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EXISTENCE AND UNIQUENESS OF LAX-TYPE SOLUTIONS TO THE RIEMANN PROBLEM OF SCALAR BALANCE LAW WITH SINGULAR SOURCE TERM

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Abstract. We give a new approach of constructing the generalized entropy solutions to the Riemann problem of scalar nonlinear balance laws with singular source terms. The source term is singular in the sense that it is a product of delta function and a discontinuous function, which is undefined in distribution. By re-formulating the source term, we study the corresponding perturbed Riemann problem. The existence and stability of perturbed Riemann solutions is established under some entropy condition so that the generalized entropy solutions of Riemann problem can be interpreted as the limit of corresponding perturbed Riemann solutions. The self-similarity of generalized entropy solutions is also obtained, which means that Lax's method in [13] can be extended to scalar nonlinear balance laws with singular source terms.

1. INTRODUCTION

In this paper we consider the Riemann problem of scalar nonlinear balance law

$$u_t + f(u)_x = a'(x)g(u), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{1.1}$$

$$u(x,0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \tag{1.2}$$

where $u = u(x,t) \in \mathbb{R}$ and u_L, u_R are constants. Also $a(x)$ is given by

$$a(x) = \begin{cases} a_L, & x < 0, \\ a_R, & x > 0, \end{cases} \tag{1.3}$$

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where a_L, a_R are also constants, and $a'(x) \equiv da/dx$ in the sense of distribution. We assume that f, g are smooth functions of u, and 0 is the only root of $f'(u)$ and $g(u)$, also $f''(u) > 0$ for all u.

It is well known that equation (1.1) only admits weak solutions, it follows that $a'g(u)$ may not be defined in distribution since it can be a product of delta function a' and discontinuous function $g(u)$. Indeed, by the results of Dal Maso- LeFloch- Murat [4], source term $a'g(u)$ is only defined as a Borel measure, which is weaker than distribution. The goal of this paper is to investigate the global existence and stability of generalized entropy solutions of Lax-type to (1.1)-(1.3).

We first review previous results related to this topic. When $g(u) \equiv 0$, equation (1.1) is reduced to a scalar conservation law

$$u_t + f(u)_x = 0. \tag{1.4}$$

To the case that (1.4) is a system with genuinely nonlinear or linear degenerate characteristic fields, the existence of weak solutions to the Riemann problem is first established by Lax [13]. In [13], Lax showed that, under some entropy condition, Riemann problem (1.4), (1.2) admits a unique admissible weak solution consisting of constant states separated by elementary waves (rarefaction waves, shock waves or contact discontinuities) when $|u_L - u_R|$ is sufficiently small. We call this kind of self-similar solutions the $weak\ solutions\ of\ Lax\ type$. Moreover, in the paper by Glimm [6], the global existence of solutions for the Cauchy problem is established by the scheme involved random choice method whose approximate solutions are constructed based on the Riemann solutions in [13]. On the other hand, the measure-valued solutions of (1.1) was studied by DiPerna [5] by using the technique of zero diffusion method and compensated compactness. To the quasilinear hyperbolic equations

$$u_t + f(t, x, u)_x = g(t, x, u), \tag{1.5}$$

the weak solutions to the Cauchy problem of scalar equation was first studied by Volpert [23] and Kruzkov [12]. For system (1.5) with $f = f(x, u),\ g = g(x, u)$, the global existence of weak solutions for the Cauchy problem was first established by Liu [16]. On the other hand, Dafermos [1] invented the technique of generalized characteristics to study the structure of solutions to general system (1.5). Furthermore, the weak solutions to the Cauchy problem of general system (1.5) were studied by Dafermos-Hsiao [3] and Hong-LeFloch [9]. We refer the readers to [19] for more details of the Riemann problem for balance laws without convexity. We notice that the results described above cannot be applied to our case since the source term in (1.1) is not defined in distribution. In addition, the solutions of (1.1)-(1.3) may not be self-similar due to the appearance of source term. Therefore, the technique in [13] cannot be applied to our problem.

In this paper we provide a new approach of constructing a unique generalized entropy solution of (1.1)-(1.3). The entropy solution, which is constructed as the limit of corresponding perturbed Riemann solution, is self-similar so that Lax's method can be extended to the Cauchy problem. The framework of this paper is described as follows. First, for given $0 < \varepsilon << 1$, we re-formulate source term $a'g(u)$ by $a'_\varepsilon(x)g(u^\varepsilon)$ where $a_\varepsilon(x)$ is chosen as a smooth monotone function in $[-\varepsilon, +\varepsilon]$ and connects constant states $a_L,\ a_R$ at $x = \pm\varepsilon$. Then, for such $a_\varepsilon(x)$ we define the corresponding $perturbed\ Riemann\ problem$ of (1.1)-(1.3):

$$u^\varepsilon_t + f(u^\varepsilon)_x = a'_\varepsilon(x)g(u^\varepsilon), \quad (x,t) \in \mathbb{R}\times\mathbb{R}^+, \tag{1.6}$$

$$u^\varepsilon(x,0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \tag{1.7}$$

when $u_L > u_R$, or

$$u^\varepsilon(x,0) = \begin{cases} u_L, & x < -\varepsilon, \\ \zeta^\varepsilon(x), & -\varepsilon \le x \le \varepsilon, \\ u_R, & x > \varepsilon, \end{cases} \tag{1.8}$$

when $u_L < u_R$ where $\zeta^\varepsilon(x)$ is the linear function connecting $u_L,\ u_R$ at $x = \pm\varepsilon$. And $a_\varepsilon(x)$ is given by

$$a_\varepsilon(x) = \begin{cases} a_L, & x < -\varepsilon, \\ \varphi^\varepsilon(x), & -\varepsilon \le x \le \varepsilon, \\ a_R, & x > \varepsilon, \end{cases} \tag{1.9}$$

where $\varphi^\varepsilon(x)$ is a C^1 monotone function connecting $a_L,\ a_R$ at $x = \pm\varepsilon$. Note that $a'_\varepsilon(x)g(u^\varepsilon)$ in (1.6) is defined in distribution. Next, we apply the characteristic method for first order partial differential equations to construct $u^\varepsilon(x,t)$ of (1.6)-(1.9). When $u^\varepsilon(x,t)$ is constructed, the generalized solutions of (1.1)-(1.3) can be interpreted as the limit of $u^\varepsilon(x,t)$, which is given in the following definition.

Definition 1.1. Suppose that $u^\varepsilon(x,t)$ is the weak solution of perturbed Riemann problem (1.6)-(1.9) for given $a_\varepsilon(x)$. Then, generalized solution $u(x,t)$ of (1.1)-(1.3) under the re-formulation of source term $a'_\varepsilon(x)g(u^\varepsilon)$ is defined as

$$u(x,t) \equiv \lim_{\varepsilon\to 0} u^\varepsilon(x,t). \tag{1.10}$$

We show that, the limit of perturbed Riemann solutions to (1.6)-(1.9) with $a_\varepsilon(x)$ piecewise linear, consist of at most three constant states separated by either rarefaction waves or shock waves, and a discontinuous standing wave from stationary field (standing shock) when $a_L \neq a_R$ (Sections 2 and 3). Thus, the generalized solutions of (1.1)-(1.3) are self-similar under Definition 1.1. In addition, by (1.10) we observe that the standing shocks appear in generalized solutions due to the bending (instead of intersection) of characteristic curves in stationary field. So Rankine-Hugoniot condition cannot be applied to this kind of discontinuous waves.

We notice that the behavior of perturbed Riemann solutions depend on the signs of source term. In this paper we focus on the cases that (A) $u_L,\ u_R > 0$, $a_L > a_R$, and $g(u) > 0$ ($(a_R - a_L)g(u) < 0$), or (B) $u_L,\ u_R > 0$, $a_L > a_R$, and $g(u) < 0$ ($(a_R - a_L)g(u) > 0$). The analysis for the rest of cases is similar. On the other hand, there are infinitely many ways to select the profile of $\varphi^\varepsilon(x)$ in (1.9), and the structure of perturbed Riemann solutions is dependent on the choice of $\varphi^\varepsilon(x)$. Therefore, the generalized solutions of (1.1)-(1.3) may not be unique under Definition 1.1. To obtain the uniqueness of generalized solutions, we give a condition to $a_\varepsilon(x)$, which can be considered as an extra entropy condition beside Lax entropy condition (Section 4). The entropy condition is given as follows. To Case (A) (resp., Case (B)), we assume that $a_\varepsilon(x)$ in (1.6) is a monotone function satisfying

$$a_\varepsilon(x) = \bar{a}_\varepsilon(x) + \delta_\varepsilon(x), \tag{1.11}$$

where $\bar{a}_\varepsilon(x)$ is the piecewise linear function (see (2.3)) and $\delta_\varepsilon(x) \in C_0^2([-\varepsilon, \varepsilon])$ satisfying

$$\delta_\varepsilon''(x) > 0 \quad \text{(resp., } \delta_\varepsilon''(x) < 0\text{)} \quad \forall\, x \in (-\varepsilon, \varepsilon), \tag{1.12}$$

and

$$\|\delta_\varepsilon'(x)\|_{L^1([-\varepsilon,\varepsilon])} \to 0 \quad \text{as } \varepsilon \to 0. \tag{1.13}$$

We show the stability of $u^\varepsilon(x,t)$ whenever $a_\varepsilon(x)$ satisfying (1.11)-(1.13) so that the uniqueness of generalized entropy solution to (1.1)-(1.3) can be established under the following definition.

Definition 1.2. Suppose that $u^\varepsilon(x,t)$ is a weak solution of (1.6)-(1.9) where $a_\varepsilon(x)$ satisfies (1.11)-(1.13). Then the generalized entropy solution of (1.1)-(1.3) is defined as

$$u(x,t) \equiv \lim_{\varepsilon \to 0} u^\varepsilon(x,t).$$

We mention that equation (1.1) can be written as the 2×2 system of balance laws

$$U_t + F(U)_x = a'(x)G(U),$$

or the hyperbolic system in non-conservative form

$$U_t + A(U) \cdot U_x = 0,$$

where $U = (a, u)^T$, $F(U) = (0, f(u))^T$, $G(U) = (0, g(u))^T$ and

$$A(U) = \begin{bmatrix} 0 & 0 \\ -g & \frac{\partial f}{\partial u} \end{bmatrix}.$$

Note that the eigenvalues of $A(U)$ are 0 and $\frac{\partial f}{\partial u}$, which implies that the resonant phenomenon (eigenvalues of $A(U)$ coincide) occurs when the initial data contains u^* satisfying $f_u(u^*) = 0$. The technique developed here provides a new direction of studying the generalized solutions to the Riemann problem of resonant systems, although we only study the non-resonance case ($u_L,\ u_R$ are nonzero and have the same signs) in this paper. We refer the readers to [4, 7, 8, 14] for more details of the resonant systems in hyperbolic balance laws.

The outline of the paper is summarized as follows. In Section 2, we apply characteristic method to study the classical perturbed Riemann solutions of (1.6)-(1.9) where $a_\varepsilon(x)$ is piecewise linear. In Section 3, we study the shock waves of (1.6)-(1.9) with $a_\varepsilon(x)$ given in Section 2. In Section 4, we first demonstrate an example to show that the type of weak solutions to (1.6)-(1.9) completely change (from rarefaction waves to shock waves) if piecewise linear function $a_\varepsilon(x)$ is perturbed into a non-monotone function in $[-\varepsilon, \varepsilon]$. Next, by imposing (1.11)-(1.13) as the extra conditions beside Lax entropy condition, we establish the stability of perturbed Riemann solutions, which leads to the uniqueness of generalized entropy solutions to (1.1)-(1.3). In the end of the paper, we give the main theorem of this paper.

2. Classical Solutions of Perturbed Riemann Problem

In this section we study the classical solutions of perturbed Rimann problem where $a_\varepsilon(x)$ is linear within $[-\varepsilon, \varepsilon]$. For given $0 < \varepsilon << 1$, we consider

$$u_t^\varepsilon + f(u^\varepsilon)_x = a_\varepsilon'(x) g(u^\varepsilon), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \tag{2.1}$$

$$u^\varepsilon(x, 0) = u_0^\varepsilon(x) \equiv \begin{cases} u_L, & x < -\varepsilon, \\ \zeta^\varepsilon(x) \equiv \big(\dfrac{u_R - u_L}{2\varepsilon}\big)\, x + \dfrac{u_R + u_L}{2}, & -\varepsilon \le x \le \varepsilon, \\ u_R, & x > \varepsilon, \end{cases} \tag{2.2}$$

where $u_L < u_R$, and

$$a_\varepsilon(x) = \begin{cases} a_L, & x < -\varepsilon, \\ \varphi^\varepsilon(x), & -\varepsilon \le x \le \varepsilon, \\ a_R, & x > \varepsilon, \end{cases} \tag{2.3}$$

where

$$\varphi^{\varepsilon}(x) = \big(\frac{a_R - a_L}{2\varepsilon}\big)\, x + \frac{a_R + a_L}{2}, \quad -\varepsilon \le x \le \varepsilon. \tag{2.4}$$

We carry out the analysis to the case $a_L > a_R$ and $u_R > u_L > 0$. In addition, to avoid the resonance case, we assume that

$$u_R > u_L > 2\gamma > 0 \tag{2.5}$$

when $g(u) > 0$ $((a_R - a_L)g(u) < 0)$ and

$$\gamma = \frac{a_L - a_R}{f'(u_L)} \max_{u\in[0,u_L]} g(u). \tag{2.6}$$

Moreover, for $g(u) < 0$ $((a_R - a_L)g(u) > 0)$, to prevent the appearance of shocks in solutions, we assume that $\delta \equiv |u_L - u_R|$ satisfies

$$\delta > \frac{a_L - a_R}{f'(u_L)} \max_{u\in[u_L,u_R]} (-g(u)). \tag{2.7}$$

Note that

$$a'_{\varepsilon}(x) \equiv b^{\varepsilon}(x) = \begin{cases} \dfrac{a_R - a_L}{2\varepsilon}, & -\varepsilon < x < \varepsilon, \\ 0, & |x| > \varepsilon. \end{cases} \tag{2.8}$$

It implies that $b^{\varepsilon}(x)g(u^{\varepsilon})$ is not defined at $x = \pm\varepsilon$. However, the solution along each characteristic curve remains continuous, so we can give the value of $u^{\varepsilon}(-\varepsilon, t)$ (resp., $u^{\varepsilon}(\varepsilon, t)$) by $u^{\varepsilon}(-\varepsilon^{+}, t)$ (resp., $u^{\varepsilon}(\varepsilon^{-}, t)$).

