TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 2, pp. 725-748, April 2013
DOI: 10.11650/tjm.17.2013.2337
This paper is available online at http://journal.taiwanmathsoc.org.tw

THE EXISTENCE OF SOLUTIONS AND WELL-POSEDNESS FOR BILEVEL MIXED EQUILIBRIUM PROBLEMS IN BANACH SPACES

Jia Wei Chen*, Zhongping Wan and Yeol Je Cho

Abstract. In this paper, a new class of bilevel mixed equilibrium problems (for short, (BMEP)) is introduced and investigated in reflexive Banach space and some topological properties of solution sets for the lower level mixed equilibrium problem and the problem (BMEP) are established without coercivity. Subsequently, we construct a new iterative algorithm which can directly compute some solutions of the problem (BMEP). Some strong convergence theorems of the sequence generated by the proposed algorithm are also presented. Finally, the well-posedness and generalized well-posedness for the problem (BMEP) are introduced by an ϵ-bilevel mixed equilibrium problem. Also, we explore the sufficient and necessary conditions for (generalized) well-posedness of the problem (BMEP) and show that, under some suitable conditions, the well-posedness and generalized well-posedness of (BMEP) are equivalent to the uniqueness and existence of its solutions, respectively. These results are new and improve some recent results in this field.

1. Introduction

The equilibrium problem, which was first introduced by Blum and Oettli [6], provides a unified model of many problems such as optimization problems, variational inequalities problems, complementarity problems and fixed point problems and so on. Subsequently, equilibrium and generalized different types of equilibrium problems were

Received August 29, 2012, accepted October 6, 2012.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: 49J40, 90C33.
Key words and phrases: Bilevel mixed equilibrium problem, ϵ-Bilevel mixed equilibrium problem, Mixed equilibrium problem, Well-posedness, Hausdorff metric.
This research was supported by the Natural Science Foundation of China (71171150, 11201039), the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2011-0021821), the Academic Award for Excellent Ph.D. Candidates Funded by Wuhan University and the Fundamental Research Fund for the Central Universities (No. 201120102020004), the PhD short-time mobility program by Wuhan University.
*Corresponding author.

intensively studied (see, for example, [8, 9, 10, 11, 12, 21, 29, 32, 33] and the reference therein). For the past decades, mathematical programs with variational inequality, equilibrium and complementarity constraints have been caused many scholars' interests (see, for example, [5, 19, 23, 24, 25, 26, 30, 31] and references therein). In 2010, Moudafi [27] introduced a class of bilevel equilibrium problem (for short, (BEP)):

Find $x \in S_F$ such that

$$H(x, y) \geq 0, \quad \forall y \in S_F,$$

where S_F is the solution set of the following equilibrium problem:

Find $u \in K$ such that

$$F(u, y) \geq 0, \quad \forall y \in K,$$

where K is a nonempty closed convex subset of a Hilbert space and $H, F : K \times K \to R$ are two functions. He pointed out that this class is absorbing since it includes hierarchical optimization problems, optimization problems with equilibrium, variational inequalities, complementarity constraints as special cases. Also, by using the proximal method, an iterative algorithm to compute approximate solution of the problem (BEP) and the weak convergence of the iterative sequence generated by the algorithm were suggested and derived, respectively.

Throughout this paper, let E be a real Banach space with its dual space E^*, the norm and the dual pair between E and E^* are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. Let K be a nonempty convex subset of E and $\Phi, \Psi : K \times K \to R \cup \{+\infty\}$ and $\phi, \psi : E \times E \to R \cup \{+\infty\}$ be functions.

Motivated by Moudafi's works [27], Ding [13, 14] considered the following *bilevel mixed equilibrium problem* (for short, (BMEP)) in reflexive Banach spaces as follows:

Find $x \in S_{\Psi,\psi}$ such that

$$\Phi(x, y) + \phi(y, x) - \phi(x, x) \geq 0, \quad \forall y \in S_{\Psi,\psi}, \tag{1.1}$$

where $S_{\Psi,\psi}$ is the solution set of the following mixed equilibrium problem:

Find $y \in K$ such that

$$\Psi(y, z) + \psi(z, y) - \psi(y, y) \geq 0, \quad \forall z \in K. \tag{1.2}$$

Ding et al. [17] also extended the problem (BEP) model to the bilevel generalized mixed equilibrium problems in reflexive Banach space. Ding [13, 14] and Ding et al. Yao [17] studied the existence results of solutions and the behavior of solution set for the mixed equilibrium problems and the bilevel mixed equilibrium problems under suitable assumptions. By using auxiliary principle technique, he/they also suggested some iterative algorithms for solving the mixed equilibrium problems and the bilevel mixed equilibrium problems. Also, some strong convergence theorems of the iterative sequences generated by the proposed algorithms was proved under suitable assumptions.

In [15, 16], Ding further studied the generalized problem (BMEP) involving generalized variational inequalities in Banach spaces: the existence and iterative algorithm aspects by the same method as in [13, 14, 17]. We observe that Ding [13, 14, 15, 16] and Ding, Liou and Yao [17] solved the bilevel mixed equilibrium problems by the iterative algorithms which divided into two stages, i.e., firstly constructed an iterative algorithm to solve the lower level mixed equilibrium problem and then constructed another iterative algorithm for the upper level mixed equilibrium problem on the solution set of the lower level one. However, (BMEP) was reduced to one-level mixed equilibrium problem under the strongly monotone assumptions which imposed on the mappings of lower level one since the solution set of the lower mixed equilibrium problem is a singleton (see, for example, Theorem 3.1 [13, 16, 17], Theorem 3.2[14, 15]). Recently, Dinh and Muu [18] extended the problem (BEP) to the bilevel pseudomonotone equilibrium problems in finite dimensional Euclidean spaces. They used a penalty function to convert a bilevel problem into one-level ones, and proved that under pseudo-∇-monotonicity property any stationary point of a regularized gap function is a solution of the penalized equilibrium problem. Chadli et al. [7] studied the existence and algorithmic aspects of a class of the problems (BMEP) in Banach spaces, introduced a suitable regularization of the problem (BMEP) by means of an auxiliary problem and then constructed an iterative algorithm by the auxiliary problem. They also proved that a sequence generated by the proposed algorithm is strongly convergent to a solution of the bilevel mixed equilibrium problem. Very recently, Anh et al. [1] analyzed the convergence of an extragradient algorithm for a class of bilevel pseudomonotone variational inequalities (for short, (BPVI)) in finite dimensional Euclidean spaces, which is a special model of the problem (BEP) in [27].

It is well-known that the well-posedness plays an important role in stability analysis and numerical method in optimization theory and applications. Many scholars studied various kinds of the well-posedness for optimization problems, variational inequalities and equilibrium problems (see, for example, [19, 22, 28] and others). In [2], Anh et al. gave some sufficient conditions for the well-posedness and unique well-posedness to the bilevel equilibrium and optimization problems with equilibrium constraints under the assumptions of existence of solutions and the relaxed level closedness and pseudocontinuity. As we know, the well-posedness is closely related to the existence of solutions. So it is necessary to study the existence of solutions for the bilevel equilibrium problems.

Motivated and inspired by the above results, we introduce and investigate the following *bilevel mixed equilibrium problem* (for short, (BMEP)):

Find $x \in S_{\Psi,B,\psi}$ such that

$$\Phi(x,y) + \langle Ax, y-x\rangle + \phi(y,x) - \phi(x,x) \geq 0, \quad \forall y \in S_{\Psi,B,\psi}, \tag{1.3}$$

where $S_{\Psi,B,\psi}$ is the solution set of the lower level mixed equilibrium problem:

Find $y^* \in K$ such that

(1.4) $\quad \Psi(y^*, z) + \langle By^*, z - y^* \rangle + \psi(z, y^*) - \psi(y^*, y^*) \geq 0, \quad \forall z \in K,$

where $A, B : E \to E^*$ are two vector-valued mappings.

Denote the solution set of the problem (BMEP) (1.3) with (1.4) by $\aleph$.

Special cases of the problem (BMEP) (1.3) with (1.4):

(1) If $A = B = 0$, then problem (BMEP) (1.3) with (1.4) reduces to the problem (BMEP) (1.1) with (1.2) studied by Ding [13, 14, 15].

(2) If E is a Hilbert space or a finite Euclidean space, $A = B = 0$ and $\phi = \psi = 0$, then the problem (BMEP) (1.3) with (1.4) reduces to the following problem (BEP):

Find $x \in S_\Psi$ such that

$$\Phi(x, y) \geq 0, \quad \forall y \in S_\Psi,$$

where S_F is the solution set of the following equilibrium problem:

Find $u \in K$ such that

$$\Psi(x, y) \geq 0, \quad \forall y \in K,$$

which was studied by Moudafi[27] and Dinh and Muu[18].

(3) If E is a finite Euclidean space and $\Phi = \Psi = \phi = \psi = 0$, then the problem (BMEP) (1.3) with (1.4) reduces to the following *bilevel variational inequalities*:

Find $x \in S_B$ such that

(1.5) $$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in S_B,$$

where S_B is the solution set of the lower level variational inequality problem:

Find $y^* \in K$ such that

(1.6) $$\langle By^*, z - y^* \rangle \geq 0, \quad \forall z \in K,$$

which was studied by Anh, Kim and Muu[1].

