$

thus,

$$\int_{A_3}F(t,U+u)dt\le\alpha.$$

This together with lemma 2.2 shows that

$$meas\ A_3\cdot\theta(\bar{\rho})\le\alpha,$$

i.e.

$$meas\ A_3\le\frac{\alpha}{\theta(\bar{\rho})}. \tag{2.17}$$

We claim that there exists a $T>0$ large enough such that $v(t)\in B_{\bar{\rho}}(\xi_2)$ for all $t>T$. $(**)$ If not, from the discussion above, we may choose $T>k+1$ such that $v(t)\bar{\in}B_{\bar{\rho}}(\xi_1)$ for all $t>T$. If $(**)$ is of not the case, there must be two sequences $\{T_i\},\{T_i'\}\subset\mathbb{R}$ such that $T_i\to\infty, T_i'\to\infty$ as $i\to\infty$. Furthermore, $T_i$ and $T_i'$ possess the following propositions:

$$T_i<T_i'<T_{i+1},\qquad v(T_i)\in\partial B_{\bar{\rho}}(\xi_2),\qquad v(T_i')\in\partial B_{\bar{\rho}}(\xi_2)$$

and

$$v(t)\bar{\in}B_{\bar{\rho}}(\xi_2) \qquad \text{if } t\in\cup_{i=1}^{\infty}(T_i,T_i').$$

By lemma 2.2 and (1.6)

$$\varphi_U(u)\ge\int_{\mathbb{R}}F(t,v(t))dt\ge\int_{\cup_{i=1}^{\infty}[T_i,T_i']}F(t,v(t))dt\ge\sum_{i=1}^{\infty}(T_i'-T_i)\theta(\bar{\rho}),$$

i.e.

$$\varphi_U(u)\ge\sum_{i=1}^{\infty}(T_i'-T_i)\theta(\bar{\rho})\to\infty.$$

This contrary to the fact $\varphi_U(u)\le\alpha$. By the same reason, there exists a $T'>0$ large enough such that $v(t)\in B_{\bar{\rho}}(\xi_1)$ for all $t<-T'$. Let $T_0=\max\{T,T'\}$, then

$$v(t)\in B_{\bar{\rho}}(\xi_1) \qquad \text{if } t\in(-\infty,-T_0)$$

and

$$v(t)\in B_{\bar{\rho}}(\xi_2) \qquad \text{if } t\in(T_0,\infty).$$

Now, we shall show that $u\in E=W^{1,2}(R,R^n)$. By (2.14) and $(F4)$

$$\begin{aligned}\alpha\ge\varphi_U(u)&=\int_{-\infty}^{\infty}\frac{1}{2}\mid\dot{U}+\dot{u}\mid^2dt+\int_{A_1\cup A_2}F(t,U+u)dt+\int_{A_3}F(t,U+u)dt\\&\ge\int_{-\infty}^{\infty}\frac{1}{2}\mid\dot{u}\mid^2dt+\int_{-1}^{1}[\langle\dot{U},\dot{u}\rangle+\frac{1}{2}\mid\dot{U}\mid^2]dt\\&+\int_{A_1\cup A_2}\mu_1\mid u\mid^2dt+\int_{A_3}F(t,U+u)dt\\&\ge\int_{-\infty}^{\infty}[\frac{1}{2}\mid\dot{u}\mid^2+\mu_1\mid u\mid^2]dt-\mu_1\int_{A_3}\mid u\mid^2dt\\&\quad-\int_{-1}^{1}\mid\langle\dot{U},\dot{u}\rangle\mid dt+\frac{1}{2}\int_{-1}^{1}\mid\dot{U}\mid^2dt.\end{aligned}$$

By lemma 2.4, (1.4) and (2.17), there exists a finite constant $M>0$ such that

$$\mu_1\int_{A_3}\mid u\mid^2dt\le M,\int_{-1}^{1}\mid\langle\dot{U}+\dot{u}\rangle\mid dt\le M \text{ and } \int_{-1}^{1}\mid\dot{U}\mid^2dt\le M.$$

These imply that

$$\int_{-\infty}^{\infty}[\frac{1}{2}\mid\dot{u}\mid^2+\mu_1\mid u\mid^2]dt\le\alpha+3M.$$

Thus $u\in E$, i.e. $u(t)\to0$ as $\mid t\mid\to\infty$, and $v(t)=U(t)+u(t)$ satisfies (2.15). This complete the proof.

*Proof of Theorem 1.2.* Let $\{u_m\} \subset E$ be a sequence such that

$$\varphi_{U_{i,j}}(u_m) \to \alpha_{i,j} \qquad \mathit{and} \qquad \varphi'_{U_{i,j}}(u_m) \to 0.$$

By the same method as in the proof of Theorem 1.1 it is easy to show that there exists a $u \in W^{1,2}_{loc}(R, R^n) \cap L^\infty(R, R^n)$ such that $u_m \rightharpoonup u$ in $W^{1,2}_{loc}(R, R^n)$ and $u_m \to u$ in $L^\infty(R, R^n)$. Furthermore, $\varphi_{U_{i,j}}(u) \leq \alpha_{i,j}$.

Let $v = u + U_{i,j}$, $\mathcal{M}_0 = \{\xi \mid \xi \in \mathcal{M}$ and there exists a $t \in \mathbb{R}$ such that $v(t) \in B_{\rho_0}(\xi)\}$. By the same way as in the proof of Lemma 2.4, we know that $\sharp\mathcal{M}_0$ is finite. As the same reason in the proof of Theorem 1.1 , for any $\xi \in \mathcal{M}\backslash\{\xi_i\}$, there exists a $\rho_1 \in (0, \rho_0]$ such that $v(t)\bar{\in} B_{\rho_1}(\xi)$ for $t < -k - 1$. Similarly, for all $\xi \in \mathcal{M}\backslash\{\xi_j\}$, there exists a $\rho_2 \in (0, \rho_0]$ such that $v(t)\bar{\in} B_{\rho_2}(\xi)$ for $t > k + 1$. Then it follows that $u \in E$ and $v(t)$ is the desired solution.


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