We construct the solutions of (2.1)-(2.4) by characteristic method. To start, by (2.8) we rewrite the perturbed Riemann problem as

$$\begin{aligned} u^{\varepsilon}_t + f(u^{\varepsilon})_x &= b^{\varepsilon}(x)g(u^{\varepsilon}), \\ u^{\varepsilon}(x,0) &= u^{\varepsilon}_0(x). \end{aligned} \tag{2.9}$$

For convenience we define the following regions and segments:

$$\begin{aligned} &\Omega_L \equiv \{(x,t) : x < -\varepsilon,\ t > 0\}, && \Omega_{\varepsilon} \equiv \{(x,t) : -\varepsilon < x < \varepsilon,\ t > 0\}, \\ &\Omega_R \equiv \{(x,t) : x > \varepsilon,\ t > 0\}, && \Gamma_L \equiv \{(x,t) : x = -\varepsilon,\ t > 0\}, \\ &\Gamma_R \equiv \{(x,t) : x = \varepsilon,\ t > 0\}, && \Gamma_{0L} \equiv \{(x,t) : x < -\varepsilon,\ t = 0\}, \\ &\Gamma_{0R} \equiv \{(x,t) : x > \varepsilon,\ t = 0\}, && \Gamma_{0\varepsilon} \equiv \{(x,t) : -\varepsilon < x < \varepsilon,\ t = 0\}. \end{aligned} \tag{2.10}$$

We let $x^{\varepsilon}(t;\tilde{x}_0)$ denote the characteristic curve starting at $\tilde{x}_0 \equiv (x_0, 0)$, and $u^{\varepsilon}(t;\tilde{x}_0)$ denote the solution along $x^{\varepsilon}(t;\tilde{x}_0)$. The notations $x^{\varepsilon}(t)$, $u^{\varepsilon}(t)$ are adopted for $x^{\varepsilon}(t;\tilde{x}_0)$,

$u^\varepsilon(t;\tilde{x}_0)$ when $\tilde{x}_0$ is specified. By characteristic method, the initial value problem of $x^\varepsilon(t;\tilde{x}_0)$ and $u^\varepsilon(t;\tilde{x}_0)$ is

$$\frac{dx^\varepsilon}{dt} = f'(u^\varepsilon), \tag{2.11}$$

$$\frac{du^\varepsilon}{dt} = b^\varepsilon(x^\varepsilon)g(u^\varepsilon), \tag{2.12}$$

$$x^\varepsilon(0) = x_0, \quad u^\varepsilon(0) = u_0^\varepsilon(x_0). \tag{2.13}$$

Note that the solution of (2.11)-(2.13) is globally bounded since the source term only affects the solution in a short period of time. Now we construct the solution of (2.9) in each region, which can be divided into the following three cases.

Case I. $\tilde{x}_0 \equiv (x_0, 0) \in \Gamma_{0L}$.

First, since there is no effect of source term in Ω_L, we obtain $u^\varepsilon(t;\tilde{x}_0) = u_L$, and $x^\varepsilon(t;\tilde{x}_0) = x_0 + f'(u_L)t$ in Ω_L. Also $x^\varepsilon(t;\tilde{x}_0)$ intersects Γ_L at $(-\varepsilon, t_1)$ where

$$t_1 = \frac{-\varepsilon - x_0}{f'(u_L)}, \tag{2.14}$$

and $u^\varepsilon(t_1;\tilde{x}_0) = u_L$. Next, to construct $u^\varepsilon(t;\tilde{x}_0)$, $x^\varepsilon(t;\tilde{x}_0)$ in Ω_ε, we give initial data $u^\varepsilon(t_1) = u_L$ and integrate (2.12) with respect to t. Then we obtain

$$G(u^\varepsilon(t;\tilde{x}_0)) = G(u_L) + b^\varepsilon(t - t_1),$$

where

$$G(u) \equiv \int_\gamma^u \frac{ds}{g(s)}, \tag{2.15}$$

and γ is given in (2.6). It follows that $u^\varepsilon(t;\tilde{x}_0)$ in Ω_ε can be expressed as

$$u^\varepsilon(t;\tilde{x}_0) = G^{-1}(G(u_L) + b^\varepsilon(t - t_1)), \tag{2.16}$$

where t_1 is in (2.14). Since $g(u)$ is nonzero when $u \geq \gamma$, we see that u^ε in (2.16) is well defined. Plugging (2.16) into (2.11) together with $x^\varepsilon(t_1;\tilde{x}_0) = -\varepsilon$, we obtain

$$\begin{aligned} x^\varepsilon(t;\tilde{x}_0) &= -\varepsilon + \int_{t_1}^t f'(G^{-1}(G(u_L) + b^\varepsilon(s - t_1)))ds \\ &= -\varepsilon + F(t;\tilde{x}_0) \end{aligned} \tag{2.17}$$

in Ω_ε where

$$F(t;\tilde{x}_0) \equiv \int_{t_1}^t f'(G^{-1}(G(u_L) + b^\varepsilon(s - t_1)))ds. \tag{2.18}$$

For fixed $\tilde{x}_0$, we define $F^{-1}(x) \equiv F^{-1}(x;\tilde{x}_0)$. Note that $F^{-1}(x)$ is well defined in $[0, 2\varepsilon]$. Then, by mean value theorem and $x^\varepsilon(t_2;\tilde{x}_0) = \varepsilon$, we have

$$t_2 = F^{-1}(2\varepsilon) = t_1 + \frac{2\varepsilon}{f'(u^\varepsilon(c_1;\tilde{x}_0))} \tag{2.19}$$

for some $c_1 \in (t_1, t_2)$. Also, by (2.19) we easily obtain

$$u^\varepsilon(t_2;\tilde{x}_0) = G^{-1}(G(u_L) + b^\varepsilon(F^{-1}(2\varepsilon) - t_1)). \tag{2.20}$$

In the case $g(u) > 0$ ($(a_R - a_L)g(u) < 0$), by (2.12) we see that u^ε is decreasing along each characteristic curve in Ω_ε. This may causes the problem that the characteristic curves do not pass through Γ_R when u^ε decreases to 0. On the other hand, if $g(u) < 0$ ($(a_R - a_L)g(u) > 0$), solution u^ε is increasing along each characteristic curve in Ω_ε, which causes the other problem that $x^\varepsilon(t;\tilde{x}_0)$ may intersect characteristic curves starting on Γ_{0R} when $u^\varepsilon(t;\tilde{x}_0) > u_R$ on Γ_R, it implies that the classical solution no longer exists. However, in Lemma 2.1 we give conditions (2.5)-(2.7) to prevent those situation taking place. Next, since the source term vanishes in Ω_R, by (2.20) it leads to

$$u^\varepsilon(t;\tilde{x}_0) = u^\varepsilon(t_2;\tilde{x}_0) = G^{-1}(G(u_L) + b^\varepsilon(F^{-1}(2\varepsilon) - t_1)), \quad t > t_2.$$

And $x^\varepsilon(t;\tilde{x}_0)$ is a straight line in Ω_R given by

$$x^\varepsilon(t;\tilde{x}_0) = \varepsilon + f'(G^{-1}(G(u_L) + b^\varepsilon(F^{-1}(2\varepsilon) - t_1)))(t - t_2), \quad t > t_2.$$

Thus, by previous construction of $u^\varepsilon(t;\tilde{x}_0)$, $x^\varepsilon(t;\tilde{x}_0)$, we obtain

$$u^\varepsilon(t;\tilde{x}_0) = \begin{cases} u_L, & 0 \le t < t_1, \\ G^{-1}(G(u_L) + b^\varepsilon(t - t_1)), & t_1 \le t < t_2, \\ G^{-1}(G(u_L) + b^\varepsilon(F^{-1}(2\varepsilon) - t_1)), & t \ge t_2, \end{cases} \tag{2.21}$$

and

$$x^\varepsilon(t;\tilde{x}_0) = \begin{cases} x_0 + f'(u_L)t, & 0 \le t < t_1, \\ -\varepsilon + F(t;\tilde{x}_0), & t_1 \le t < t_2, \\ \varepsilon + f'(G^{-1}(G(u_L) + b^\varepsilon(F^{-1}(2\varepsilon) - t_1)))(t - t_2), & t \ge t_2, \end{cases} \tag{2.22}$$

where F, G, t_1 and t_2 are given in (2.18), (2.15), (2.14), (2.19) respectively. By simple calculation, we observe that, if $g(u) > 0$ (resp., $g(u) < 0$), then $x^\varepsilon(t;\tilde{x}_0)$ is increasing, concave down (resp., up) with respect to t in Ω_ε, and $u^\varepsilon(t;\tilde{x}_0)$ is decreasing (resp., increasing) along $x^\varepsilon(t;\tilde{x}_0)$. We have the following lemma regarding to the behavior of

$u^\varepsilon(t;\tilde{x}_0)$ and $x^\varepsilon(t;\tilde{x}_0)$ issued from Γ_{0L}.

Lemma 2.1 *Consider perturbed Riemann Problem* (2.9). *Suppose that* $\tilde{x}_0,\ \tilde{x}_1 \in \Gamma_{0L}$.
(*1*) *Characteristic curves* $x^\varepsilon(t;\tilde{x}_0)$, $x^\varepsilon(t;\tilde{x}_1)$ *are parallel in* Ω_ε.
(*2*) *If* $g(u) > 0$, *then* $\gamma < u^\varepsilon(t;\tilde{x}_0) < u_R$ *for* $t \geq 0$ *under conditions* (2.5), (2.6).
(*3*) *If* $g(u) < 0$, *then* $2\gamma < u^\varepsilon(t;\tilde{x}_0) < u_R$ *for* $t \geq 0$ *under condition* (2.7).
(*4*) *Functions* $u^\varepsilon(t;\tilde{x}_0)$, $x^\varepsilon(t;\tilde{x}_0)$ *tend to piecewise linear functions of* t *as* ε *approaches 0, that is,*

$$\lim_{\varepsilon\to 0} x^\varepsilon(t;\tilde{x}_0) = \begin{cases} x_0 + f'(u_L)t, & 0 \leq t < -\frac{x_0}{f'(u_L)}, \\ f'(u^*)(t + \frac{x_0}{f'(u_L)}), & t > -\frac{x_0}{f'(u_L)}, \end{cases} \tag{2.23}$$

$$\lim_{\varepsilon\to 0} u^\varepsilon(t;\tilde{x}_0) = \begin{cases} u_L, & 0 \leq t < -\frac{x_0}{f'(u_L)}, \\ u^*, & t > -\frac{x_0}{f'(u_L)}, \end{cases} \tag{2.24}$$

where

$$u^* \equiv \lim_{\varepsilon\to 0} \tilde{u}^\varepsilon(a_R) \tag{2.25}$$

with $\tilde{u}^\varepsilon(a_\varepsilon)$ *solving initial value problem*

$$\frac{d\tilde{u}^\varepsilon}{da_\varepsilon} = \frac{g(\tilde{u}^\varepsilon)}{f'(\tilde{u}^\varepsilon)}, \quad \tilde{u}^\varepsilon(a_L) = u_L. \tag{2.26}$$

Proof. First, we show statement (1). By the monotonicity of $x^\varepsilon(t;\tilde{x}_i)$ in Ω_ε, $i = 0,\ 1$, we can rewrite initial value problems (2.11)-(2.13) of $u^\varepsilon(t;\tilde{x}_i)$ into

$$\begin{cases} \dfrac{du^\varepsilon(t;\tilde{x}_i)}{dx^\varepsilon(t;\tilde{x}_i)} = b^\varepsilon \dfrac{g(u^\varepsilon(t;\tilde{x}_i))}{f'(u^\varepsilon(t;\tilde{x}_i))}, \\ u^\varepsilon(t_1^i;\tilde{x}_i) = u_L, \quad i = 0,\ 1, \end{cases}$$

where $t_1^i = \dfrac{-\varepsilon - x_i}{f'(u_L)}$, $i = 0,\ 1$. By change of variable $z = t - \dfrac{x_0 - x_1}{f'(u_L)}$ to the problem of $u^\varepsilon(t;\tilde{x}_1)$ above, we obtain

$$\begin{cases} \dfrac{du^\varepsilon(z;\tilde{x}_1)}{dx^\varepsilon(z;\tilde{x}_1)} = b^\varepsilon \dfrac{g(u^\varepsilon(z;\tilde{x}_1))}{f'(u^\varepsilon(z;\tilde{x}_1))}, \\ u^\varepsilon(t_1^0;\tilde{x}_1) = u_L. \end{cases}$$

It implies that $u^\varepsilon(t;\tilde{x}_0)$, $u^\varepsilon(t;\tilde{x}_1)$ solve the same initial value problem. By the uniqueness theorem of ordinary differential equations (ODEs), it leads to $u^\varepsilon(t;\tilde{x}_0) = u^\varepsilon(z;\tilde{x}_1)$, and consequently $\dfrac{dx^\varepsilon(t;\tilde{x}_0)}{dt} = \dfrac{dx^\varepsilon(z;\tilde{x}_1)}{dz}$. Next, we observe that

$$x^\varepsilon(\frac{-\varepsilon - x_0}{f'(u_L)}; \tilde{x}_0) = -\varepsilon,$$

and

$$x^\varepsilon(t = \frac{-\varepsilon - x_1}{f'(u_L)}; \tilde{x}_1) = x^\varepsilon(z = \frac{-\varepsilon - x_0}{f'(u_L)}; \tilde{x}_1) = -\varepsilon.$$

Thus, by the uniqueness theorem of ODEs again, we obtain $x^\varepsilon(t; \tilde{x}_0) = x^\varepsilon(z; \tilde{x}_1)$, which is sufficient to imply that $x^\varepsilon(t; \tilde{x}_0)$, $x^\varepsilon(z; \tilde{x}_1)$ are parallel in Ω_ε.