(4) Let $W : K \to R$ be functional. If the mappings $A = 0$, $\phi = \psi = 0$ and $\Phi(x, y) = W(y) - W(x)$, then the problem (BMEP) (1.3) with (1.4) reduces to the following *optimization problem* with mixed equilibrium constraints:

(1.7) $$\min \quad W(x) \quad \text{subject to} \quad x \in S_{\Psi,B,\psi},$$

where $S_{\Psi,B,\psi}$ is the solution set of the lower level mixed equilibrium problem (1.4).

We now recall some definitions and lemmas which are needed in the sequel.

Definition 1.1. ([4]). Let K be a closed convex subset of E and $\Psi : K \times K \to R$ be functional. Ψ is said to be:

(1) *monotone* if $\Psi(z, y) + \Psi(y, z) \leq 0$ for all $(y, z) \in K \times K$;

(2) *strictly monotone* if $\Psi(z, y) + \Psi(y, z) < 0$ for all $(y, z) \in K \times K$ and $y \neq z$;

(3) α-*strongly monotone* if there exists a positively constant α such that

$$\Psi(z,y)+\Psi(y,z)\le -\alpha\|y-z\|^2,\quad \forall(y,z)\in K\times K;$$

(4) β-Lipschitz if there exists a positively constant β such that

$$|\Psi(y,z)|\le \beta\|y-z\|,\quad \forall(y,z)\in K\times K.$$

Definition 1.2. ([4]). Let $A:E\to E^*$ be a vector-valued mapping. A is said to be

(1) *monotone* if $\langle Ay-Az,y-z\rangle\ge 0$ for all $(y,z)\in E\times E$;
(2) *strictly monotone* if $\langle Ay-Az,y-z\rangle>0$ for all $(y,z)\in E\times E, y\ne z$;
(3) ζ-*strongly monotone* if there exists a positively constant ζ such that

$$\langle Ay-Az,y-z\rangle\ge \zeta\|y-z\|^2,\quad \forall(y,z)\in E\times E;$$

(4) ι-*Lipschitz* if there exists a positively constant ι such that

$$\|Ay-Az\|\le \iota\|y-z\|,\quad \forall(y,z)\in E\times E.$$

Remark 1.1. It is easy to see that, if $A:E\to E^*$ is ι-Lipschitz, then, for each $(y,z)\in E\times E$, the mapping $\Psi(y,z)=\langle Ay-Az,y-z\rangle$ is also ι-Lipschitz.

Definition 1.3. ([3]). A function $\psi:E\times E\to R$ is said to be *skew symmetric* if

$$\psi(y,y)-\psi(y,z)-\psi(z,y)+\psi(z,z)\ge 0,\quad \forall(y,z)\in E\times E.$$

Remark 1.2. If a function $\psi:E\times E\to R$ is skew symmetric, then, for any $x,y\in E$, the function $\Upsilon(x,y)=\psi(x,y)-\psi(x,x)$ is monotone. Indeed, for each $x,y\in E$,

$$\begin{aligned}\Upsilon(x,y)+\Upsilon(y,x) &= \psi(x,y)-\psi(x,x)+\psi(y,x)-\psi(y,y)\\ &= -[\psi(x,x)-\psi(x,y)-\psi(y,x)+\psi(y,y)]\le 0.\end{aligned}$$

Remark 1.3. The skew symmetric functions have the properties which can be considered an analogous results of monotonicity of gradient and nonnegativity of a second derivative for the convex function. For the applications and properties of this function, the readers refer to Antipin [3].

The remaining of this paper is organized as follows: Section 2 investigates the existence of solutions and the behavior of solution sets to the upper level mixed equilibrium problem (1.3) and (BMEP) (1.3) with (1.4). Section 3 proposes an iterative algorithm which directly compute some solutions of the problem (BMEP) and analyze the strong convergence of the sequence generated by the proposed algorithm. Also, we explore the sufficient and necessary conditions of the well-posedness for thye problem (BMEP) and establish the equivalence between the well-posedness (generalized well-posedness) of the problem (BMEP) and the uniqueness and existence of its solutions under some suitable conditions in Section 4.

2. The Existence Results for (BMEP)

In this section, we investigate the sufficient optimality conditions for (BMEP) (1.3) with (1.4) and the lower level mixed equilibrium problem, and discuss some topological properties of their solutions.

Lemma 2.1. ([14]). *Let K be a bounded closed convex subset of reflexive Banach space E. Let $\Psi : K \times K \to R$ and $\psi : E \times E \to R$ be two bifunctions. Suppose that the following conditions are satisfied:*

(a) $\Psi(y, y) \geq 0$ *for each $y \in K$;*

(b) *for each $z \in K$, $y \mapsto \Psi(y, z)$ is weakly upper semicontinuous and, for each $y \in K$, $z \mapsto \Psi(y, z)$ is convex;*

(c) *ψ is weakly continuous and ψ is convex in the first argument.*

Then $S_{\Psi,\psi} \neq \emptyset$.

If, further, assume that

(d) *Ψ is monotone and ψ is skew symmetric and, for each $y \in K$, $z \mapsto \Psi(y, z)$ is lower semicontinuous.*

Then $S_{\Psi,\psi}$ is nonempty closed convex.

Lemma 2.2. ([14]). *Let K be a closed convex subset of reflexive Banach space E with int$K \neq \emptyset$. Let $\Psi : K \times K \to R$ and $\psi : E \times E \to R$ be two bifunctions. Suppose that the following conditions are satisfied:*

(a) *Ψ is α-strongly monotone and δ-Lipschitz continuous such that $\Psi(y, y) \geq 0$ for each $y \in K$;*

(b) *for each $z \in K$, $y \mapsto \Psi(y, z)$ is weakly upper semicontinuous and, for each $y \in K$, $z \mapsto \Psi(y, z)$ is convex and weakly lower semicontinuous;*

(c) *ψ is skew symmetric and weakly continuous and ψ is convex in the first argument.*

Then $S_{\Psi,\psi}$ is nonempty closed convex.

Next, we establish the behavior of solution set of lower level mixed equilibrium problem (1.4).

Theorem 2.1. *Let K be a bounded closed convex subset of reflexive Banach space E, $B : K \to E^*$ be monotone and weakly upper semicontinuous and $\Psi : K \times K \to R$, $\psi : E \times E \to R$ be two bifunctions. Assume that Ψ and ψ satisfy the conditions (a)-(c) of Lemma 2.1. Then $S_{\Psi,B,\psi} \neq \emptyset$. Moreover, if Ψ and ψ also satisfy the condition (d) of Lemma 2.1, then $S_{\Psi,B,\psi}$ is weakly compact convex.*

Proof. Define the mapping $F : K \times K \to R$ by

$$F(y,z) = \Psi(y,z) + \langle By, z-y\rangle, \quad \forall (y,z) \in K \times K.$$

From $\Psi(z,z) \geq 0$ for all $z \in K$, one has $F(z,z) \geq 0$. Since $B : K \to E^*$ is weakly upper semicontinuous and, from the condition (b) of Lemma 2.1, it follows that F is weakly upper semicontinuous with respect to the first argument. We declare that F is also convex with respect to the second argument. Indeed, for any $z_1, z_2 \in K, t \in (0,1)$, let $z_t = tz_1 + (1-t)z_2$. Then $z_t \in K$ for all $t \in (0,1)$. By the convexity of Ψ with respect to the second argument, we have, for each $y \in K$,

$$\begin{aligned} F(y,z_t) &= \Psi(y,z_t) + \langle By, z_t - y\rangle \\ &= \Psi(y, tz_1 + (1-t)z_2) + \langle By, t(z_1 - y) + (1-t)(z_2 - y)\rangle \\ &\leq t[\Psi(y,z_1) + \langle By, z_1 - y\rangle] + (1-t)[\Psi(y,z_2) + \langle By, z_2 - y\rangle] \\ &= tF(y,z_1) + (1-t)F(y,z_2). \end{aligned}$$

Therefore, by Lemma 2.1, there exists $y^* \in K$ such that

$$F(y^*,z) + \psi(z,y^*) - \psi(y^*,y^*) \geq 0, \quad \forall z \in K.$$

Moreover, one has

$$\Psi(y^*,z) + \langle By^*, z-y^*\rangle + \psi(z,y^*) - \psi(y^*,y^*) \geq 0, \quad \forall z \in K.$$

This implies that $S_{\Psi,B,\psi} \neq \emptyset$.

Let us show that F is monotone and F is lower semicontinuous in the second argument. We first prove that F is monotone. For any $y,z \in K$, we have

$$\begin{aligned} F(y,z) + F(z,y) &= \Psi(y,z) + \langle By, z-y\rangle + \Psi(z,y) + \langle Bz, y-z\rangle \\ &= [\Psi(y,z) + \Psi(z,y)] + [\langle By, z-y\rangle + \langle Bz, y-z\rangle] \\ &\leq -\langle Bz - By, z-y\rangle \\ &\leq 0, \end{aligned}$$

i.e., F is monotone. From the lower semicontinuity of Ψ with respect to the second argument, it follows that F is lower semicontinuous in the second argument. Again, from Lemma 2.1, $S_{\Psi,B,\psi}$ is nonempty closed convex. Taking into account of the boundedness of K, $S_{\Psi,B,\psi}$ is nonempty bounded closed convex. Since E is a reflexive Banach space, we conclude that $S_{\Psi,B,\psi}$ is nonempty weakly compact convex. This completes the proof. ∎

Example 2.1. Let $E = R = (-\infty, +\infty)$ and $K = [-1,1]$. For each $y,z \in E$, let $\Psi(y,z) = z - y$, $\psi(y,z) = y - z$ and

$$B(y) = \begin{cases} -2, & \text{if } y \leq 0, \\ -2 + y, & \text{if } y > 0. \end{cases}$$

Clearly, Ψ, B are monotone and ψ is skew symmetric. Moreover, all the conditions of Theorem 2.1 are satisfied. Simple computation allows that $S_{\Psi,B,\psi} = [-1, 0]$.