Next, we show statement (2). Since $b^\varepsilon g(u^\varepsilon) < 0$ and $u_L < u_R$, it is easy to see that $u^\varepsilon(t; \tilde{x}_0) < u_R$. To show $\gamma < u^\varepsilon(t; \tilde{x}_0)$, by statement (1) and (2.21), it is equivalent to show that

$$u^\varepsilon(t_2; (-\varepsilon, 0)) = G^{-1}(G(u_L) + b^\varepsilon F^{-1}(2\varepsilon)) > \gamma, \tag{2.27}$$

where G, F are in (2.15), (2.18). Since $g(u) > 0$, we see that G^{-1} is increasing. Then by the fact $G(\gamma) = 0$, it implies that (2.27) is equivalent to

$$G(u_L) + b^\varepsilon F^{-1}(2\varepsilon) > 0.$$

By mean value theorem, we see that the last inequality is equivalent to

$$u_L - \gamma > (a_L - a_R)\frac{g(u_1)}{f'(u_2)} \tag{2.28}$$

for some $u_1 \in (\gamma, u_L)$ and $u_2 \in (u_L, u_R)$. Then, by (2.5) and comparing (2.6) with (2.28), we complete the proof of statement (2).

To show statement (3), it is equivalent to show

$$u^\varepsilon(t_2; (-\varepsilon, 0)) = G^{-1}(G(u_L) + b^\varepsilon F^{-1}(2\varepsilon)) < u_R. \tag{2.29}$$

Since $g(u) < 0$, by mean value theorem and G^{-1} is decreasing, we obtain that (2.29) is equivalent to

$$\int_{u_R}^{u_L} \frac{ds}{g(s)} > \frac{a_L - a_R}{f'(u_3)} \tag{2.30}$$

for some $u_3 \in (u_L, u_R)$. Applying mean value theorem to the LHS of (2.30), we see that condition (2.7) is sufficient to imply (2.29).

Next we show statement (4). First, by statement (1) we observe that the value of $u^\varepsilon(t; \tilde{x}_0)$ at $x = \varepsilon$ is equal to some constant for any $\tilde{x}_0 \in \Gamma_{0L}$. Let c denote the constant. Then, by replacing b^ε by da_ε/dx in (2.12) and the re-scaling technique to (2.11)-(2.13), we observe that $u^\varepsilon(x, t)$ along each characteristic curve in Ω_ε can be regarded as a function of a_ε, and constant c can be solved by (2.26). Finally, we simply let ε approach 0 in (2.21), (2.22) and use the fact that $t_2 \to t_1$ and $t_1 \to -\dfrac{x_0}{f'(u_L)}$ as $\varepsilon \to 0$, we complete the proof of statement (4). ∎

Case II. $x_0 \in \Gamma_{0\varepsilon}$.

By similar analysis in Case I and (2.2), we obtain

$$u^\varepsilon(t;\tilde{x}_0) = \begin{cases} G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon t), & 0 \le t < t_2^1, \\ G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon F_1^{-1}(\varepsilon - x_0)), & t \ge t_2^1, \end{cases} \tag{2.31}$$

and

$$\begin{aligned} &x^\varepsilon(t;\tilde{x}_0) \\ &= \begin{cases} x_0 + F_1(t;\tilde{x}_0), & 0 \le t < t_2^1, \\ \varepsilon + f'(G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon F_1^{-1}(\varepsilon - x_0)))(t - t_2^1), & t \ge t_2^1, \end{cases} \end{aligned} \tag{2.32}$$

where G is in (2.15) and $F_1,\ t_2^1$ are given by

$$F_1(t;\tilde{x}_0) \equiv \int_0^t f'(G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon s))ds, \tag{2.33}$$

$$t_2^1 \equiv F_1^{-1}(\varepsilon - x_0) = \frac{\varepsilon - x_0}{f'(u(c_2^1;\tilde{x}_0))} \tag{2.34}$$

for some $c_2^1 \in (0, t_2^1)$, and $F_1^{-1}(x) \equiv F_1^{-1}(x;\tilde{x}_0)$ for fixed $\tilde{x}_0$. Note that F_1^{-1} is well defined on $[0, 2\varepsilon]$. On the other hand, for $x_0 \in \Gamma_{0\varepsilon}$, there exists a constant $k \in (-1, 1)$ such that $x_0 = k\varepsilon$. Therefore $u^\varepsilon(t;\tilde{x}_0)$, $x^\varepsilon(t;\tilde{x}_0)$ in (2.31), (2.32) can be written as

$$u^\varepsilon(t;\tilde{x}_0) = \begin{cases} G^{-1}(G(\zeta^\varepsilon(k\varepsilon)) + b^\varepsilon t), & 0 \le t < t_2^1, \\ G^{-1}(G(\zeta^\varepsilon(k\varepsilon)) + b^\varepsilon F_1^{-1}((1-k)\varepsilon)), & t \ge t_2^1, \end{cases} \tag{2.35}$$

and

$$\begin{aligned} &x^\varepsilon(t;\tilde{x}_0) \\ &= \begin{cases} k\varepsilon + F_1(t), & 0 \le t < t_2^1, \\ \varepsilon + f'(G^{-1}(G(\zeta^\varepsilon(k\varepsilon)) + b^\varepsilon F_1^{-1}((1-k)\varepsilon)))(t - t_2^1), & t \ge t_2^1, \end{cases} \end{aligned} \tag{2.36}$$

where $t_2^1 = F_1^{-1}((1-k)\varepsilon)$ (see Figure 1).

Next, we let $D \equiv \bigcup_{\tilde{x}_0 \in \Gamma_{0\varepsilon}} \{(x^\varepsilon(t;\tilde{x}_0), t) :\ t \ge 0\}$, and $L_{-\varepsilon^+},\ L_{\varepsilon^-}$ be the characteristic curves starting at $(-\varepsilon^+, 0)$, $(\varepsilon^-, 0)$ respectively. We also let $D_\varepsilon \equiv D \bigcap \Omega_\varepsilon$, and $D_R \equiv D \bigcap \Omega_R$.

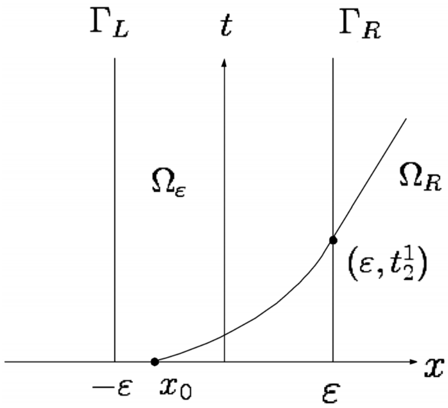

Fig. 1. The characteristic curve starting at $\tilde{x}_0 \in \Gamma_{0\varepsilon}$.

Lemma 2.2. *Consider perturbed Riemann problem* (2.9) *where* $a_R < a_L$ *and* $u_L,\ u_R$ *satisfy* (2.5)-(2.7). *Then all the characteristic curves in* D *do not intersect. Furthermore, given any* $(x, t) \in D$*, there exists a unique* $x_0 = x_0(x, t) \in (-\varepsilon, \varepsilon)$ *such that characteristic curve* $x^\varepsilon(t; (x_0, 0))$ *passes through* (x, t)*, namely* $u^\varepsilon(x, t)$ *can be determined uniquely in* D .

Proof. First we consider $g(u) < 0$. Given $\tilde{x}_0 = (x_0, 0),\ \tilde{x}_1 = (x_1, 0)$ where $x_0,\ x_1 \in (-\varepsilon, \varepsilon)$ and $x_1 > x_0$, we see that $G(\zeta^\varepsilon(x_0)) > G(\zeta^\varepsilon(x_1))$ since $\zeta^\varepsilon(x_0) < \zeta^\varepsilon(x_1)$ and G is decreasing. Therefore, we have

$$G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon t) < G^{-1}(G(\zeta^\varepsilon(x_1)) + b^\varepsilon t) \tag{2.37}$$

for $0 \le t \le t_1^*$ where t_1^* is the time when $x^\varepsilon(t; \tilde{x}_1)$ intersects Γ_R. Then, by (2.31) and (2.37), it implies that $u^\varepsilon(t; \tilde{x}_0) < u^\varepsilon(t; \tilde{x}_1)$ for $0 \le t \le t_1^*$. Following $f'' > 0$ we obtain $f'(u^\varepsilon(t; \tilde{x}_0)) < f'(u^\varepsilon(t; \tilde{x}_1))$ for $0 \le t \le t_1^*$, which implies that $x^\varepsilon(t; \tilde{x}_0),\ x^\varepsilon(t; \tilde{x}_1)$ do not intersect when $0 \le t \le t_1^*$.

Next, for $t \ge t_1^*$ we see that $x^\varepsilon(t; \tilde{x}_1)$ is a straight line, so it is necessary to show $u^\varepsilon(t; \tilde{x}_1)|_{x=\varepsilon} > u^\varepsilon(t; \tilde{x}_0)|_{x=\varepsilon}$, or equivalently

$$G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon F_1^{-1}(\varepsilon - x_0)) < G^{-1}(G(\zeta^\varepsilon(x_1)) + b^\varepsilon F_1^{-1}(\varepsilon - x_1)). \tag{2.38}$$

To show (2.38), given $x \in (-\varepsilon, \varepsilon)$ we define

$$H(x) \equiv u^\varepsilon(t; (x, 0))|_{x=\varepsilon} = G^{-1}(G(\zeta^\varepsilon(x)) + b^\varepsilon F_1^{-1}(\varepsilon - x)). \tag{2.39}$$

Differentiating H with respect to x, we have

$$\frac{dH}{dx} = \frac{g(\vartheta)}{2\varepsilon}\left[\frac{u_R - u_L}{g(\zeta^\varepsilon(x))} + \frac{a_L - a_R}{f'(G^{-1}(\vartheta))}\right], \tag{2.40}$$

where $\vartheta = G(\zeta^\varepsilon(x)) + b^\varepsilon F_1^{-1}(\varepsilon - x)$. Then, by (2.7), (2.40) and $g(\vartheta) < 0$, it leads to $\frac{dH}{dx} > 0$, which implies that (2.38) holds. We complete the proof for case $g(u) < 0$.

Next, we consider $g(u) > 0$. By the similar analysis given in $g(u) < 0$, we also obtain $u^\varepsilon(t; \tilde{x}_0) < u^\varepsilon(t; \tilde{x}_1)$ for $0 \le t \le t_1^*$. In addition, by $b^\varepsilon g(u) \le 0$ we have $u^\varepsilon(t_1^*; \tilde{x}_0) > u^\varepsilon(t; \tilde{x}_0)$ for all $t > t_1^*$. It follows that $u^\varepsilon(t; \tilde{x}_0) < u^\varepsilon(t; \tilde{x}_1)$ for all $t \ge 0$. Finally, by $f'' > 0$ again, we show that $f'(u^\varepsilon(t; \tilde{x}_0)) < f'(u^\varepsilon(t; \tilde{x}_1))$ for $t \ge 0$, which implies that $x^\varepsilon(t; \tilde{x}_0),\ x^\varepsilon(t; \tilde{x}_1)$ do not intersect for all $t \ge 0$.

Next, we show $u^\varepsilon(x, t)$ can be determined uniquely in D. First, given $(x, t) \in D_\varepsilon$, by (2.32), (2.33), it is equivalent to show that there exists a unique $x_0 \in (-\varepsilon, \varepsilon)$ such that

$$h(x, t, x_0) \equiv -x + x_0 + \int_0^t f'(G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon s))ds = 0. \tag{2.41}$$

Differentiating $h(x, t, x_0)$ with respect to x_0, we obtain in both cases ($g(u) < 0$ or $g(u) > 0$) that

$$\begin{aligned}
&\frac{\partial h(x, t, x_0)}{\partial x_0} \\
&= 1 + \int_0^t f''(G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon s)) \cdot \frac{\partial}{\partial x_0} G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon s)ds \\
&= 1 + \int_0^t f''(G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon s)) \cdot \frac{1}{G'(G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon s))} \cdot \frac{\partial}{\partial x_0} G(\zeta^\varepsilon(x_0))ds \\
&= 1 + \frac{(\zeta^\varepsilon(x_0))'}{g(\zeta^\varepsilon(x_0))} \cdot \int_0^t f''(G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon s)) \cdot g(G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon s))ds \\
&> 0.
\end{aligned}$$

Therefore, by implicit function theorem there exists a unique $x_0 = x_0(x, t)$ such that (2.41) holds. Since all the characteristic curves in D_ε do not intersect, it follows that $x_0 \in (-\varepsilon, \varepsilon)$.

Next, suppose that $(x, t) \in D_R$. Since all the characteristic curves are straight lines in D_R, there exists a unique $(\varepsilon^-, t^*) \in D_\varepsilon$ such that the characteristic curve passing through (ε^-, t^*) also passes through (x, t). Therefore, by previous analysis we obtain that there exists a unique $x_0 \in (-\varepsilon, \varepsilon)$ such that characteristic curve $x^\varepsilon(t; (x_0, 0))$ passes through $(x, t) \in D_R$. We complete the proof. ■

Case III. $\tilde{x}_0 \in \Gamma_{0R}$.

Due to the vanishing of source term in this case, we can easily obtain that

$$u^\varepsilon(t; \tilde{x}_0) = u_R, \tag{2.42}$$

$$x^\varepsilon(t; \tilde{x}_0) = x_0 + f'(u_R)t,\ t \ge 0. \tag{2.43}$$

Next, we define the following regions:

$$
\begin{aligned}
&\tilde{\Omega}_\varepsilon \equiv \Omega_\varepsilon \backslash D_\varepsilon, \\
&\Omega_R^1 \equiv \bigcup \{(x^\varepsilon(t;\tilde{x}_0),t):\ \tilde{x}_0 \in \Gamma_{0L}\} \bigcap \Omega_R, \qquad (2.44)\\
&\Omega_R^2 \equiv \{(x^\varepsilon(t;\tilde{x}_0),t):\ \tilde{x}_0 \in \Gamma_{0R}\}.
\end{aligned}
$$

By previous cases study of $u^\varepsilon(x,t)$, $x^\varepsilon(x,t)$, we obtain the global classical solution of perturbed Riemann problem (2.1)-(2.4) where $u_R > u_L > \gamma > 0$ and $a_R < a_L$, which is given in the following theorem.

Theorem 2.3. *Consider perturbed Riemann problem* (2.1)-(2.4) *where* $a_R < a_L$ *and* $u_L,\ u_R$ *satisfy* (2.5)-(2.7). *Then the classical solution of* (2.1)-(2.4) *exists for* $t \geq 0$. *Furthermore, given* $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, *there exists a unique* $x_0 = x_0(x,t) \in \mathbb{R}$ *such that the classical solution can be expressed as*

$$
u^\varepsilon(x,t) = \begin{cases}
u_L, & (x,t) \in \Omega_L, \\
G^{-1}(G(u_L) + b^\varepsilon(t + \dfrac{\varepsilon + x_0}{f'(u_L)})), & (x,t) \in \tilde{\Omega}_\varepsilon, \\
u^*, & (x,t) \in \Omega_R^1, \\
G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon t), & (x,t) \in D_\varepsilon, \\
G^{-1}(G(\zeta^\varepsilon(x_0)) + b^\varepsilon F_1^{-1}(\varepsilon - x_0)), & (x,t) \in D_R, \\
u_R, & (x,t) \in \Omega_R^2,
\end{cases} \qquad (2.45)
$$

where $G,\ F_1$ *and* u^* *are given in* (2.15), (2.33) *and* (2.25) *respectively (see Figure 2).*

Next, we study the behavior of perturbed Riemann solutions as ε approaches 0. The following theorem indicates that the limit function $u(x,t) \equiv \lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ is a discontinuous function consisting of three constant states separated by a standing shock and a rarefaction wave (a C^1 function of $\frac{x}{t}$) when $a_L \neq a_R$.

Theorem 2.4. *Suppose that* $u^\varepsilon(x,t)$ *is the classical solution of* (2.1)- (2.4) *which is given in* (2.45). *Then function* $u(x,t) \equiv \lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ *consists of three constant states* $u_L,\ u^*,\ u_R$ *separated by a discontinuity on t-axis and a rarefaction wave in region* $\{(x,t) : f'(u^*)t \leq x \leq f'(u_R)t, t > 0\}$ *(Figure 3). That is, function* $u(x,t)$ *can be expressed as*

$$
u(x,t) = \begin{cases}
u_L, & (x,t) \in \Omega_1, \\
u^*, & (x,t) \in \Omega_2, \\
(f')^{-1}(\dfrac{x}{t}), & (x,t) \in \Omega_3, \\
u_R, & (x,t) \in \Omega_4,
\end{cases} \qquad (2.46)
$$

where u^* *is in* (2.25) *and* $\{\Omega_i :\ i = 1,\ 2,\ 3,\ 4\}$ *are given by*

$$\begin{aligned}\Omega_1 &\equiv \{(x,t) : x < 0, t > 0\},\\ \Omega_2 &\equiv \{(x,t) : 0 < x < f'(u^*)t, t > 0\},\\ \Omega_3 &\equiv \{(x,t) : f'(u^*)t \le x \le f'(u_R)t, t > 0\},\\ \Omega_4 &\equiv \{(x,t) : x > f'(u_R)t, t > 0\}.\end{aligned} \tag{2.47}$$

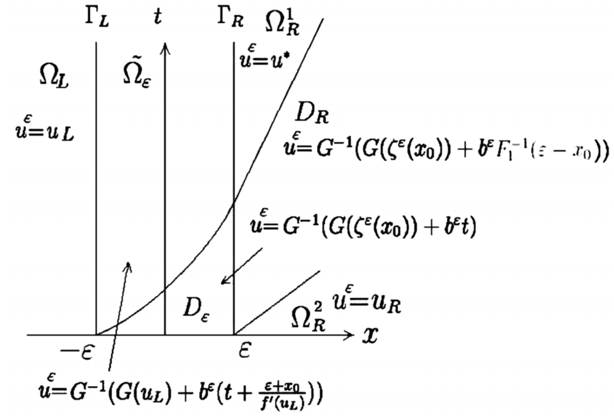

Fig. 2. Classical solution $u^\varepsilon(x,t)$ of (2.1)-(2.4) ($g(u) > 0$).

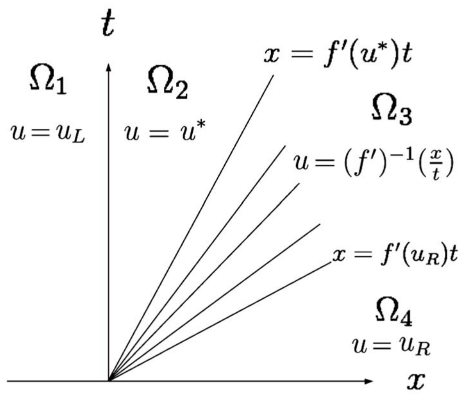

Fig. 3. Limit function $u(x,t)$ in $\Omega_i, i = 1, 2, 3, 4$ when $u_R > u_L > \gamma > 0$ and $a_R < a_L$.

Proof. First, we observe that all the points in D_ε tend to $(0,0)$ as $\varepsilon \to 0$. Moreover, letting $\varepsilon \to 0$ in (2.45), we see that $u(0^-,t) = u_L$ and $u(0^+,t) = u^*$ for all $t > 0$ where u^* is in (2.25). Therefore $u(x,t)$ has a discontinuity on t-axis.

Next, let $L_{-\varepsilon^+}$ (resp., L_{ε^-}) denote the characteristic curve starting at $(-\varepsilon^+, 0)$ (resp., $(\varepsilon^-, 0)$). Suppose that $L_{-\varepsilon^+}$, L_{ε^-} intersect $x = \varepsilon$ at (ε, t_1^*), (ε, t_2^*). Then, by plugging $x_0 = -\varepsilon^+,\ \varepsilon^-$ into (2.45) for $u^\varepsilon(x,t)$ in D_R, we obtain

$$u^\varepsilon(\varepsilon, t_1^*) = u^*,\ u^\varepsilon(\varepsilon, t_2^*) = u_R,$$

where u^* is given in (2.45). In addition, by (2.41), (2.40), we see that the values of $u^\varepsilon(x,t)|_{\tilde{L}}$, $\tilde{L} = \{(\varepsilon, t);\ t \in [t_2^*, t_1^*]\}$, increase from u^* to u_R in both cases $g(u) > 0$, $g(u) < 0$, and all the points in $\tilde{L}$ tend to $(0,0)$ as $\varepsilon \to 0$. Thus, region D_R tends to Ω_3 given in (2.47), and all the characteristic curves in Ω_3 are straight lines with slopes varying from $f'(u^*)$ to $f'(u_R)$ as $\varepsilon \to 0$. Moreover, since source term vanishes in D_R, we observe that the value of $u^\varepsilon(x,t)$ along each characteristic curve in Ω_R is a constant determined by the value of u^ε at some point in $\tilde{L}$. This is sufficient to say that $u(x,t) = v(\dfrac{x}{t})$ in Ω_3 for some C^1 function v. To obtain $u(x,t)$ in Ω_3, we just solve $\dfrac{x}{t} = f'(v)$ to get $u(x,t) = (f')^{-1}(\dfrac{x}{t})$. Finally, letting ε approach 0 in (2.45), we complete the proof. ■

3. Shock Waves of the Riemann and Perturbed Riemann Problems

In this section we study the discontinuous solutions (shock waves) $u^\varepsilon(x,t)$ to the perturbed Riemann problem (2.1), (2.3) and (2.4) where $u_L > u_R > 0$ and $a_R < a_L$. In addition, we study the self-similarity of $u(x,t) \equiv \lim\limits_{\varepsilon\to 0} u^\varepsilon(x,t)$. For convenience, we choose the initial data to be

$$u^\varepsilon(x,0) = u_0^\varepsilon(x) \equiv \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases} \tag{3.1}$$

We notice that, the structure of admissible shocks for (2.1), (2.3), (2.4) and (3.1) is determined by Rankine-Hugoniot condition and Lax entropy condition, which is more complicated than the one in homogeneous equations due to the appearance of source term. To study the shock waves, the first difficulty is to track the location of shocks, especially the ones in region Ω_ε. Secondly, the shocks may disappear in finite time when $g(u)$ is positive and sufficiently large, while u_L, u_R are sufficiently close. Furthermore, even the shock exists globally, the states on both sides of the shock may not be constants as ε approaches 0, which implies that the speed of shock may not be a constant. In this section we first give a condition to $|u_L - u_R|$ when $g(u) > 0$ (but no condition of $|u_L - u_R|$ required when $g(u) < 0$) so that the shock exists for all $t \geq 0$. In addition, we show that the limit of perturbed Riemann solution is self-similar, i.e., $\lim\limits_{\varepsilon\to 0} u^\varepsilon(x,t) = v(\frac{x}{t})$ for some discontinuous function v.

To start, we define

$$\delta \equiv |u_L - u_R| > 0. \tag{3.2}$$

The following lemma indicates that the solution of (2.1), (2.3), (2.4) and (3.1) consists of a shock for all $t \geq 0$ under some condition of δ.

Lemma 3.1. *Consider perturbed Riemann problem* (2.1)*,* (2.3)*,* (2.4) *and* (3.1) *with* $g(u) > 0$*,* $u_L > u_R > 0$ *and* $a_R < a_L$*. Suppose that* δ *in* (3.2) *satisfies*

$$\delta \equiv |u_L - u_R| > \frac{a_L - a_R}{f'(u_R)} \max_{u \in [u_R, u_L]} g(u). \tag{3.3}$$

Then the solution of (2.1)*,* (2.3)*,* (2.4) *and* (3.1) *consists of a shock for all* $t \geq 0$*. If* $g(u) < 0$*, then the result holds for any* $\delta > 0$*.*

Proof. First, we consider $g(u) > 0$. For all $x_0 \leq -\varepsilon$, by Lemma 2.1 we see that $u^\varepsilon(t; (x_0, 0)) = u^*$ at $x = \varepsilon$ where u^* is in (2.25). On the other hand, by Lemma 2.2 it follows that $u^* < u^\varepsilon(t; (x_1, 0))$ at $x = \varepsilon$ for any $x_1 \in (-\varepsilon, 0)$. Moreover, for $x_2 \in (0, \varepsilon)$, we observe that $u^\varepsilon(t; (x_2, 0)) < u_R$ at $x = \varepsilon$. Therefore, by previous facts and that $f'(u)$ stands for the slope of characteristic curves, we want to show that $u^* > u_R$ under condition (3.3).

Next, we see that $u^* > u_R$ implies

$$G^{-1}(G(u_L) + \frac{a_R - a_L}{f'(u_c)}) > u_R,$$

where $u_c \in [u_L, u_R]$ and G is in (2.15). Since $(G^{-1})' > 0$, the last inequality is equivalent to

$$G(u_L) + \frac{a_R - a_L}{f'(u_c)} > G(u_R). \tag{3.4}$$

Therefore, by the definition of G in (2.15) and (3.4), we observe that $u^* > u_R$ if

$$\int_{u_L - \delta}^{u_L} \frac{ds}{g(s)} > \frac{a_L - a_R}{f'(u_c)}. \tag{3.5}$$

Applying mean value theorem to the LHS of (3.5) and the fact that $f'(u_c) \geq f'(u_R)$, we show that $u^* > u_R$ if δ satisfies (3.3).

Next, we consider $g(u) < 0$. By $G' < 0$, $(G^{-1})' < 0$ and $b^\varepsilon g(u) > 0$, we easily obtain

$$G^{-1}(G(u_L) + b^\varepsilon t) > G^{-1}(G(u_R) + b^\varepsilon t)$$

for $u_L > u_R$ and $t \geq 0$, which implies that $u^\varepsilon(t;(x_0,0)) > u^\varepsilon(t;(x_1,0))$, and consequently $f'(u^\varepsilon(t;(x_0,0))) > f'(u^\varepsilon(t;(x_1,0)))$ in Ω_ε for any $x_0 \in (-\varepsilon, 0)$ and $x_1 \in (0,\varepsilon)$. Therefore, the shock occurs in Ω_ε. Next, for all $t \geq 0$ we have

$$u^\varepsilon(t;(x_2,0)) > u^\varepsilon(t;(x_0,0))$$

for any $x_2 \leq -\varepsilon$, $x_0 \in (-\varepsilon, 0)$, and

$$u^\varepsilon(t;(x_1,0)) > u^\varepsilon(t;(x_3,0))$$

for any $x_3 \geq \varepsilon$, $x_0 \in (-\varepsilon, 0)$, which is sufficient to imply that the shock occurs outside Ω_ε. We complete the proof. ∎

Taking the limit to $u^\varepsilon(x,t)$, we obtain the following theorem.