If the monotonicity of Ψ is strengthened, then the solution set $S_{\Psi,B,\psi}$ is a singleton.

Theorem 2.2. *Suppose that all the conditions of Theorem 2.1 hold and Ψ or B is strictly monotone. Then $S_{\Psi,B,\psi}$ is a singleton.*

Proof. By Theorem 2.1, the solution set $S_{\Psi,B,\psi}$ is nonempty weakly compact convex. Suppose that there exist $y_1^*, y_2^* \in S_{\Psi,B,\psi}$ and $y_1^* \neq y_2^*$. Then we have

$$\Psi(y_1^*, z) + \langle By_1^*, z - y_1^* \rangle + \psi(z, y_1^*) - \psi(y_1^*, y_1^*) \geq 0, \quad \forall z \in K, \tag{2.8}$$

and

$$\Psi(y_2^*, z) + \langle By_2^*, z - y_2^* \rangle + \psi(z, y_2^*) - \psi(y_2^*, y_2^*) \geq 0, \quad \forall z \in K. \tag{2.9}$$

Substituting $z = y_2^*$ and $z = y_1^*$ into (2.1) and (2.2), respectively, one has

$$\Psi(y_1^*, y_2^*) + \langle By_1^*, y_2^* - y_1^* \rangle + \psi(y_2^*, y_1^*) - \psi(y_1^*, y_1^*) \geq 0, \tag{2.10}$$

and

$$\Psi(y_2^*, y_1^*) + \langle By_2^*, y_1^* - y_2^* \rangle + \psi(y_1^*, y_2^*) - \psi(y_2^*, y_2^*) \geq 0. \tag{2.11}$$

Adding up (2.3) and (2.4), one has

$$\begin{aligned} 0 \ &\leq \ [\Psi(y_1^*, y_2^*) + \langle By_1^*, y_2^* - y_1^* \rangle + \psi(y_2^*, y_1^*) - \psi(y_1^*, y_1^*)] \\ &\quad + [\Psi(y_2^*, y_1^*) + \langle By_2^*, y_1^* - y_2^* \rangle + \psi(y_1^*, y_2^*) - \psi(y_2^*, y_2^*)] \\ &= \ [\Psi(y_1^*, y_2^*) + \Psi(y_2^*, y_1^*)] + [\langle By_1^*, y_2^* - y_1^* \rangle + \langle By_2^*, y_1^* - y_2^* \rangle] \\ &\quad - [\psi(y_1^*, y_1^*) - \psi(y_1^*, y_2^*) - \psi(y_2^*, y_1^*) + \psi(y_2^*, y_2^*)] \\ &\leq \ [\Psi(y_1^*, y_2^*) + \Psi(y_2^*, y_1^*)] - \langle By_2^* - By_1^*, y_2^* - y_1^* \rangle. \end{aligned}$$

Since Ψ or B is strictly monotone, by the above inequality, we get

$$0 \leq [\Psi(y_1^*, y_2^*) + \Psi(y_2^*, y_1^*)] - \langle By_2^* - By_1^*, y_2^* - y_1^* \rangle < 0,$$

which is a contradiction. Therefore, $S_{\Psi,B,\psi}$ is a singleton. This completes the proof. ■

Example 2.2. Let $E = R^2 = (-\infty, +\infty) \times (-\infty, +\infty)$ and $K = [0, 1] \times [0, 1]$. For each $x, y \in E$, let $\Psi(x, y) = \langle y + x, y - x \rangle$, $\psi(x, y) = \langle x, y \rangle$ and $B(x) = (e^{x_1}, e^{x_2})$, where $x = (x_1, x_2), y = (y_1, y_2)$. Note that

$$\Psi(x, y) + \Psi(y, x) = \langle y + x, y - x \rangle + \langle y + x, x - y \rangle \leq 0, \quad \forall x, y \in E,$$

$$\psi(x,x)-\psi(x,y)-\psi(y,x)+\psi(y,y) = \langle x,x\rangle-\langle x,y\rangle-\langle y,x\rangle+\langle y,y\rangle \geq 0, \quad \forall x,y \in E$$

and

$$\begin{aligned}\langle Bx - By, x-y\rangle &= (x_1-y_1)(e^{x_1}-e^{y_1}) \\ &\quad +(x_2-y_2)(e^{x_2}-e^{y_2}) > 0, \quad \forall x,y \in E,\, x \neq y.\end{aligned}$$

Therefore, all the conditions of Theorem 2.2 are satisfied. From a simple computation, we have $S_{\Psi,B,\psi} = \{(0,0)\}$.

Corollary 2.1. *Assume that all the conditions of Theorem 2.2 are satisfied and $\Phi(y,y) \geq$ for all $y \in K$. Then (BMEP) (1.3) with (1.4) has a unique solution.*

Proof. It directly follows from Theorem 2.2 and $\Phi(y,y) \geq 0$ for all $y \in K$. ■

If $B \equiv 0$, then, from Theorem 2.2, we have the following:

Corollary 2.2. *Assume that K, Ψ and ψ satisfy all the conditions of Theorem 2.2 and Ψ be strictly monotone. Then $S_{\Psi,\psi}$ is a singleton.*

If $A = B \equiv 0$, then, from Corollary 2.1, we obtain the following:

Corollary 2.3. *Assume that K, Ψ, Φ and ψ satisfy all the conditions of Corollary 2.1. Then (BMEP) (1.1) with (1.2)] has a unique solution.*

Theorem 2.3. *Let K be a nonempty closed convex subset of a reflexive Banach space E with int$K \neq \emptyset$, $B : K \to E^*$ be monotone and weakly continuous such that $\|By\| \leq l$ for all $y \in K$ and $l > 0$ and $\Psi : K \times K \to R$, $\psi : E \times E \to R$ be two bifunctions. Assume that Ψ and ψ satisfy the conditions (a)-(c) of Lemma 2.2. Then $S_{\Psi,B,\psi}$ is nonempty closed convex. Moreover, $S_{\Psi,B,\psi}$ is a singleton.*

Proof. Define the mapping $F : K \times K \to R$ by

$$F(y,z) = \Psi(y,z) + \langle By, z-y\rangle, \quad \forall (y,z) \in K \times K.$$

Since $B : K \to E^*$ is weakly continuous, this together with the condition (b) of Lemma 2.2 yields that, for each $z \in K$, $y \mapsto F(y,z)$ is weakly upper semicontinuous and for each $y \in K$, $z \mapsto F(y,z)$ is convex and weakly lower semicontinuous. By the monotonicity of B and Ψ, for any $y, z \in K$, we have

$$0 \leq \langle By - Bz, y-z\rangle = -\langle Bz - By, y-z\rangle \tag{2.12}$$

and

$$\Psi(y,z) + \Psi(z,y) \leq -\alpha\|y-z\|^2. \tag{2.13}$$

From both (2.5) and (2.6), we have

$$\begin{aligned} F(y,z)+F(z,y) &= \Psi(y,z)+\langle By, z-y\rangle+\Psi(z,y)+\langle Bz, y-z\rangle \\ &= \Psi(y,z)+\Psi(z,y)+\langle Bz-By, y-z\rangle \\ &\le -\alpha\|y-z\|^2. \end{aligned}$$

Therefore, F is α-strongly monotone. In view of $\Psi(z,z)\ge 0$ for all $z\in K$, we have

$$F(z,z)=\Psi(z,z)+\langle Bz, z-z\rangle=\Psi(z,z)\ge 0.$$

Now, we show that F is Lipschitz continuous. Noticing that Ψ is δ-Lipschtiz continuous, we have

$$|\Psi(y,z)|\le\delta\|y-z\|,\quad \forall y,z\in K.$$

Thus, for any $y,z\in K$, one has

$$\begin{aligned} |F(y,z)|=|\Psi(y,z)+\langle By, z-y\rangle| &\le |\Psi(y,z)|+|\langle By, z-y\rangle| \\ &\le \delta\|y-z\|+\|By\|\cdot\|z-y\| \\ &\le (\delta+l)\|y-z\|, \end{aligned}$$

that is, F is $(\delta+l)$-Lipschitz continuous. By Lemma 2.2, it follows that the set

$$\{y^*\in K: F(y^*,z)+\psi(z,y^*)-\psi(y^*,y^*)\ge 0,\quad \forall z\in K\}$$

is nonempty and closed convex. This shows that

$$S_{\Psi,B,\psi}=\{y^*\in K:\Psi(y^*,z)+\langle By^*, z-y^*\rangle+\psi(z,y^*)-\psi(y^*,y^*)\ge 0,\quad \forall z\in K\}$$

is nonempty closed and convex. As in the proof of Theorem 2.2, for any $y_1^*, y_2^*\in S_{\Psi,B,\psi}$, we have

$$0\le[\Psi(y_1^*,y_2^*)+\Psi(y_2^*,y_1^*)]-\langle By_2^*-By_1^*, y_2^*-y_1^*\rangle\le-\alpha\|y_1^*-y_2^*\|^2\le 0$$

and hence $\|y_1^*-y_2^*\|^2=0$. Therefore, $y_1^*=y_2^*$, which yields that $S_{\Psi,B,\psi}$ is a singleton. This completes the proof. ■

Remark 2.1. If $B\equiv 0$, then Theorem 2.3 reduces to Theorem 3.2 of Ding [14]. The proofs of Theorem 2.3 illustrate that $S_{\Psi,\psi}$ is a singleton under the assumptions of Lemma 2.2.

In the sequel, we investigate the sufficient optimality conditions for (BMEP) (1.3) with (1.4).

Theorem 2.4. *Assume that all the conditions of Theorem 2.1 hold. Let $A: K\to E^*$ be monotone and weakly upper semicontinuous and $\Phi: K\times K\to R$ and $\phi: E\times E\to R$ be two bifunctions. If Ψ and ψ satisfy the following conditions:*

(a) $\Phi(y,y) \geq 0$ *for each* $y \in K$;

(b) *for each* $y \in K$, $x \mapsto \Phi(x,y)$ *is weakly upper semicontinuous and, for each* $x \in K$, $y \mapsto \Phi(x,y)$ *is convex;*

(c) ϕ *is weakly continuous and* ϕ *is convex in the first argument.*

Then (BMEP)[(1.3)-(1.4)] is solvable.

Moreover, if Φ is monotone, ϕ is skew symmetric and, for each $x \in K$, $y \mapsto \Phi(x,y)$ is lower semicontinuous, then the solution set of (BMEP) (1.3) with (1.4) is weakly compact convex.

Proof. It immediately follows from the proofs of Theorem 2.1 that $S_{\Psi,B,\psi}$ is a nonempty bounded closed convex subset of K. By the similar methods to the proof of Theorem 2.1, we know that the solution set of (BMEP) (1.3) with (1.4) is a nonempty bounded closed convex subset of $S_{\Psi,B,\psi}$ and so it is weakly compact convex. This completes the proof. ■

Example 2.3. Let E, K, Ψ, B and ψ be the same as Example 2.1. For each $x, y \in E$, let $\Phi(x,y) = y - x$, $\phi(x,y) = x - y$ and

$$A(x) = \begin{cases} -2, & \text{if } x < 0, \\ -2 + x, & \text{if } x \geq 0. \end{cases}$$

It is easy to verify that $\Phi, \Psi, \phi, \psi, A$ and B satisfy all the conditions of Theorem 2.4 and so, $\aleph = [-1, 0]$.

Theorem 2.5. *Assume that all the conditions of Theorem 2.4 hold and* Φ *or* A *is strictly monotone. Then (BMEP) (1.3) with (1.4) has a unique solution.*

Proof. From Theorem 2.4, we know that the solution set of (BMEP) (1.3) with (1.4) is nonempty and weakly compact convex. By the similar proofs of Theorem 2.2, (BMEP) (1.3) with (1.4) has a unique solution. ■

Example 2.4. Let E, K, Φ, Ψ, ϕ, B and ψ be the same as Example 2.3. For each $x \in E$, let $A(x) = x$. It is easy to verify that Φ, Ψ, ϕ, ψ, A and B satisfy all the conditions of Theorem 2.5 and so $\aleph = \{0\}$.

The following results are direct consequences of Theorems 2.4 and 2.5.

Corollary 2.4. *Let* K, Φ, ϕ, Ψ *and* ψ *be the same as Theorem 2.4. Then the solution set of* (*BMEP*) (*1.1*) *with* (*1.2*) *is nonempty and weakly compact convex.*

Corollary 2.5. *Let* K, Φ, ϕ, Ψ *and* ψ *be the same as Theorem 2.4 and* Φ *is strictly monotone. Then the solution set of* (*BMEP*) (*1.1*) *with* (*1.2*) *is a singleton.*

Theorem 2.6. *Let* K *be a closed convex subset of a reflexive Banach space* E

with int$K \neq \emptyset$, $A : K \to E^*$ be a vector-valued mapping, $\Phi : K \times K \to R$ and $\phi : E \times E \to R$ be two bifunctions such that $\Phi(x, x) \geq 0$. Assume that all the conditions of Theorem 2.3 are satisfied. Then (BMEP) (1.3) with (1.4) has a unique solution.

Proof. By Theorem 2.3, we know that $S_{\Psi,B,\psi}$ is a singleton. Set the solution set $S_{\Psi,B,\psi} = \{y^*\}$. Then we have

$$\Phi(y^*, y^*) + \langle Ay^*, y^* - y^*\rangle + \phi(y^*, y^*) - \phi(y^*, y^*) = \Phi(y^*, y^*) \geq 0.$$

Therefore, y^* is the unique solution of (BMEP) (1.3) with (1.4). This completes the proof. ■

If the mappings $A = B \equiv 0$, then, from Theorem 2.6, we have the following:

Corollary 2.6. *Let K, Φ, ϕ, Ψ and ψ be the same as Theorem 2.6. Then (BMEP) (1.1) with (1.2) has a unique solution.*

Remark 2.2. Compared with Theorem 3.3 of Ding [14], the conditions of Corollary 2.6 is weaker than those of Theorem 3.3 in [14]. Since Φ and ϕ do not involve the strong monotonicity, Lipschitz continuity, convexity, weak upper (lower) semicontinuity and skew symmetry, respectively.

3. Algorithms and Convergence Analysis for (BMEP)

In this section, let E be a Hilbert space. We suggest an iterative algorithm to directly compute the solution of the problem (BMEP) (1.3) with (1.4) from the perspective of the theoretical analysis and analyze the convergence of the proposed algorithm. We firstly consider the following mixed variational inequalities.

For any $\rho, \beta > 0$ and $x \in E$, we consider the following mixed variational inequalities (MVI):

Find $y^* \in K$ such that

$$\begin{aligned}&\langle y^* - x, z - y^*\rangle + \rho[\Phi(y^*, z) + \langle Ay^*, z - y^*\rangle + \phi(z, y^*) - \phi(y^*, y^*)]\\ &+ \beta[\Psi(y^*, z) + \langle By^*, z - y^*\rangle + \psi(z, y^*) - \psi(y^*, y^*)] \geq 0, \quad \forall z \in K.\end{aligned}$$

Now, we show that (MVI) is solvable.

Lemma 3.1. *Assume that all the conditions of Theorem 2.4 hold. Then, for each $\rho, \beta > 0$ and $x \in E$, the problem (MVI) has a unique solution.*

Proof. For the sake of brevity, let $T(y, z) = \Phi(y, z) + \langle Ay, z - y\rangle + \phi(z, y) - \phi(y, y)$ and $H(y, z) = \Psi(y, z) + \langle By, z - y\rangle$ and $M(y, z) = \rho T(y, z) + \beta H(y, z)$ for all

$y, z \in K$. It is easy to verify that, for each $z \in K$, $M(z,z) \geq 0$, $y \mapsto M(y,z)$ is weakly upper semicontinuous and so, for each $y \in K$, $z \mapsto M(y,z)$ is convex. For any $y, z \in K$, we have

$$\begin{aligned}
& M(y,z) + M(z,y) \\
= \ & \rho[\Phi(y,z) + \langle Ay, z-y\rangle + \phi(z,y) - \phi(y,y)] + \beta[\Psi(y,z) + \langle By, z-y\rangle] \\
& + \rho[\Phi(z,y) + \langle Az, y-z\rangle + \phi(y,z) - \phi(z,z)] + \beta[\Psi(z,y) + \langle Bz, y-z\rangle] \\
= \ & \rho[\Phi(y,z) + \Phi(z,y) + \langle Ay - Az, z-y\rangle - (\phi(y,y) - \phi(y,z) - \phi(z,y) + \phi(z,z))] \\
& + \beta[\Psi(y,z) + \Psi(z,y) + \langle By - Bz, z-y\rangle] \leq 0,
\end{aligned}$$

which implies that M is monotone on $K \times K$. Since ψ satisfies the conditions of Lemma 2.1, $\beta\psi$ still satisfies the conditions (c) and (d) of Lemma 2.1. Set $P(y,z) = \langle y-x, z-y\rangle$ for all $y, z \in K$. Then, for any $y, z \in K$, $P(z,z) = P(y,y) = 0$ and

$$P(y,z) + P(z,y) = \langle y-x, z-y\rangle + \langle z-x, y-z\rangle = \langle y-z, z-y\rangle \leq -\|y-z\|^2,$$

that is, P is 1-strongly monotone on $K \times K$. So, $M + P$ is 1-strongly monotone on $K \times K$. Moreover, for each $z \in K$, $M(z,z) + P(z,z) \geq 0$ and $y \mapsto M(y,z) + P(y,z)$ is weakly upper semicontinuous and so, for each $y \in K$, $z \mapsto M(y,z) + P(y,z)$ is convex. By Corollary 2.2, there exists a unique $y^* \in K$ such that

$$M(y^*,z) + P(y^*,z) + \beta\psi(z,y^*) - \beta\psi(y^*,y^*) \geq 0, \quad \forall z \in K,$$

that is, there exists a unique $y^* \in K$ such that

$$\begin{aligned}
& \langle y^* - x, z - y^*\rangle + \rho[\Phi(y^*,z) + \langle Ay^*, z-y^*\rangle + \phi(z,y^*) - \phi(y^*,y^*)] \\
& + \beta[\Psi(y^*,z) + \langle By^*, z-y^*\rangle + \psi(z,y^*) - \psi(y^*,y^*)] \geq 0, \quad \forall z \in K.
\end{aligned}$$

Therefore, for each $\rho, \beta > 0$ and $x \in E$, the problem (MVI) has a unique solution. This completes the proof. ■

Remark 3.1. The monotonicity of Φ, Ψ, A and B can not be relaxed for the pseudomonotonicity. Since the sums of two pseudomonotone mappings is not necessarily pseudomonotone. Moreover, if the bifunctions f and g are pseudomonotone and monotone, respectively, then the function $f + g$ is also not necessarily monotone (see, Examples 3.1 and 3.2).