Theorem 3.2. *Suppose $u^\varepsilon(x,t)$ is the solution of* (2.1)*,* (3.1) *and* (2.4) *with $a_R < a_L$, $u_L > u_R > 0$ and $\delta \equiv |u_L - u_R|$ satisfying* (3.3)*. Then $u(x,t) \equiv \lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ consists of at most three constant states u_L, u^*, u_R separated by a standing discontinuity and a shock (Figure 4). Namely,*

$$u(x,t) = \begin{cases} u_L, & (x,t) \in \tilde{\Omega}_1, \\ u^*, & (x,t) \in \tilde{\Omega}_2, \\ u_R, & (x,t) \in \tilde{\Omega}_3, \end{cases} \tag{3.6}$$

where u^* is solved by (2.25) *and* (2.26)*, and $\tilde{\Omega}_i$, $i = 1,2,3$, are given by*

$$\begin{aligned} \tilde{\Omega}_1 &\equiv \{(x,t) : x < 0,\ t > 0\}, \\ \tilde{\Omega}_2 &\equiv \{(x,t) : 0 < x < \frac{f(u_R) - f(u^*)}{u_R - u^*}\, t,\ t > 0\}, \\ \tilde{\Omega}_3 &\equiv \{(x,t) : x > \frac{f(u_R) - f(u^*)}{u_R - u^*}\, t,\ t > 0\}. \end{aligned}$$

Proof. First, we study the shock wave of (2.1), (3.1) and (2.4) when $g(u) \equiv 0$. Let $S^\varepsilon(x(t),t)$; $t \geq 0$, denote the location of shock when $g(u) \equiv 0$. Then, by Rankine-Hugoniot condition, we have

$$S^\varepsilon(x(t),t) = \{(x(t),t) :\ x(t) = \frac{f(u_R) - f(u_L)}{u_R - u_L} t,\ t \geq 0\}.$$

Also, characteristic curve $x^\varepsilon(t;(\varepsilon,0))$ can be expressed as

$$x^\varepsilon(t;(\varepsilon,0)) = \{(x(t),t) :\ x(t) = \varepsilon + f'(u_R)t,\ t \geq 0\}.$$

We notice that $x^\varepsilon(t;(\varepsilon,0))$ impinges to $S^\varepsilon(x(t),t)$ at $(x^*,t^*) = (\varepsilon + \dfrac{f'(u_R)\varepsilon}{m - f'(u_R)}, \dfrac{\varepsilon}{m-f'(u_R)})$ where $m = \dfrac{f(u_R)-f(u_L)}{u_R-u_L}$. Note that $x^*,\ t^*$ are of order $O(\varepsilon)$. In addition, characteristic curve $x^\varepsilon(t;(x_0,0))$, $x_0 = \varepsilon + \dfrac{f'(u_R)-f'(u_L)}{m-f'(u_R)}\varepsilon$, impinges to $S^\varepsilon(x(t),t)$ on the left at (x^*,t^*). By $f''>0$ we observe that $x_0<0$. Moreover, the region bounded by $x^\varepsilon(t;(\varepsilon,0))$, $x^\varepsilon(t;(x_0,0))$ and $\{(x,0):\ x\in[x_0,\varepsilon]\}$ vanishes as $\varepsilon\to 0$.

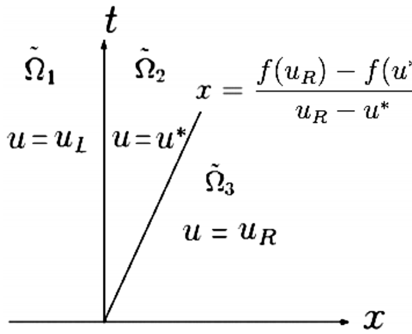

Fig. 4. Solution in regions $\tilde\Omega_i, i=1,2,3$ for $u_L>u_R>0$ and $a_R<a_L$.

Next, we consider $g(u)>0$. Let $\tilde S^\varepsilon(x(t),t)$, $t\ge 0$, be the location of shock in this case. Since the characteristic curves are affected by the source term, curve $\tilde S^\varepsilon(x(t),t)$ can be obtained by perturbing $S^\varepsilon(x(t),t)$ under Rankine-Hugoniot condition. More precisely, let $u^-(t)$, $u^+(t)$ be the states on the left and right of $\tilde S^\varepsilon(x(t),t)$ at time $t>0$, then by Rankine-Hugoniot condition we obtain that $(\tilde x(t),t)\in S^\varepsilon(x(t),t)$ satisfies

$$\frac{d\tilde x(t)}{dt} = \frac{f(u^+(t))-f(u^-(t))}{u^+(t)-u^-(t)},\quad t>0. \tag{3.7}$$

Therefore, by (3.7) and the facts that $|f(u_L)-f(u^-(t))| = O(1)$, $|f(u_R)-f(u^+(t))| = O(1)$ for $0<t<t^* = \dfrac{\varepsilon}{m-f'(u_R)}$ and $\tilde S^\varepsilon(x(t),t)$, $S^\varepsilon(x(t),t)$ all start at $(0,0)$, we obtain

$$|\tilde S^\varepsilon(x(t),t) - S^\varepsilon(x(t),t)| = O(1)\varepsilon \tag{3.8}$$

for $0<t<t^*$. Let $(\tilde x^*,\tilde t^*)$ be the point where $x^\varepsilon(t;(\varepsilon,0))$ impinges to $\tilde S^\varepsilon(x(t),t)$ on the right. Also let $x^\varepsilon(t;(x_1,0))$ for some $x_1<0$ be the characteristic curve impinging to $\tilde S^\varepsilon(x(t),t)$ on the left at $(\tilde x^*,\tilde t^*)$. Then, by (3.8) we obtain that $\tilde x^*,\ \tilde t^*,\ x_1\to 0$

as $\varepsilon \to 0$. It follows that region $\tilde{D}^{\varepsilon}$ bounded by $x^{\varepsilon}(t;(\varepsilon,0))$, $x^{\varepsilon}(t;(x_1,0))$ and $\{(x,0):\ x \in [x_1,\varepsilon]\}$ vanishes as $\varepsilon \to 0$.

Next, we see that $u^{\varepsilon}(x,t)$ outside $\Omega_{\varepsilon} \bigcup \tilde{D}^{\varepsilon}$ is piecewise constant. Furthermore, by (3.8) we see that $\tilde{S}^{\varepsilon}(x(t),t) \to S^{\varepsilon}(x(t),t)$ as $\varepsilon \to 0$, which means that $x_1,\ x_0 \to 0$, and in particular $x_1 \to x_0$ as $\varepsilon \to 0$. Then , by statement (4) of Lemma 2.1, it leads to $u^{\varepsilon}(t;(x_1,0)) = u^*$ at $x = 0^+$, and $u^{\varepsilon}(t;(x_1,0)) = u_L$ at $x = 0^-$ as $\varepsilon \to 0$ where u^* is solved by (2.25), (2.26). Finally, by letting ε approach 0, we complete the proof for $g(u) > 0$. The proof for $g(u) < 0$ is similar, we will omit it. The proof is complete. ■

By previous results in Sections 2 and 3, we see that the standing shock (standing wave discontinuity) occurs due to the "bending" of characteristic curves as ε approaches 0, but not by the intersection of characteristic curves. It means that the discontinuity is not governed by the Rankine-Hugoniot condition but by taking the ε-limit to solutions of ODEs as described in the following theorem.

Theorem 3.3. *Let $u(x,t)$ be the generalized solution of* (1.1)-(1.3) *constructed by Definition 1.1, and $u_L,\ u_R$ are nonzero and of the same sign. If $u_L,\ u_R > 0$ (resp., $u_L,\ u_R < 0$), then the standing wave discontinuity in $u(x,t)$ connects constant states $u_L,\ u^*$ (resp., $u_R,\ u_*$) where u^* (resp., u_*) is the ε-limit of $\tilde{u}^{\varepsilon}(a_R)$ (resp., $\tilde{u}^{\varepsilon}(a_L)$) with $\tilde{u}^{\varepsilon}$ solving*

$$\frac{d\tilde{u}^{\varepsilon}}{da_{\varepsilon}} = \frac{g(\tilde{u}^{\varepsilon})}{f'(\tilde{u}^{\varepsilon})}, \quad \tilde{u}^{\varepsilon}(a_L) = u_L \ (resp.,\ \tilde{u}^{\varepsilon}(a_R) = u_R).$$

4. Stability of Perturbed Riemann Solutions and Generalized Entropy Solutions

In Sections 2 and 3 we constructed solution $u^{\varepsilon}(x,t)$ of perturbed Riemann problem when $a_{\varepsilon}(x)$ is piecewise linear, and we study the self-similarity of $u(x,t) = \lim\limits_{\varepsilon \to 0} u^{\varepsilon}(x,t)$. To construct the generalized entropy solution of (1.1)-(1.3) based on Definition 1.2, it is necessary to study the stability of $u^{\varepsilon}(x,t)$. We see that the behavior of $u^{\varepsilon}(x,t)$ depends on the choice of $a_{\varepsilon}(x)$, which means that $u^{\varepsilon}(x,t)$ may not be stable under the general profile of $a_{\varepsilon}(x)$. Consequently, it may lead to the non-uniqueness of generalized entropy solutions to (1.1)-(1.3).

In the first part of the section, we give an example to demonstrate the non-stability of perturbed Riemann solutions under general profile of $a_{\varepsilon}(x)$. In the second part of the section, we show that the stability of $u^{\varepsilon}(x,t)$ can be established with respect to some set $A_{\varepsilon} \equiv \{\tilde{a}^{\varepsilon}(x)\}$ in which every $\tilde{a}^{\varepsilon}(x) \in A_{\varepsilon}$ can be expressed as

$$\tilde{a}^{\varepsilon}(x) = a_{\varepsilon}(x) + \delta_{\varepsilon}(x),$$

where $a_{\varepsilon}(x)$ is in (2.3), and $\delta_{\varepsilon}(x)$ is a perturbation satisfying some properties given later in this section.

To start, for $0 < \varepsilon \ll 1$, we set $u_L = 1,\ u_R = 2,\ a_L = 3,\ a_R = 1,\ f(u) = \frac{1}{2}u^2$ and $g(u) = 1$ in (2.1)-(2.2). Then we consider the following perturbed Riemann problem

$$\begin{cases} u_t^\varepsilon + (\frac{1}{2}(u^\varepsilon)^2)_x = (a_\varepsilon(x) + \delta_\varepsilon(x))', \\ u^\varepsilon(x,0) = u_0^\varepsilon(x), \end{cases} \tag{4.1}$$

where

$$a_\varepsilon(x) = \begin{cases} 3 & x < -\varepsilon, \\ \dfrac{-x}{\varepsilon} + 2, & -\varepsilon \le x \le \varepsilon, \\ 1 & x > \varepsilon, \end{cases} \tag{4.2}$$

$$u_0^\varepsilon(x) = \begin{cases} 1, & x < -\varepsilon, \\ \dfrac{x}{2\varepsilon} + \dfrac{3}{2}, & -\varepsilon \le x \le \varepsilon, \\ 2, & x > \varepsilon, \end{cases} \tag{4.3}$$

and

$$\delta_\varepsilon(x) = \begin{cases} \dfrac{2}{\varepsilon^2}\, x + 2, & -\varepsilon^2 < x \le 0, \\ -\frac{2}{\varepsilon^2}\, x + 2, & 0 < x < \varepsilon^2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Note that

$$a_\varepsilon'(x) + \delta_\varepsilon'(x) = \begin{cases} \dfrac{2}{\varepsilon^2} - \dfrac{1}{\varepsilon}, & -\varepsilon^2 < x < 0, \\ -\dfrac{2+\varepsilon}{\varepsilon^2}, & 0 < x < \varepsilon^2, \\ -\dfrac{1}{\varepsilon}, & -\varepsilon < x < -\varepsilon^2 \text{ or } \varepsilon^2 < x < \varepsilon, \\ 0, & x < -\varepsilon \text{ or } x > \varepsilon, \end{cases} \tag{4.5}$$

which means that $a_\varepsilon(x) + \delta_\varepsilon(x)$ is not a monotone function in $(-\varepsilon, \varepsilon)$.

When $\delta_\varepsilon(x) \equiv 0$, by the results in Section 2, we see that there exists a global classical solution of (4.1). In the following we show that, when $\delta_\varepsilon(x)$ is given by (4.4), the characteristic curves starting at $\{(x_0, 0) : x_0 < \varepsilon^2\}$ never reach line $x = \varepsilon^2$. It follows that there exists a region where the information of $u^\varepsilon(x,t)$ is missing. In addition, those characteristic curves intersect in Ω_ε so that shocks occur with negative speeds. This implies that the behavior of perturbed Riemann solution completely changes when $a_\varepsilon(x)$ in (2.3) is perturbed by $\delta_\varepsilon(x)$.