Example 3.1. (Example 2.8, [18]). Let $C = \{(x_1, x_2) : x_1 \geq -1, \frac{x_1 - 9}{10} \leq x_2 \leq 10x_1 + 9\}$, $f(x,y) = (x_1y_2 - x_2y_1)e^{x_1}$ and $g(x,y) = (x_2y_1 - x_1y_2)e^{x_2}$ for all $x, y \in C$. It is easy to check that f and g are pseudomonotone on $C \times C$. However, for all $\rho, \beta > 0$, the function $\rho f(x,y) + \beta g(x,y)$ is not pseudomonotone.

Example 3.2. Let $f(x,y) = x^2 + xy - 2y^2$ and $g(x,y) = -3x^2y + xy^2 + 2y^3$ for all $x, y \in C = [1, +\infty)$. For each $x, y \in K$, we have

$$f(x,y) + f(y,x) = x^2 + xy - 2y^2 + y^2 + xy - 2x^2 = -(x-y)^2 \le 0.$$

So, f is monotone on $C \times C$. Now, we verify that g is pseudomonotone. Suppose that, for any $x, y \in C$, $g(x,y) \ge 0$. Then we have

$$g(x,y) = -3x^2y + xy^2 + 2y^3 = y(3x+2y)(y-x) \ge 0,$$

that is, $y \ge x$. From this, we have

$$g(y,x) = -3y^2x + yx^2 + 2x^3 = x(3y+2x)(x-y) \le 0.$$

Consequently, g is pseudomonotone on $C \times C$. However, for any $x, y \in C$,

$$\begin{aligned}
& f(x,y) + g(x,y) + f(y,x) + g(y,x) \\
= \ & -(x-y)^2 + y(3x+2y)(y-x) + x(3y+2x)(x-y) \\
= \ & -(x-y)^2 + 2(x+y)(x-y)^2 \\
= \ & (2x+2y-1)(x-y)^2 \\
\ge \ & 0.
\end{aligned}$$

Therefore, the function $f(x,y) + g(x,y)$ is not monotone on $C \times C$.

Remark 3.2. We observe that, if $x \in E$ is a solution of the problem (BMEP) (1.3) with (1.4), then x remains a solution of the problem (MVI). Conversely, if $x \in E$ is a solution of (MVI) and the problem (1.4) has a unique solution y^*, then x is also a solution of the problem (BMEP) (1.3) with (1.4). Motivated by this, we construct the following iterative algorithm for solving the problem (BMEP) (1.3) with (1.4).

Algorithm I.

Step 1. Take $\{\beta_k\}, \{\rho_k\} \subset (0, +\infty)$ and choose $y_0 \in E$ arbitrarily. Let $k = 0$ and go to Step 2.

Step 2. For any given $y_k \in E$, compute $y_{k+1} \in K$ such that, for each $z \in K$,

$$\begin{aligned}
& \langle y_{k+1} - y_k, z - y_{k+1} \rangle + \beta_{k+1}\rho_{k+1}[\Phi(y_{k+1}, z) \\
& + \langle Ay_{k+1}, z - y_{k+1} \rangle + \phi(z, y_{k+1}) - \phi(y_{k+1}, y_{k+1})] \\
& + \beta_{k+1}[\Psi(y_{k+1}, z) + \langle By_{k+1}, z - y_{k+1} \rangle + \psi(z, y_{k+1}) - \psi(y_{k+1}, y_{k+1})] \ge 0.
\end{aligned}$$

Step 3. States updating: let $k := k + 1$ and go to Step 2.

Theorem 3.1. *Assume that all the conditions of Theorem 2.4 hold and there exists* $\sigma > 0$ *such that*

$$\Phi(y,z) + \langle Ay, z-y\rangle + \phi(z,y) - \phi(y,y) \geq -\sigma\|y-z\|^2, \quad \forall y,z \in K.$$

If the following conditions are satisfied:

(a) $\beta_k \to +\infty$ *and* $\rho_k \to 0$ *such that* $\frac{1}{\rho_k\beta_k}$ *is finite in Algorithm I;*

(b) $0 \leq \frac{\beta_k}{\beta_{k+1}} + (\rho_{k+1}-\rho_k)\beta_k\sigma < \tau$, *where* $\tau < 1$,

then the sequence $\{y_k\}$ *generated by Algorithm I converges strongly to a solution of the problem (BMEP) (1.3) with (1.4).*

Proof. For the sake of brevity, let

$$T(y_{k+1},z) = \Phi(y_{k+1},z) + \langle Ay_{k+1}, z-y_{k+1}\rangle + \phi(z,y_{k+1}) - \phi(y_{k+1},y_{k+1})$$

and

$$H(y_{k+1},z) = \Psi(y_{k+1},z) + \langle By_{k+1}, z-y_{k+1}\rangle.$$

By Algorithm I, one has

$$\langle y_k - y_{k-1}, z-y_k\rangle + \beta_k\rho_k T(y_k,z) + \beta_k[H(y_k,z) + \psi(z,y_k) - \psi(y_k,y_k)] \geq 0$$

and

$$\begin{aligned}&\langle y_{k+1} - y_k, z-y_{k+1}\rangle + \beta_{k+1}\rho_{k+1} T(y_{k+1},z)\\ &+\beta_{k+1}[H(y_{k+1},z) + \psi(z,y_{k+1}) - \psi(y_{k+1},y_{k+1})] \geq 0.\end{aligned}$$

Moreover, it follows that

$$\begin{aligned}&\frac{1}{\beta_k}\langle y_k - y_{k-1}, z-y_k\rangle + \rho_k T(y_k,z)\\ &+[H(y_k,z) + \psi(z,y_k) - \psi(y_k,y_k)] \geq 0\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}&\frac{1}{\beta_{k+1}}\langle y_{k+1} - y_k, z-y_{k+1}\rangle + \rho_{k+1} T(y_{k+1},z)\\ &+[H(y_{k+1},z) + \psi(z,y_{k+1}) - \psi(y_{k+1},y_{k+1})] \geq 0.\end{aligned} \tag{3.15}$$

Substituting $z = y_{k+1}$ and $z = y_k$ into (3.1) and (3.2), respectively, we have

$$\begin{aligned}&\frac{\langle y_k - y_{k-1}, y_{k+1}-y_k\rangle}{\beta_k} + \rho_k T(y_k,y_{k+1})\\ &+[H(y_k,y_{k+1}) + \psi(y_{k+1},y_k) - \psi(y_k,y_k)] \geq 0\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}(3.17)\qquad &\frac{\langle y_{k+1}-y_k, y_k-y_{k+1}\rangle}{\beta_{k+1}}+\rho_{k+1}T(y_{k+1},y_k)\\ &+[H(y_{k+1},y_k)+\psi(y_k,y_{k+1})-\psi(y_{k+1},y_{k+1})]\geq 0.\end{aligned}$$

Since Φ, Ψ, A, B are monotone and ϕ, ψ are skew symmetric,

$$(3.18)\qquad T(y_k,y_{k+1})+T(y_{k+1},y_k)\leq 0,\quad H(y_k,y_{k+1})+H(y_{k+1},y_k)\leq 0$$

and so,

$$\begin{aligned}(3.19)\qquad &[H(y_k,y_{k+1})+\psi(y_{k+1},y_k)-\psi(y_k,y_k)]\\ &+[H(y_{k+1},y_k)+\psi(y_k,y_{k+1})-\psi(y_{k+1},y_{k+1})]\leq 0.\end{aligned}$$

From (3.3)-(3.6), it follows that

$$\begin{aligned}\frac{\langle y_k-y_{k-1},y_{k+1}-y_k\rangle}{\beta_k} &\geq (\rho_k-\rho_{k+1})T(y_{k+1},y_k)+\frac{1}{\beta_{k+1}}\|y_{k+1}-y_k\|^2\\ &\geq (\rho_k-\rho_{k+1})(-\sigma)\|y_{k+1}-y_k\|^2+\frac{1}{\beta_{k+1}}\|y_{k+1}-y_k\|^2\\ &\geq [\frac{1}{\beta_{k+1}}-(\rho_k-\rho_{k+1})\sigma]\|y_{k+1}-y_k\|^2\end{aligned}$$

and so,

$$\begin{aligned}\|y_k-y_{k-1}\|\|y_{k+1}-y_k\| &\geq \langle y_k-y_{k-1},y_{k+1}-y_k\rangle\\ &\geq [\frac{\beta_k}{\beta_{k+1}}-(\rho_k-\rho_{k+1})\beta_k\sigma]\|y_{k+1}-y_k\|^2.\end{aligned}$$

Consequently, one has

$$\|y_k-y_{k-1}\|\geq[\frac{\beta_k}{\beta_{k+1}}+(\rho_{k+1}-\rho_k)\beta_k\sigma]\|y_{k+1}-y_k\|.$$

This together with the condition (b) yields that $\{y_k\}$ is a Cauchy sequence. Without loss of generality, let $y_k\to\bar{y}\in K$. Since Ψ and B are upper semicontinuous, ψ is continuous, it follows from (3.2) and (a) that

$$\Psi(\bar{y},z)+\langle B\bar{y},z-\bar{y}\rangle+\psi(z,\bar{y})-\psi(\bar{y},\bar{y})\geq 0,\quad \forall z\in K,$$

that is, $\bar{y}\in S_{\Psi,B,\psi}$.