To show this, we set $\tilde{x}_0 \equiv (\varepsilon^4, 0)$, $\tilde{x}_1 \equiv (-\varepsilon^4, 0)$. We also define regions $\Omega_\varepsilon^L \equiv \{(x,t) : -\varepsilon^2 < x < 0, t > 0\}$, $\Omega_\varepsilon^R \equiv \{(x,t) : 0 < x < \varepsilon^2, t > 0\}$. Then, by (4.5) and the results in Section 2, we obtain

$$\tilde{u}^\varepsilon(t;\tilde{x}_0) = (\frac{\varepsilon^3}{2} + \frac{3}{2}) - \frac{2+\varepsilon}{\varepsilon^2}\, t, \quad \tilde{x}^\varepsilon(t;\tilde{x}_0) = \varepsilon^4 + \tilde{u}^\varepsilon(t;\tilde{x}_0)\, t, \quad t \geq 0.$$

It is easy to see that the graph of $\Gamma_0 \equiv \{(\tilde{x}^\varepsilon(t;\tilde{x}_0), t) :\ t \geq 0\}$ is a parabola concave down with respect to t. Moreover, for sufficiently small ε we have

$$\max_{t\geq 0} \tilde{x}^\varepsilon(t;\tilde{x}_0) = \tilde{x}^\varepsilon(t_0^*;\tilde{x}_0) = \varepsilon^4 + [\varepsilon^2(\frac{\varepsilon^3}{2} + \frac{3}{2})^2]\ (2(2+\varepsilon))^{-1} < \varepsilon^2,$$

where $t_0^* \equiv [\frac{\varepsilon^5}{2} + \frac{3}{2}\,\varepsilon^2](2+\varepsilon)^{-1}$. It implies that Γ_0 does not intersect line $x = \varepsilon^2$. Next, by direct calculation we obtain that Γ_0 intersects t-axis at $(0, t_1)$ where

$$t_1 \equiv \left[(\frac{\varepsilon^5}{2} + \frac{3\varepsilon^2}{2}) + \varepsilon^2\sqrt{(\frac{\varepsilon^3}{2} + \frac{3}{2})^2 - 4(-\frac{2+\varepsilon}{2\varepsilon^2})\,\varepsilon^4}\,\right] (2+\varepsilon)^{-1}.$$

On the other hand, we have

$$\tilde{u}^\varepsilon(t;\tilde{x}_1) = (\frac{3}{2} - \frac{\varepsilon^3}{2}) + \frac{2-\varepsilon}{\varepsilon^2}\, t, \quad \tilde{x}^\varepsilon(t;\tilde{x}_1) = x_1 + \tilde{u}^\varepsilon(t;\tilde{x}_1)\, t$$

for $0 \leq t \leq \hat{t}_1 \equiv \left[\frac{\varepsilon^5}{2} - \frac{3}{2}\,\varepsilon^2 + \varepsilon^2\sqrt{(\frac{3}{2} - \frac{\varepsilon^3}{2})^2 + 2\varepsilon^2(2-\varepsilon)}\,\right](2-\varepsilon)^{-1}$ where $\hat{t}_1$ is the first intersection time when curve $\tilde{x}^\varepsilon(t;\tilde{x}_1)$ intersects line $x = 0$. We observe that $t_1 > \hat{t}_1$ for sufficiently small ε. Next, by the results in Section 2, we obtain

$$\tilde{u}^\varepsilon(t;\tilde{x}_1) = \sqrt{(\frac{3}{2} - \frac{\varepsilon^3}{2})^2 + 2\varepsilon^2(2-\varepsilon)} - \frac{2+\varepsilon}{\varepsilon^2}\,(t - \hat{t}_1),$$

$$\tilde{x}^\varepsilon(t;\tilde{x}_1) = \sqrt{(\frac{3}{2} - \frac{\varepsilon^3}{2})^2 + 2\varepsilon^2(2-\varepsilon)}(t - \hat{t}_1) - \frac{2+\varepsilon}{2\varepsilon^2}\,(t - \hat{t}_1)^2$$

for $\hat{t}_1 \leq t \leq \hat{t}_2$ where $\hat{t}_2 \equiv \hat{t}_1 + \left[2\varepsilon^2\sqrt{(\frac{3}{2} - \frac{\varepsilon^3}{2})^2 + 2\varepsilon^2(2-\varepsilon)}\,\right](2+\varepsilon)^{-1}$ is the second intersection time when $\tilde{x}^\varepsilon(t;\tilde{x}_1)$ intersects $x = 0$. We see that the graph of $\Gamma_1 \equiv \{(\tilde{x}^\varepsilon(t;\tilde{x}_1), t) :\ 0 \leq t \leq \hat{t}_2\}$ is also a parabola concave down with respect to t. Also, for sufficiently small ε, we have

$$\max_{t\geq 0} \tilde{x}^\varepsilon(t;\tilde{x}_1) = \tilde{x}^\varepsilon(t_1^*;\tilde{x}_1) = [\varepsilon^2[(\frac{3}{2} - \frac{\varepsilon^3}{2})^2 + 2\varepsilon^2(2-\varepsilon)]]\ (2(2+\varepsilon))^{-1} < \varepsilon^2,$$

where $t_1^* \equiv \hat{t}_1 + \left[\varepsilon^2\sqrt{(\frac{3}{2}-\frac{\varepsilon^3}{2})^2+2\varepsilon^2(2-\varepsilon)}\right](2+\varepsilon)^{-1}$. It follows that Γ_1 does not intersect line $x=\varepsilon^2$. Finally, by taking Taylor expansion to $\hat{t}_2$, we obtain $\hat{t}_2 > t_1 > \hat{t}_1$ for sufficiently small ε. It implies that $\Gamma_0,\ \Gamma_1$ intersect at some point $(\bar{x},\bar{t}\ ) \in \Omega_\varepsilon^R$ (Figure 5). Therefore, shock wave occurs in $\Omega_\varepsilon^L \bigcup \Omega_\varepsilon^R$. It indicates that the perturbed Riemann solutions are unstable with respect to general profile of $a_\varepsilon(x)$.

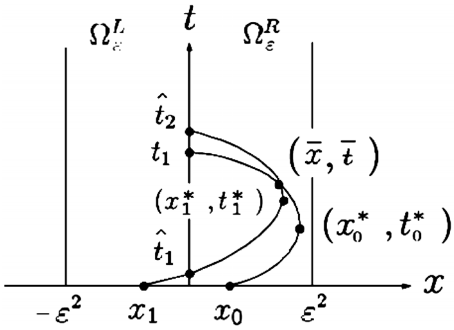

Fig. 5. Shock occurs in $\Omega_\varepsilon^R \subset \Omega_\varepsilon^L \bigcup \Omega_\varepsilon^R$.

In the rest of this section we establish the stability of perturbed Riemann solutions when $a_\varepsilon(x)$ is restricted to some class of functions. To start, given $0<\varepsilon\ll 1$, we consider the following perturbed Riemann problems

$$\left\{\begin{array}{l} u_t^\varepsilon + f(u^\varepsilon)_x = a_\varepsilon'(x)g(u^\varepsilon), \\ u^\varepsilon(x,0) = u_0^\varepsilon(x), \end{array}\right. \tag{4.6}$$

and

$$\left\{\begin{array}{l} \tilde{u}_t^\varepsilon + f(\tilde{u}^\varepsilon)_x = \tilde{a}_\varepsilon'(x)g(\tilde{u}^\varepsilon), \\ \tilde{u}^\varepsilon(x,0) = u_0^\varepsilon(x) \end{array}\right. \tag{4.7}$$

with $a_\varepsilon(x)$ given in (2.3), and $\tilde{a}_\varepsilon(x) = a_\varepsilon(x) + \delta_\varepsilon(x)$ where $\delta_\varepsilon(x) \in C_0^2([-\varepsilon,\varepsilon])$ satisfies

$$\tilde{a}_\varepsilon'(x) = b^\varepsilon(x) + \delta_\varepsilon'(x) < 0, \quad x\in(-\varepsilon,\varepsilon), \tag{4.8}$$

$$\delta_\varepsilon''(x) > 0, \quad x\in(-\varepsilon,\varepsilon), \tag{4.9}$$

$$\|\delta_\varepsilon'(x)\|_{L^1([-\varepsilon,\varepsilon])} \to 0 \quad as\ \varepsilon\to 0. \tag{4.10}$$

Here $b^\varepsilon(x)$ is given in (2.8). Also $g(u)$, u_L, u_R and $u_0^\varepsilon(x)$ are restricted to either one of the following four cases:

(i) $g(u) > 0$ for $u > 0$, $u_0^\varepsilon(x)$ is given by (2.2) and $u_R > u_L > \gamma^*(\varepsilon) > 0$ where

$$\gamma^*(\varepsilon) \equiv (2\,\frac{a_L - a_R}{f'(u_L)} + \|\delta_\varepsilon'(x)\|_{L^1([-\varepsilon,\varepsilon])}) \max_{u\in[0,u_L]} g(u). \tag{4.11}$$

(ii) $g(u) > 0$ for $u > 0$, $u_0^\varepsilon(x)$ is given by (3.1) and $u_L > u_R > 0$ with $\delta \equiv |u_L - u_R|$ satisfying

$$\delta > (\frac{a_L - a_R}{f'(u_L)} + \|\delta_\varepsilon'(x)\|_{L^1([-\varepsilon,\varepsilon])}) \max_{u\in[u_R,u_L]} g(u). \tag{4.12}$$

(iii) $g(u) < 0$ for $u > 0$, $u_0^\varepsilon(x)$ is given by (2.2), and $u_R > u_L > 0$ with $\delta \equiv |u_L - u_R|$ satisfying

$$\delta > (\frac{a_L - a_R}{f'(u_L)} + \|\delta_\varepsilon'(x)\|_{L^1([-\varepsilon,\varepsilon])}) \max_{u\in[u_L,u_R]} (-g(u)). \tag{4.13}$$

(iv) $g(u) < 0$ for $u > 0$, $u_0^\varepsilon(x)$ is given by (3.1), and $u_L > u_R > 0$.

By (4.10) it follows that $\gamma^*(\varepsilon) \to 2\gamma$ and condition (4.12) becomes (3.3) as $\varepsilon \to 0$ where γ is in (2.6). Furthermore, by similar analysis in the proof of Lemma 2.1 and Lemma 3.1, we obtain $\tilde{u}^\varepsilon(x,t) > \gamma > 0$ for $t \geq 0$ in Case (i), and $\tilde{u}^\varepsilon(x,t)$ consists of a shock globally in Case (ii). Also, condition (4.13) ensures that $\tilde{u}^\varepsilon(x,t)$ is a global classical solution in Case (iii).

First, we consider Cases (i) and (iii). We wish to show the solution of (4.7) converges pointwise to the solution of (4.6) for some assumptions of $\tilde{a}_\varepsilon(x)$. Note that there exists a unique global classical solution of (4.6) by the results in Section 2. In the following lemma we show that the characteristic curves of problem (4.7) in Cases (i) and (iii) do not intersect so that the classical solution of (4.7) also exists globally.

Lemma 4.1. *We consider perturbed Riemann problem* (4.7). *Set* $\tilde{x}_1 \equiv (x_1, 0)$, $\tilde{x}_2 \equiv (x_2, 0)$ *and* $x_1 < x_2$.

(a) *If* u_L, u_R *and* $u_0^\varepsilon(x)$ *are in Case (i), and* $\delta_\varepsilon(x)$ *satisfies* (4.8)-(4.10), *then characteristic curves* $\tilde{x}^\varepsilon(t;\tilde{x}_1)$, $\tilde{x}^\varepsilon(t;\tilde{x}_2)$ *do not intersect for* $t \geq 0$.

(b) *If* u_L, u_R *and* $u_0^\varepsilon(x)$ *are in Case (iii), and* $\delta_\varepsilon(x)$ *satisfies* (4.8), (4.10) *and*

$$\delta_\varepsilon''(x) < 0 \tag{4.14}$$

for $x \in (-\varepsilon, \varepsilon)$, *then statement (1) still holds.*

Proof. We only show statement (1). The proof of statement (2) is similar. First we consider the case that $\tilde{x}_1,\ \tilde{x}_2 \in \Gamma_{0\varepsilon}$ where $\Gamma_{0\varepsilon}$ is in (2.10). Then $\tilde{u}^\varepsilon(t;\tilde{x}_i)$, $i = 1,\ 2$, satisfy

$$\begin{cases} \dfrac{d\tilde{u}^\varepsilon(t;\tilde{x}_i)}{dt} = [b^\varepsilon + \delta_\varepsilon'(\tilde{x}^\varepsilon(t;\tilde{x}_i))]g(\tilde{u}^\varepsilon(t;\tilde{x}_i)), \\ \tilde{u}^\varepsilon(0;\tilde{x}_i) = u_0^\varepsilon(x_i), \quad i = 1, 2. \end{cases}$$

It leads to

$$G(\tilde{u}^\varepsilon(t;\tilde{x}_i)) = G(u_0^\varepsilon(x_i)) + b^\varepsilon\, t + \int_0^t \delta_\varepsilon'(\tilde{x}^\varepsilon(s;\tilde{x}_i))ds,\ i=1,2, \tag{4.15}$$

where

$$u_0^\varepsilon(x_i) = (\frac{u_R - u_L}{2\varepsilon})\ x_i + \frac{1}{2}(u_R + u_L),\ i = 1, 2,$$

and G is in (2.15). Subtracting equations in (4.15) and applying mean-value theorem to $G(\tilde{u}^\varepsilon(t;\tilde{x}_2)) - G(\tilde{u}^\varepsilon(t;\tilde{x}_1))$ and $\delta_\varepsilon'(\tilde{x}^\varepsilon(s;\tilde{x}_2)) - \delta_\varepsilon'(\tilde{x}^\varepsilon(s;\tilde{x}_1))$, we obtain

$$\begin{aligned} \tilde{u}^\varepsilon(t;\tilde{x}_2) - \tilde{u}^\varepsilon(t;\tilde{x}_1) \,&= g(c_1(t))[G(u_0^\varepsilon(x_2)) - G(u_0^\varepsilon(x_1))] \\ &\quad + g(c_1(t)) \int_0^t \delta_\varepsilon''(c_2(s))(\tilde{x}^\varepsilon(s;\tilde{x}_2) - \tilde{x}^\varepsilon(s;\tilde{x}_1))ds, \end{aligned} \tag{4.16}$$

where $c_1(t)$ lies between $\tilde{u}^\varepsilon(t;\tilde{x}_1)$ and $\tilde{u}^\varepsilon(t;\tilde{x}_2)$ for $t > 0$, and $c_2(s)$ lies between $\tilde{x}^\varepsilon(s;\tilde{x}_1)$ and $\tilde{x}^\varepsilon(s;\tilde{x}_2)$ for $s \in (0,t]$. On the other hand, characteristic curves $\tilde{x}^\varepsilon(t;\tilde{x}_i)$, $i = 1,\ 2$, satisfy

$$\begin{cases} \dfrac{d\tilde{x}^\varepsilon(t;\tilde{x}_i)}{dt} = f'(\tilde{u}^\varepsilon(t;\tilde{x}_i)), \\ \tilde{x}^\varepsilon(0;\tilde{x}_i) = x_i, \quad i = 1, 2. \end{cases} \tag{4.17}$$

It follows that

$$\begin{aligned} &\tilde{x}^\varepsilon(t;\tilde{x}_2) - \tilde{x}^\varepsilon(t;\tilde{x}_1) \\ &= (x_2 - x_1) + \int_0^t [f'(\tilde{u}^\varepsilon(s;\tilde{x}_2)) - f'(\tilde{u}^\varepsilon(s;\tilde{x}_1))]ds \\ &= (x_2 - x_1) + \int_0^t f''(c_3(s))(\tilde{u}^\varepsilon(s;\tilde{x}_2) - \tilde{u}^\varepsilon(s;\tilde{x}_1))ds, \end{aligned} \tag{4.18}$$

where $c_3(s)$ lies between $\tilde{u}^\varepsilon(s;\tilde{x}_1)$ and $\tilde{u}^\varepsilon(s;\tilde{x}_2)$ for $s \in (0,t]$. Substituting (4.18) into (4.16), it leads to

$$w(t) = g(c_1(t))[A(t) + \int_0^t \int_0^s B(\tau,s)w(\tau)d\tau ds],$$

where

$$\begin{aligned} w(t) \,&\equiv\, \tilde{u}^\varepsilon(t;\tilde{x}_2) - \tilde{u}^\varepsilon(t;\tilde{x}_1), \\ A(t) \,&\equiv\, [G(u_0^\varepsilon(x_2)) - G(u_0^\varepsilon(x_1))] + (x_2 - x_1)\int_0^t \delta_\varepsilon''(c_2(s))ds, \\ B(\tau,s) \,&\equiv\, \delta_\varepsilon''(c_2(s))f''(c_3(\tau)) \end{aligned}$$

for $t > 0,\ s \in (0, t]$ and $\tau \in (0, s]$. We notice that

$$w(0) = u_0^\varepsilon(x_2) - u_0^\varepsilon(x_1) = \frac{u_R - u_L}{2\varepsilon}\,(x_2 - x_1) > 0. \tag{4.19}$$

In addition, by (2.15), (4.19) and that $g(u),\ \delta_\varepsilon''$ are positive, we obtain $A(t) > 0$ for $t \geq 0$.

Next, we claim that $w(t) > 0$ for all $t > 0$. Suppose not, then there exists $t^* > 0$ such that $w(t^*) \leq 0$. If $w(t^*) < 0$, then by the fact that $w(0) > 0$ and intermediate value theorem, there exists at least one point $\tilde{t} \in (0, t^*)$ such that $w(\tilde{t}\,) = 0$. Define

$$t_1 \equiv \inf\,\{\tilde{t} \in (0, t^*] : w(\tilde{t}\,) = 0\}. \tag{4.20}$$

Then $w(t) > 0$ for all $t \in (0, t_1)$, and

$$w(t_1) = g(c_1(t_1))[A(t_1) + \int_0^{t_1}\int_0^s B(\tau, s)w(\tau)d\tau ds] = 0,$$

which implies

$$\int_0^{t_1}\int_0^s B(\tau, s)w(\tau)d\tau ds = -A(t_1) < 0.$$

On the other hand, by (4.9) and $f'' > 0$ we have $B(\tau, s) > 0$ for $\tau \in (0, s]$ and $s \in (0, t_1)$, it follows that there exists some $\tau \in (0, s] \subset (0, t_1)$ such that $w(\tau) < 0$, which contradicts (4.20). The proof for $w(t^*) = 0$ is similar. By the analysis above and the fact that $w(0) > 0$ again, we obtain that $w(t) = \tilde{u}^\varepsilon(t; \tilde{x}_2) - \tilde{u}^\varepsilon(t; \tilde{x}_1) > 0$ for all $t > 0$. We complete the proof of the claim. Finally, by (4.17) and $f'' > 0$, we conclude that $\dfrac{dx^\varepsilon(t; \tilde{x}_1)}{dt} < \dfrac{dx^\varepsilon(t; \tilde{x}_2)}{dt}$ for $t > 0$. It implies that $\tilde{x}^\varepsilon(t; \tilde{x}_1),\ \tilde{x}^\varepsilon(t; \tilde{x}_2)$ do not intersect for $t \geq 0$ when $\tilde{x}_1,\ \tilde{x}_2 \in \Gamma_{0\varepsilon}$.

Next, if $\tilde{x}_1,\ \tilde{x}_2 \in \Gamma_{0L}$, then by the same analysis given in Lemma 2.1, we obtain that $\tilde{x}^\varepsilon(t; \tilde{x}_1),\ \tilde{x}^\varepsilon(t; \tilde{x}_2)$ are parallel for $t > 0$. On the other hand, if $\tilde{x}_1,\ \tilde{x}_2 \in \Gamma_{0R}$, since there is no effect of the source term, we also obtain that $\tilde{x}^\varepsilon(t; \tilde{x}_1),\ \tilde{x}^\varepsilon(t; \tilde{x}_2)$ are parallel for $t > 0$. The proof is complete. ■

Next, we show that the classical solution of (4.7) in Cases (i) (resp., Case (iii)) converges pointwise to the classical solution of (4.6) if perturbation $\delta_\varepsilon(x)$ satisfies (4.8)-(4.10) (resp., (4.8),(4.10) and (4.14)). First, given $\tilde{x}_0 \equiv (x_0, 0)$, for the convenience we let $x^\varepsilon(t)$ (resp., $\tilde{x}^\varepsilon(t)$) denote the characteristic curve of (4.6) (resp., (4.7)) starting at $\tilde{x}_0$, and let $u^\varepsilon(t)$ (resp., $\tilde{u}^\varepsilon(t)$) denote the solution along $x^\varepsilon(t)$ (resp., $\tilde{x}^\varepsilon(t)$). We define

$$U^\varepsilon(t) \equiv \begin{pmatrix} x^\varepsilon(t) \\ u^\varepsilon(t) \end{pmatrix},\quad \tilde{U}^\varepsilon(t) \equiv \begin{pmatrix} \tilde{x}^\varepsilon(t) \\ \tilde{u}^\varepsilon(t) \end{pmatrix},\quad U_0^\varepsilon(x) \equiv \begin{pmatrix} x \\ u_0^\varepsilon(x) \end{pmatrix}, \tag{4.21}$$

and

$$F(U^\varepsilon(t)) \equiv \begin{pmatrix} f'(u^\varepsilon(t)) \\ b^\varepsilon g(u^\varepsilon(t)) \end{pmatrix}, \tag{4.22}$$

$$\tilde{F}(\tilde{U}^\varepsilon(t)) \equiv \begin{pmatrix} f'(\tilde{u}^\varepsilon(t)) \\ (b^\varepsilon + \delta'_\varepsilon(\tilde{x}^\varepsilon(t)))g(\tilde{u}^\varepsilon(t)) \end{pmatrix}. \tag{4.23}$$

We have the following theorem regarding to the stability of classical perturbed Riemamm solutions.

Theorem 4.2. *We consider perturbed Riemann problems* (4.6), (4.7) *where* $g(u)$, $u_0^\varepsilon(x)$, u_L, u_R *and* $\delta_\varepsilon(x)$ *satisfy the hypotheses in Lemma 4.1. Given* $\tilde{x}_0 \equiv (x_0, 0)$, *suppose* $U^\varepsilon(t)$, $\tilde{U}^\varepsilon(t)$ *are the vector functions given in* (4.21). *Then* $U^\varepsilon(t) \to \tilde{U}^\varepsilon(t)$ *as* $\varepsilon \to 0$ *for each* $t > 0$. *It means that the classical perturbed Riemann solution of* (4.6) *in Case (i) (resp., Case (iii)) is stable with respect to the perturbation satisfying* (4.8)-(4.10) *(resp.,* (4.8), (4.10) *and* (4.14)*).*

Proof. According to the characteristic method to (4.6), (4.7), we see that $U^\varepsilon(t)$, $\tilde{U}^\varepsilon(t)$ satisfy initial value problems

$$\begin{cases} \dot{U}^\varepsilon = F(U^\varepsilon), \\ U^\varepsilon(0) = U_0^\varepsilon(x_0), \end{cases} \tag{4.24}$$

$$\begin{cases} \dot{\tilde{U}}^\varepsilon = \tilde{F}(\tilde{U}^\varepsilon), \\ \tilde{U}^\varepsilon(0) = U_0^\varepsilon(x_0), \end{cases} \tag{4.25}$$

where " $\cdot$" denotes the derivative with respect to t, and $U_0^\varepsilon(x_0)$ is given in (4.21).

When $x_0 > \varepsilon$, since $b^\varepsilon = \delta'_\varepsilon(x) = 0$, the proof is trivial. Next we consider $x_0 \in [-\varepsilon, \varepsilon]$. Suppose that $x^\varepsilon(t; \tilde{x}_0)$, $\tilde{x}^\varepsilon(t; \tilde{x}_0)$ intersect the line $x = \varepsilon$ at $t = t_\varepsilon$, $t = \tilde{t}_\varepsilon$ respectively. Then we define $\beta \equiv \min\{t_\varepsilon, \tilde{t}_\varepsilon\}$, $\alpha \equiv \max\{t_\varepsilon, \tilde{t}_\varepsilon\}$. Note that t_ε, $\tilde{t}_\varepsilon$, α and β are of order ε. Next, by change of variable $z = \dfrac{t}{\varepsilon}$ and letting $U^\varepsilon(t) \equiv V^\varepsilon(z)$, $\tilde{U}^\varepsilon(t) \equiv \tilde{V}^\varepsilon(z)$, initial value problems (4.24), (4.25) can be transformed into

$$\begin{cases} \dfrac{dV^\varepsilon}{dz} = \varepsilon F_1(V^\varepsilon), \\ V^\varepsilon(0) = U_0^\varepsilon(x_0), \end{cases} \tag{4.26}$$

$$\begin{cases} \dfrac{d\tilde{V}^\varepsilon}{dz} = \varepsilon \tilde{F}_1(\tilde{V}^\varepsilon), \\ \tilde{V}^\varepsilon(0) = U_0^\varepsilon(x_0), \end{cases} \tag{4.27}$$

where $F_1(V^\varepsilon) \equiv F(U^\varepsilon)$ and $\tilde{F}_1(\tilde{V}^\varepsilon) \equiv \tilde{F}(\tilde{U}^\varepsilon)$. Then by (4.26), (4.27) and that $g(\tilde{V}^\varepsilon(z))$ is uniformly bounded for $z \in [0, \frac{\beta}{\varepsilon}]$, we obtain

$$
\begin{aligned}
&|V^\varepsilon(z) - \tilde{V}^\varepsilon(z)| \\
&\le \int_0^z |\varepsilon F_1(V^\varepsilon(s)) - \varepsilon F_1(\tilde{V}^\varepsilon(s))| ds + \int_0^z |\varepsilon F_1(\tilde{V}^\varepsilon(s)) - \varepsilon \tilde{F}_1(\tilde{V}^\varepsilon(s))| ds \\
&\le \int_0^z \phi(s) \cdot |V^\varepsilon(s) - \tilde{V}^\varepsilon(s)| ds + \int_0^z |\varepsilon \delta'_\varepsilon(\tilde{x}^\varepsilon(s)) g(\tilde{V}^\varepsilon(s))| ds \\
&\le \int_0^z \phi(s) \cdot |V^\varepsilon(s) - \tilde{V}^\varepsilon(s)| ds + K \|\varepsilon \delta'_\varepsilon(\tilde{x}^\varepsilon(s))\|_{L^1([0,\frac{\beta}{\varepsilon}])} \\
&\le \int_0^z \phi(s) \cdot |V^\varepsilon(s) - \tilde{V}^\varepsilon(s)| ds + O(1) K \|\varepsilon \delta'_\varepsilon(x)\|_{L^1([-\varepsilon,\varepsilon])}
\end{aligned} \tag{4.28}
$$

for $z \in [0, \frac{\beta}{\varepsilon}]$. Here $K \equiv \max_{z \in [0,\frac{\beta}{\varepsilon}]} |g(\tilde{V}^\varepsilon(z))|$, and $\phi(s) \equiv \varepsilon \|DF_1(\bar{V}^\varepsilon(s))\|$ where $DF_1 \equiv \begin{bmatrix} 0 & f'' \\ 0 & b^\varepsilon g' \end{bmatrix}$ and $\bar{V}^\varepsilon(s)$ lies between $V^\varepsilon(s)$, $\tilde{V}^\varepsilon(s)$ for $s \in [0, \frac{\beta}{\varepsilon}]$. The last inequality in (4.28) holds due to the fact that $\tilde{x}^\varepsilon(t) \in [-\varepsilon, \varepsilon]$ is monotone in t. Then, by Gronwall's inequality and that $\phi(s)$ is uniformly bounded in $[0, \frac{\beta}{\varepsilon}]$, we obtain

$$
\begin{aligned}
&|V^\varepsilon(z) - \tilde{V}^\varepsilon(z)| \\
&\le K_1 \varepsilon \|\delta'_\varepsilon(x)\|_{L^1([-\varepsilon,\varepsilon])} + K_1 \varepsilon \int_0^z \|\delta'_\varepsilon(x)\|_{L^1([-\varepsilon,\varepsilon])} \phi(s) e^{\int_s^z \phi(\tau) d\tau} ds \\
&\le K_1 \, (1 + \frac{M\beta}{\varepsilon} e^{\frac{M\beta}{\varepsilon}}) \, \varepsilon \|\delta'_\varepsilon(x)\|_{L^1([-\varepsilon,\varepsilon])}
\end{aligned} \tag{4.29}
$$

for $z \in [0, \frac{\beta}{\varepsilon}]$ and some positive constants K_1, M. Since $\frac{M\beta}{\varepsilon}$ is of order $O(1)$, it follows that $|V^\varepsilon(z) - \tilde{V}^\varepsilon(z)| \to 0$ as $\varepsilon \to 0$ for $z \in [0, \frac{\beta}{\varepsilon}]$ if $\delta_\varepsilon(x)$ satisfies (4.10). It is sufficient to say that $U^\varepsilon(t) \to \tilde{U}^\varepsilon(t)$ as $\varepsilon \to 0$ for each $t \in [0, \beta]$ if $\delta_\varepsilon(x)$ satisfies (4.10). Next, since α, β are of order ε, we have $\alpha \to \beta$ (or $t_\varepsilon \to \tilde{t}_\varepsilon$) as $\varepsilon \to 0$. Finally, by the vanishing of source term when $t > \alpha$, it implies that $U^\varepsilon(t) \to \tilde{U}^\varepsilon(t)$ as $\varepsilon \to 0$ for each $t > \alpha$. We just proved the case of $x_0 \in [-\varepsilon, \varepsilon]$. To case $x_0 < -\varepsilon$, the proof is similar. We complete the proof. ■

We have established the stability of classical perturbed Riemann solutions. Next, we study the stability of shocks for perturbed Riemann problem of Cases (ii) and (iv), which is given in the following theorem.