Now, we show that $\bar{y}\in S_{\Psi,B,\psi}$ is a solution of the problem (1.3). For any $y\in S_{\Psi,B,\psi}\subseteq K$, from Algorithm I, we have

$$\begin{aligned}&\langle y_{k+1}-y_k,y-y_{k+1}\rangle+\beta_{k+1}\rho_{k+1}T(y_{k+1},y)\\ &+\beta_{k+1}[H(y_{k+1},y)+\psi(y,y_{k+1})-\psi(y_{k+1},y_{k+1})]\geq 0,\end{aligned}$$

which implies that, for any $y \in S_{\Psi,B,\psi}$,

$$\begin{aligned} & T(y_{k+1}, y) + \frac{1}{\beta_{k+1}\rho_{k+1}}\langle y_{k+1} - y_k, y - y_{k+1}\rangle \\ \geq & -\frac{1}{\rho_{k+1}}(H(y_{k+1}, y) + \psi(y, y_{k+1}) - \psi(y_{k+1}, y_{k+1})) \\ \geq & \frac{1}{\rho_{k+1}}(H(y, y_{k+1}) + \psi(y_{k+1}, y) - \psi(y, y)) \\ \geq & \ 0, \end{aligned}$$

that is,

$$\begin{aligned} & \Phi(y_{k+1}, y) + \langle Ay_{k+1}, y - y_{k+1}\rangle + \phi(y, y_{k+1}) - \phi(y_{k+1}, y_{k+1}) \\ + & \frac{1}{\beta_{k+1}\rho_{k+1}}\langle y_{k+1} - y_k, y - y_{k+1}\rangle \geq 0. \end{aligned}$$

Since Φ, A are upper semicontinuous and ϕ is continuous, by the condition (a), one has

$$\Phi(\bar{y}, y) + \langle A\bar{y}, y - \bar{y}\rangle + \phi(y, \bar{y}) - \phi(\bar{y}, \bar{y}) \geq 0, \quad \forall y \in S_{\Psi,B,\psi}.$$

Therefore, the sequence $\{y_k\}$ generated by Algorithm I converges strongly to a solution of the problem (BMEP) (1.3) with (1.4). This completes the proof. ■

The following result is a direct consequence of Theorem 3.1.

Corollary 3.1. *Assume that all the conditions of Theorem 2.5 hold and there exists $\sigma > 0$ such that*

$$\Phi(y, z) + \langle Ay, z - y\rangle + \phi(z, y) - \phi(y, y) \geq -\sigma\|y - z\|^2, \quad \forall y, z \in K.$$

If the conditions (*a*) *and* (*b*) *of Theorem 3.1 are satisfied, then the sequence $\{y_k\}$ generated by Algorithm I converges strongly to the unique solution of the problem* (*BMEP*) (*1.3*) *with* (*1.4*).

Remark 3.3. If we let $\rho_{k+1} = \frac{1}{\beta_{k+1}}$, then the condition (a) of Theorem 3.1 holds.

For the condition (b) of Theorem 3.1, we give the following example.

Example 3.3. Let $E = (-\infty, +\infty) = K$, $\Phi(y, z) = z - y, A(y) = -2$ and $\phi(y, z) = y - z$ for all $y, z \in E$. It is easy to check that Φ, A and ϕ satisfy all the conditions of Theorems 2.4 and 2.5. Simple computation allows that, for any positive number σ,

$$\Phi(y, z) + \langle Ay, z - y\rangle + \phi(z, y) - \phi(y, y) \geq -\sigma\|y - z\|^2, \quad \forall y, z \in E.$$

Particularly, we put $\sigma \in (0,1)$ and $\tau = 1-\sigma$. If $\beta_{k+1} = 2^k$ and $\rho_k = \frac{1}{2^k}$, then we have

$$\frac{\beta_k}{\beta_{k+1}} + (\rho_{k+1} - \rho_k)\beta_k\sigma = \frac{1-\sigma}{2} < 1-\sigma = \tau, \quad \forall k \in N \cup \{0\}.$$

4. The Well-posedness for (BMEP)

In this section, we firstly consider a class of ϵ-bilevel mixed equilibrium problem, discuss the behavior of solution set to this class of (BMEP). Further, the concepts of the well-posedness and generalized well-posedness for the problem (BMEP) (1.3) with (1.4) are introduced. The relationships between the well-posedness (generalized well-posedness) for the problem (BMEP) (1.3) with (1.4) and the uniqueness and existence of its solution are established.

For any $\epsilon > 0$, consider the following *ϵ-bilevel mixed equilibrium problem* (for short, (BMEP)) (4.1) with (4.2):

Find $x \in S_{\Psi,B,\psi}(\epsilon)$ such that

$$\Phi(x,y) + \langle Ax, y-x\rangle + \phi(y,x) - \phi(x,x) + \epsilon \geq 0, \quad \forall y \in S_{\Psi,B,\psi}(\epsilon), \tag{4.1}$$

where $S_{\Psi,B,\psi}(\epsilon)$ is the solution set of the following mixed equilibrium problem:

Find $y^* \in K$ such that

$$\Psi(y^*,z) + \langle By^*, z-y^*\rangle + \psi(z,y^*) - \psi(y^*,y^*) + \epsilon \geq 0, \quad \forall z \in K. \tag{4.2}$$

Denote the solution set of the problem (BMEP) (4.1) with (4.2) by $\aleph(\epsilon)$. It is easy to see that $\aleph \subseteq \aleph(\epsilon)$ and $\aleph(\epsilon_1) \subseteq \aleph(\epsilon_2)$ for any $\epsilon_1, \epsilon_2 > 0$ with $\epsilon_1 \leq \epsilon_2$.

Lemma 4.1. *Assume that all the conditions of Theorem 2.1 are satisfied. Then the following statements hold:*

(1) for each $\epsilon > 0$, $S_{\Psi,B,\psi}(\epsilon)$ is nonempty and weakly compact convex;

(2) $S_{\Psi,B,\psi} = \bigcap_{\epsilon>0} S_{\Psi,B,\psi}(\epsilon)$.

Proof. (1) The proofs are similar to those of Theorem 2.1 and so it is omitted here.

(2) We know that $S_{\Psi,B,\psi} \subseteq \bigcap_{\epsilon>0} S_{\Psi,B,\psi}(\epsilon)$. Now, we only need to prove that $S_{\Psi,B,\psi} \supseteq \bigcap_{\epsilon>0} S_{\Psi,B,\psi}(\epsilon)$. Let $y^* \in \bigcap_{\epsilon>0} S_{\Psi,B,\psi}(\epsilon)$. Then, for each $\epsilon > 0, y^* \in S_{\Psi,B,\psi}(\epsilon)$. Without loss of generality, suppose that $0 < \epsilon_n \to 0$. Then we have $y^* \in S_{\Psi,B,\psi}(\epsilon_n)$, that is, $y^* \in K$ such that

$$\Psi(y^*,z) + \langle By^*, z-y^*\rangle + \psi(z,y^*) - \psi(y^*,y^*) + \epsilon_n \geq 0, \quad \forall z \in K.$$

Taking the limit in the above inequality, one has

$$\Psi(y^*,z) + \langle By^*, z-y^*\rangle + \psi(z,y^*) - \psi(y^*,y^*) \geq 0, \quad \forall z \in K.$$

Therefore, $y^* \in S_{\Psi,B,\psi}$. This completes the proof. ■

Lemma 4.2. *Assume that all the conditions of Theorem 2.4 hold. Then the following statements hold:*
(1) for each $\epsilon > 0$, $\aleph(\epsilon)$ *is nonempty and weakly compact convex;*
(2) $\aleph = \bigcap_{\epsilon>0} \aleph(\epsilon)$.

Proof. The proofs of (1) and (2) are similar to those of Theorem 2.4 and Lemma 4.1 (2), respectively, and so it is omitted here. ■

Lemma 4.3. *Assume that all the conditions of Theorem 2.5 hold. Then, for each* $\epsilon > 0$, $\aleph(\epsilon)$ *is a singleton.*

Proof. The proofs are similar to those of Theorem 2.5 and so it is omitted here. ■

Definition 4.1. A sequence $\{x_n\} \subseteq K$ is called an *approximation solution sequence* of the problem (BMEP) (1.3) with (1.4) if there exists a sequence $\{\epsilon_n\}$ with $0 < \epsilon_n \to 0$ and $x_n \in S_{\Psi,B,\psi}(\epsilon_n)$ such that

(4.3) $$\Phi(x_n, y) + \langle Ax_n, y - x_n\rangle + \phi(y, x_n) - \phi(x_n, x_n) + \epsilon_n \geq 0, \quad \forall y \in S_{\Psi,B,\psi}(\epsilon_n).$$

Remark 4.1. By Definition 4.1, for any approximation solution sequence $\{x_n\}$ of the problem (BMEP) (1.3) with (1.4), there exists a sequence $\{\epsilon_n\}$ of nonnegative real numbers with $\epsilon_n \to 0$ such that $x_n \in \aleph(\epsilon_n)$.

Definition 4.2. The problem (BMEP) (1.3) with (1.4) is said to be:
(1) *well-posed* if the problem (BMEP) (1.3) with (1.4) has a unique solution and any approximation solution sequence $\{x_n\}$ of the problem (BMEP) (1.3) with (1.4) converges strongly to the unique solution.
(2) *generalized well-posed* if the solution set $\aleph$ of the problem (BMEP) (1.3) with (1.4) is nonempty and any approximation solution sequence $\{x_n\}$ of the problem (BMEP) (1.3) with (1.4) has a subsequence which converges strongly to some point of $\aleph$.

Remark 4.2. By Definition 4.2, we know that the well-posedness and generalized well-posedness of the problem (BMEP) (1.3) with (1.4) imply that $\aleph$ is compact.

Definition 4.3. Let Ω and Ξ be nonempty subsets of E.
(1) The *Hausdorff distance* $D(\Omega, \Xi)$ between Ω and Ξ defined by

$$D(\Omega, \Xi) = \max\{e(\Omega, \Xi), e(\Xi, \Omega)\},$$

where $e(\Omega, \Xi) = \sup_{\omega\in\Omega} d(\omega, \Xi)$ with $d(\omega, \Xi) = \inf_{\xi\in\Xi} \|\omega - \xi\|$.
(2) For a given $x^* \in \aleph$, we define

$$\vartheta(\epsilon) = \sup\{\|x^* - x\| : x \in \aleph(\epsilon)\}.$$

Theorem 4.1. *Let $x^* \in \aleph$. Then the problem (BMEP) (1.3) with (1.4) is well-posed if and only if $\vartheta(\epsilon) \to 0$ as $\epsilon \to 0$.*

Proof. Suppose that $\vartheta(\epsilon) \not\to 0$ as $\epsilon \to 0$. Then there exist $\varrho > 0$ and $0 < \epsilon_n \to 0$ such that $\vartheta(\epsilon_n) > \varrho$. This implies that there exists $x_n \in \aleph(\epsilon_n)$ such that $\|x^* - x_n\| > \varrho$. In view of $x_n \in \aleph(\epsilon_n)$, it follows that $x_n \in S_{\Psi,B,\psi}(\epsilon_n)$ and

$$\Phi(x_n, y) + \langle Ax_n, y - x_n\rangle + \phi(y, x_n) - \phi(x_n, x_n) + \epsilon_n \geq 0, \quad \forall y \in S_{\Psi,B,\psi}(\epsilon_n). \tag{4.4}$$

This implies that $\{x_n\}$ is an approximation solution sequence of the problem (BMEP) (1.3) with (1.4). By Definition 4.2, it follows that $x_n \to x^*$, which contradicts with $\|x^* - x_n\| > \varrho$.

Conversely, suppose that $\vartheta(\epsilon) \to 0$ as $\epsilon \to 0$. Clearly, $\aleph = \{x^*\}$. For any approximation solution sequence $\{x_n\}$ of the problem (BMEP) (1.3) with (1.4). Then there exists $0 < \epsilon_n \to 0$ such that $x_n \in \aleph(\epsilon_n)$. Take into account of $\aleph \subseteq \aleph(\epsilon_n)$, one has

$$\|x_n - x^*\| \leq \vartheta(\epsilon_n) \to 0.$$

Moreover, the sequence $\{x_n\}$ converges strongly to x^*. Therefore, the problem (BMEP) (1.3) with (1.4) is well-posed. This completes the proof. ■

Theorem 4.2. *The problem (BMEP) (1.3) with (1.4) is generalized well-posed if and only if $\aleph$ is nonempty compact and $D(\aleph, \aleph(\epsilon)) \to 0$ as $\epsilon \to 0$.*

Proof. Assume that the problem (BMEP) (1.3) with (1.4) is generalized well-posed. Then $\aleph$ is compact. As a matter of fact, for any sequence $\{x_n\} \subseteq \aleph$, there exists $0 < \epsilon_n \to 0$ such that $x_n \in \aleph(\epsilon_n)$ and so $\{x_n\}$ is an approximation solution sequence of the problem (BMEP) (1.3) with (1.4). Then $\{x_n\}$ has a subsequence which converges strongly to some point of $\aleph$. So, $\aleph$ is a compact subset of K.

Let us show that $D(\aleph, \aleph(\epsilon)) \to 0$ as $\epsilon \to 0$. We only need to prove that $e(\aleph(\epsilon), \aleph) \to 0$ as $\epsilon \to 0$. Since $\aleph \subseteq \aleph(\epsilon)$ for any $\epsilon > 0$, $e(\aleph, \aleph(\epsilon)) = 0$. Suppose that $0 < \epsilon \to 0$ and $e(\aleph(\epsilon), \aleph) \not\to 0$. Then there exist $b > 0$, $0 < \epsilon_n \to 0$ and $x_n \in \aleph(\epsilon_n)$ such that $d(x_n, \aleph) \geq b$. Again, from $x_n \in \aleph(\epsilon_n)$, $\{x_n\}$ is an approximation solution sequence of the problem (BMEP) (1.3) with (1.4). By the generalized well-posedness of the problem (BMEP) (1.3) with (1.4), $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges strongly to a point of $\aleph$. So, $\lim_{k\to+\infty} d(x_{n_k}, \aleph) = 0$, which contradicts $d(x_n, \aleph) \geq b$. Therefore, $D(\aleph, \aleph(\epsilon)) \to 0$ as $\epsilon \to 0$.

Conversely, assume that $\aleph$ is nonempty compact and $D(\aleph, \aleph(\epsilon)) \to 0$ as $\epsilon \to 0$. For any approximation solution sequence $\{x_n\}$ of the problem (BMEP) (1.3) with (1.4), there exists $0 < \epsilon_n \to 0$ such that $x_n \in \aleph(\epsilon_n)$. Consequently, we have

$$d(x_n, \aleph) \leq D(\aleph, \aleph(\epsilon_n)) \to 0.$$

Due to the compactness of $\aleph$, there exists $y_n \in \aleph$ such that

$$d(x_n, \aleph) = \|x_n - y_n\| \to 0.$$

It follows from the compactness of $\aleph$ that y_n has a subsequence $\{y_{n_k}\}$ which converges strongly to a point $\tilde{y} \in \aleph$. Thus there exists the corresponding sequence x_{n_k} of $\{x_n\}$ such that

$$\|x_{n_k} - \tilde{y}\| \leq \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - \tilde{y}\| \to 0,$$

that is, $\{x_{n_k}\}$ converges strongly to the point $\tilde{y}$. Therefore, the problem (BMEP) (1.3) with (1.4) is generalized well-posed. This completes the proof. ■

Theorem 4.3. *Assume that all the conditions of Theorem 2.5 are satisfied. Then the problem* (*BMEP*) (*1.3*) *with* (*1.4*) *is well-posed.*

Proof. From Theorem 2.5, one has the solution set $\aleph$ of the problem (BMEP) (1.3) with (1.4) is a singleton. Without loss of generality, let $\aleph = \{x^*\}$. For any approximation solution sequence $\{x_n\}$ of the problem (BMEP) (1.3) with (1.4), there exists $0 < \epsilon_n \to 0$ and $x_n \in S_{\Psi,B,\psi}(\epsilon_n)$ such that (4.3) holds. Moreover, $x_n \in \aleph(\epsilon_n)$. By Lemmas 4.2 and 4.3, $x_n = x^*$, that is, $\{x_n\}$ is strongly convergent to x^*. Therefore, the problem (BMEP) (1.3) with (1.4) is well-posed. This completes the proof. ■

Theorem 4.4. *Assume that all the conditions of Theorem 2.4 are satisfied. Then the problem* (*BMEP*) (*1.3*) *with* (*1.4*) *is generalized well-posed.*

Proof. By Lemma 4.2, we know that, for each $\epsilon > 0$, $\aleph(\epsilon)$ is nonempty and weakly compact convex. For any approximation solution sequence $\{x_n\}$ of the problem (BMEP) (1.3) with (1.4), there exist $0 < \epsilon_n \to 0$ and $x_n \in S_{\Psi,B,\psi}(\epsilon_n)$ such that (4.3) holds. Moreover, $x_n \in \aleph(\epsilon_n)$. Since $\aleph(\tilde{\epsilon}) \subseteq \aleph(\hat{\epsilon})$ for $0 < \tilde{\epsilon} \leq \hat{\epsilon}$, there exists $\bar{\epsilon} > 0$ such that $\aleph(\epsilon_n) \subseteq \aleph(\bar{\epsilon})$. So, $\{x_n\} \subseteq \aleph(\bar{\epsilon})$. By the compactness of $\aleph(\bar{\epsilon})$, there exists a subsequence x_{n_k} of $\{x_n\}$ such that $x_{n_k} \to x_0 \in \aleph(\bar{\epsilon})$. Again, from Definition 4.1, $x_{n_k} \in S_{\Psi,B,\psi}(\epsilon_{n_k})$ such that

$$\begin{aligned} &\Phi(x_{n_k}, y) + \langle Ax_{n_k}, y - x_{n_k}\rangle + \phi(y, x_{n_k}) \\ &-\phi(x_{n_k}, x_{n_k}) + \epsilon_{n_k} \geq 0, \quad \forall y \in S_{\Psi,B,\psi}(\epsilon_{n_k}). \end{aligned} \tag{4.5}$$

Observe that $S_{\Psi,B,\psi}(\epsilon_{n_k}) \to S_{\Psi,B,\psi}$ as $\epsilon_{n_k} \to 0$. By virtue of the conditions of Theorem 2.4, $x_0 \in S_{\Psi,B,\psi}$ and

$$\Phi(x_0, y) + \langle Ax_0, y - x_0\rangle + \phi(y, x_0) - \phi(x_0, x_0) \geq 0, \quad \forall y \in S_{\Psi,B,\psi},$$

that is, $x_0 \in \aleph$. Therefore, the problem (BMEP) (1.3) with (1.4) is generalized well-posed. This completes the proof. ■

Remark 4.3. Theorem 4.3 illustrate not only the well-posedness of the problem (BMEP) (1.3) with (1.4), but also the equivalence between the well-posedness of the problem (BMEP) (1.3) with (1.4) and the existence and uniqueness of its solution; Theorem 4.4 implies that the generalized well-posedness of the problem (BMEP) (1.3) with (1.4) is equivalent to the existence of its solution.

ACKNOWLEDGMENTS

The authors are indebted to the referees and the associate editor for their insightful and pertinent comments. The first author would like to gratefully thank Professor Heinz H. Bauschke and Shawn Xianfu Wang, Mathematics, Irving K. Barber School, University of British Columbia, Kelowna, British Columbia, Canada, for their hospitality and providing excellent research facilities.

REFERENCES

1. P. N. Anh, J. K. Kim and L. D. Muu, An extragradient algorithm for solving bilevel pseudomonotone variational inequalities, *J. Glob. Optim.*, doi:10.1007/s10898-012-9870-y.
2. L. Q. Anh, P. Q. Khanh and D. T. M. Van, Well-posedness under relaxed semicontinuity for bilevel equilibrium and optimization problems with equilibrium constraints, *J. Optim. Theory Appl.*, doi:10.1007/s10957-011-9963-7.
3. A. S. Antipin, Iterative gradient prediction-type methods for computing fixed-point of extremal mappings. in: *Parametric Optimization and Related Topics IV[C]*, (J. Guddat, H. Th. Jonden, F. Nizicka, G. Still and F. Twitt, eds.), Peter Lang, Frankfurt Main, 1997, pp. 11-24.
4. J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley and Sons, New York, 1984.
5. S. I. Birbil, G. Bouza, J. B. G. Frenk and G. Still, Equilibrium constrained optimization problems, *Europ. J. Oper. Research*, **169** (2006), 1108-1127.
6. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Mathematics Stud.*, **63** (1994), 123-145.
7. O. Chadli, H. Mahdioui and J. C. Yao, Bilevel mixed equilibrium problems in Banach spaces: existence and algorithmic aspects, *Numer. Algebra Cont. Optim.*, **1(3)** (2011), 549-561.
8. J. W. Chen, Y. J. Cho and Z. P. Wan, Shrinking projection algorithms for equilibrium problems with a bifunction defined on the dual space of a Banach space, *Fixed Point Theory Appl.*, **2011**, 2011:91.
9. J. W. Chen and Z. Wan, Existence of solutions and convergence analysis for a system of quasi-variational inclusions in Banach spaces, *J. Inequal. Appl.*, 2011:49, doi:10.1186/1029-242X-2011-49.

10. J. W. Chen, Z. Wan and Y. J. Cho, Levitin-Polyak well-posedness by perturbations for systems of set-valued vector quasi-equilibrium problems, *Math. Meth. Oper. Res.*, Doi 10.1007/s00186-012-0414-5.

11. J. W. Chen, Z. P. Wan and Y. Zou, Strong convergence theorems for firmly nonexpansive-type mappings and equilibrium problems in Banach spaces, *Optim.*, (2011), doi:10.1080/02331934.2011.626779.

12. Y. J. Cho, X. Qin and J. I. Kang, Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems, *Nonlinear Anal.*, **71** (2009), 4203-4214.

13. X. P. Ding, Auxiliary principle and algorithm for mixed equilibrium problems and bilevel mixed equilibrium problems in Banach spaces, *J. Optim. Theory Appl.*, **146** (2010), 347-357.

14. X. P. Ding, Existence and algorithm of solutions for mixed equilibrium problems and bilevel mixed equilibrium problems in Banach spaces, *Acta Math. Sin. English. Ser.*, **28** (2011), 503-514.

15. X. P. Ding, Bilevel generalized mixed equilibrium problems involving generalized mixed variational-like inequality problems in reflexive Banach spaces, *Appl. Math. Mech.-Engl. Ed.*, **32(11)** (2011), 1457-1474.

16. X. P. Ding, Existence and iterative algorithm of solutions for a class of bilevel generalized mixed equilibrium problems in Banach spaces, *J. Glob. Optim.*, 2011, doi:10.1007/s10898-011-9724-z.

17. X. P. Ding, Y. C. Liou and J. C. Yao, Existence and algorithms for bilevel generalized mixed equilibrium problems in Banach spaces, *J. Glob. Optim.*, doi:10.1007/s10898-011-9712-3.

18. B. V. Dinh and L. D. Muu, *On penalty and gap function methods for bilevel equilibrium problems*, J. Appl. Math., Vol. 2011, Article ID 646452, 14 pp.

19. Y. P. Fang, R. Hu and N. J. Huang, Well-posedness for equilibrium problems and for optimization problems with equilibrium constraints, *Comput. Math. Appl.*, **55** (2008), 89-100.

20. S. M. Guu and J. Li, Vector variational-like inequalities with generalized bifunctions defined on nonconvex sets, *Nonlinear Anal.*, **71(7-8)** (2009), 2847-2855.

21. N. J. Huang, J. Li and B. H. Thompson, Stability for parametric implicit vector equilibrium problems, *Math. Comput. Model.*, **43** (2006), 1267-1274.

22. M. B. Lignola and J. Morgan, Well-posedness for optimization problems with constraints defined by variational inequalities having a unique solution, *J. Global Optim.*, **16(1)** (2000), 57-67.

23. L. J. Lin, Mathematical programming with system of equilibrium constraints, *J. Glob. Optim.*, **37** (2007), 275-286.

24. L. J. Lin, Existence theorems for bilevel problem with applications to mathematical program with equilibrium constraint and semi-infinite problem, *J. Optim. Theory Appl.*, **137** (2008), 27-40.

25. Z. Q. Luo, J. S. Pang and D. Ralph, *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, 1996.
26. Y. Lv, Z. Chen and Z. Wan, A neural network approach for solving mathematical programs with equilibrium constraints, *Expert Sys. Appl.*, **38** (2011), 231-234.
27. A. Moudafi, Proximal methods for a class of bilevel monotone equilibrium problems, *J. Glob. Optim.*, **47** (2010), 287-292.
28. J. Salamon, Closedness and Hadamard well-posedness of the solution map for parametric vector equilibrium problems, *J. Glob. Optim.*, **47** (2010), 173-183.
29. Z. Wan, J. W. Chen, H. Sun and L. Yuan, A new system of generalized mixed quasivariational inclusions with relaxed cocoercive operators and applications, *J. Appl. Math.*, **2011** (2011), 1-22.
30. H. F. Xu and J. J. Ye, Necessary optimality conditions for two-stage stochastic programs with equilibrium constraints, *SIAM J. Optim.*, **20(4)** (2010), 1685-1715.
31. J. J. Ye, Necessary and sufficient optimality conditions for mathematical programs with equilibrium constraints, *J. Math. Anal. Appl.*, **307** (2005), 305-369.
32. Y. Yao, Y. J. Cho and Y. Liou, Algorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems, *Europ. J. Operat. Research*, **212** (2011), 242-250.
33. R. Y. Zhong, N. J. Huang and Y. J. Cho, *Boundedness and nonemptiness of solution sets for set-valued vector equilibrium problems with an application*, J. Inequal. Appl., Vol. 2011, Article ID 936428, 15 pp.

Jia Wei Chen
School of Mathematics and Statistics
Wuhan University
Wuhan, Hubei 430072
P. R. China
E-mail: J.W.Chen713@163.com

Zhongping Wan
School of Mathematics and Statistics
Wuhan University
Wuhan, Hubei 430072
P. R. China
E-mail: zpwan-whu@126.com

Yeol Je Cho
Department of Mathematics Education and the RINS
Gyeongsang National University
Chinju 660-701
Korea
E-mail: yjcho@gnu.ac.kr