Theorem 4.3. *Consider perturbed Riemann problems* (4.6) *and* (4.7) *where* $g(u)$*,* u_L*,* u_R *and* $u_0^\varepsilon(x)$ *are either in Case (ii) or (iv), and* $\delta_\varepsilon(x)$ *in* (4.7) *satisfies* (4.8)-(4.10) *(resp.,* (4.8), (4.10) *and* (4.14)*) in Case (ii) (resp., Case (iv)). Let* $S^\varepsilon(x(t),t)$*,* $\tilde{S}^\varepsilon(x(t),t)$*,* $t \geq 0$*, denote the location of shocks to problems* (4.6), (4.7) *respectively. Then* $S^\varepsilon(x(t),t)$ *tends to* $\tilde{S}^\varepsilon(x(t),t)$ *as* $\varepsilon \to 0$*. Furthermore, we have* $\lim\limits_{\varepsilon\to 0} |u^\varepsilon(x,t) - \tilde{u}^\varepsilon(x,t)| = 0$ *a.e. in* $\mathbb{R}^+ \times \mathbb{R}$*.*

Proof. We only prove Case (ii). The proof of Case (iv) is similar. First, since $\delta = |u_L - u_R|$ satisfies (4.12), by Lemma 3.1 we obtain that the shock waves of (4.6), (4.7) exist globally.

Let L^ε denote the characteristic curve of (4.6), (4.7) starting at $(\varepsilon, 0)$, and $x^\varepsilon(t;(\bar{x}_0,0))$ (resp., $\tilde{x}^\varepsilon(t;(\tilde{x}_0,0))$) be the characteristic curve intersecting L^ε and $S^\varepsilon(x(t),t)$ (resp., $\tilde{S}^\varepsilon(x(t),t)$) at some point $(\bar{x},\bar{t})$ (resp., $(\tilde{x},\tilde{t})$). Also, let $\bar{R}^\varepsilon$ (resp., $\tilde{R}^\varepsilon$) be the region bounded by L^ε, $x^\varepsilon(t;(\bar{x}_0,0))$ and line $\{(x,0) : \bar{x}_0 \leq x \leq \varepsilon\}$ (resp., L^ε, $\tilde{x}^\varepsilon(t;(\tilde{x}_0,0))$ and line $\{(x,0) : \tilde{x}_0 \leq x \leq \varepsilon\}$). Then, by the proof in Theorem 3.2 and (4.10), we obtain that $\bar{x}$, $\bar{t}$, $\tilde{x}$, $\tilde{t}$ are of order ε, which means that $\{S^\varepsilon(x(t),t) : 0 < t < \bar{t}\}$, $\{\tilde{S}^\varepsilon(x(t),t) : 0 < t < \tilde{t}\}$ and regions $\bar{R}^\varepsilon$, $\tilde{R}^\varepsilon$ tend to $(0,0)$ as $\varepsilon \to 0$.

Next, by Theorem 3.2 we observe that $u^\varepsilon(t;(x_1,0)) = u^*$ at $x = \varepsilon$ for any $x_1 < \bar{x}_0$ where u^* is in (2.25). Now we claim that $\tilde{u}^\varepsilon(t;(x_1,0)) = u^*$ at $x = \varepsilon$ for any $x_1 < \tilde{x}_0$. Indeed, by Theorem 3.3 and that $a_\varepsilon(x)$, $\tilde{a}_\varepsilon(x)$ are monotone functions, we obtain that $u^\varepsilon(t;(x_1,0)) = v^\varepsilon(a_\varepsilon(x))$, $\tilde{u}^\varepsilon(t;(x_1,0)) = \tilde{v}^\varepsilon(\tilde{a}_\varepsilon(x))$ in Ω_ε for some C^1 functions v^ε and $\tilde{v}^\varepsilon$ satisfying

$$\begin{cases} \dfrac{dv^\varepsilon}{da_\varepsilon} = \dfrac{g(v^\varepsilon)}{f'(v^\varepsilon)}, \\ v^\varepsilon(a_L) = u_L, \end{cases} \tag{4.30}$$

$$\begin{cases} \dfrac{d\tilde{v}^\varepsilon}{d\tilde{a}_\varepsilon} = \dfrac{g(\tilde{v}^\varepsilon)}{f'(\tilde{v}^\varepsilon)}, \\ \tilde{v}^\varepsilon(a_L) = u_L. \end{cases} \tag{4.31}$$

Since problems (4.30) and (4.31) are identical, by existence and uniqueness theorem of ODEs, it follows that $v^\varepsilon = \tilde{v}^\varepsilon$. Then, by $a_\varepsilon(\varepsilon) = \tilde{a}_\varepsilon(\varepsilon) = a_R$ and Lemma 2.1, we obtain that

$$\tilde{u}^\varepsilon(t;(x_1,0))|_{\{x=\varepsilon\}} = \tilde{v}^\varepsilon(a_R) = v^\varepsilon(a_R) = u^*,$$

where u^* is in (2.25). It follows that the states on both sides of $\{S^\varepsilon(x(t),t) : t > \bar{t}\}$ and $\{\tilde{S}^\varepsilon(x(t),t) : t > \tilde{t}\}$ are the same. Therefore, by Rankine-Hugoniot condition and the fact that $(\tilde{x},\tilde{t}) \to (\bar{x},\bar{t})$, we obtain that $S^\varepsilon(x(t),t)$ tends to $\tilde{S}^\varepsilon(x(t),t)$ as $\varepsilon \to 0$. Finally, letting ε approach 0, we have $\lim\limits_{\varepsilon\to 0} |u^\varepsilon(x,t) - \tilde{u}^\varepsilon(x,t)| = 0$ for any (x,t) not in $S^\varepsilon(x(t),t) \bigcup \tilde{S}^\varepsilon(x(t),t)$. The proof is complete. ■

We have established the stability of perturbed Riemann solutions. Therefore, the generalized entropy solution of (1.1)-(1.3) under Definition 1.2 is unique with respect to $\{\tilde{a}_\varepsilon(x)\}$. Here we emphasize that, in the proof of Theorem 4.2, condition (4.10) can be relaxed to

$$\varepsilon\|\delta'_\varepsilon(x)\|_{L^1([-\varepsilon,\varepsilon])} \to 0 \quad \text{as } \varepsilon \to 0$$

in the case of classical solutions (see (4.29)). But (4.10) is required in the case of shocks.

To state the main theorem of this paper, we define the following constants for given $0<\varepsilon\ll 1$, $f(u)$, $g(u)$, $\delta_\varepsilon(x)$ and constants $a_L,\ a_R,\ u_L, u_R$:

$$\delta \equiv |u_L - u_R|,\ \kappa_1 \equiv |\frac{a_L-a_R}{f'(u_L)}|,\ \kappa_2 \equiv |\frac{a_L-a_R}{f'(u_R)}|, \tag{4.32}$$

$$\beta_1 \equiv (2\kappa_1 + \|\delta'_\varepsilon(x)\|_{L^1([-\varepsilon,\varepsilon])}) \max_{u\in[0,u_L]} |g(u)|, \tag{4.33}$$

$$\beta_2 \equiv (\kappa_2 + \|\delta'_\varepsilon(x)\|_{L^1([-\varepsilon,\varepsilon])}) \max_{u\in[u_L,u_R]} |g(u)|, \tag{4.34}$$

$$\beta_3 \equiv (\kappa_2 + \|\delta'_\varepsilon(x)\|_{L^1([-\varepsilon,\varepsilon])}) \max_{u\in[u_R,u_L]} |g(u)|, \tag{4.35}$$

$$\beta_4 \equiv (\kappa_1 + \|\delta'_\varepsilon(x)\|_{L^1([-\varepsilon,\varepsilon])}) \max_{u\in[u_L,u_R]} |g(u)|, \tag{4.36}$$

$$\beta_5 \equiv (2\kappa_2 + \|\delta'_\varepsilon(x)\|_{L^1([-\varepsilon,\varepsilon])}) \max_{u\in[u_R,0]} |g(u)|, \tag{4.37}$$

$$\beta_6 \equiv (\kappa_1 + \|\delta'_\varepsilon(x)\|_{L^1([-\varepsilon,\varepsilon])}) \max_{u\in[u_R,u_L]} |g(u)|. \tag{4.38}$$

Main Theorem. *We consider Riemann problem* (1.1)-(1.3) *where f, g are smooth functions satisfying $f'(0) = g(0) = 0$ and $f'(u) \neq 0$, $f''(u) > 0$ for $u \neq 0$. Also, given $0<\varepsilon\ll 1$, let* (4.7) *be a corresponding perturbed Riemann problem of* (1.1)-(1.3) *with*

$$\tilde{a}_\varepsilon(x) = a_\varepsilon(x) + \delta_\varepsilon(x),$$

where $a_\varepsilon(x)$ is in (2.3) *and $\delta_\varepsilon(x)$ satisfies* (4.8)-(4.10) *(resp.,* (4.8)*,* (4.10) *and* (4.14)*) when $(a_R - a_L)g(u) < 0$ (resp., $(a_R - a_L)g(u) > 0$) for $u \neq 0$. We assume that u_L, u_R and $u_0^\varepsilon(x)$ in* (4.7) *are in either one of the following cases:*

(1) *$(a_R - a_L)g(u) < 0$, $u_R > u_L$ and $u_0^\varepsilon(x)$ is in* (2.2)*. Also $u_L > \beta_1$ (resp., $\delta > \beta_2$) when $u_L > 0$ (resp., $u_R < 0$).*

(2) *$(a_R - a_L)g(u) < 0$, $u_L > u_R$ and $u_0^\varepsilon(x)$ is in* (3.1)*. Also $\delta > \beta_3$ when $u_R > 0$.*

(3) $(a_R - a_L)g(u) > 0$, $u_R > u_L$ *and* $u_0^\varepsilon(x)$ *is in* (2.2). *Also* $\delta > \beta_4$ (*resp.*, $|u_R| > \beta_5$) *when* $u_L > 0$ (*resp.*, $u_R < 0$).

(4) $(a_R - a_L)g(u) > 0$, $u_L > u_R$ *and* $u_0^\varepsilon(x)$ *is in* (3.1). *Also* $\delta > \beta_6$ *when* $u_L < 0$.

Here $\{\beta_i\}$, $i = 1, \cdots, 6$, *are given in* (4.33)-(4.38), *and* δ, κ_1, κ_2 *are in* (4.32). *Then there exists a unique generalized entropy solution* $u(x,t)$ *of Lax-type to Riemann problem* (1.1)-(1.3), *which is given by*

$$u(x,t) = \lim_{\varepsilon \to 0} u^\varepsilon(x,t),$$

where $u^\varepsilon(x,t)$ *is the solution of* (4.7). *Furthermore, solution* $u(x,t)$ *consists of at most three constant states separated by a standing wave discontinuity and an elementary wave* (*rarefaction wave or shock wave*).

5. Appendix

In the appendix, we construct the generalized entropy solutions without proof to some Riemann problems (1.1)-(1.3) when u_L, u_R or δ in (4.32) are not in the cases given in Main Theorem. The generalized entropy solutions are constructed based on the corresponding perturbed Riemann problem where $\tilde{a}_\varepsilon(x)$ in Main Theorem is a piecewise linear function ($\delta_\varepsilon(x) = 0$). We consider the following cases:

(A) If $(a_R - a_L)g(u) < 0$, $u_R > \gamma_1 > u_L > 0$ where $\gamma_1 \equiv \dfrac{a_L - a_R}{f'(u_R)} \max\limits_{[0,u_L]} g(u)$.

In this case, perturbed Riemann solution $u^\varepsilon(x,t)$ is a classical solution. When u_L is sufficiently close to 0, there exists some $x_0 < \varepsilon$ such that characteristic curves $\{x^\varepsilon(t;(x,0)) : \ x < x_0\}$ do not intersect Γ_R, but $\{x^\varepsilon(t;(x,0)) : \ x_0 < x < \varepsilon\}$ still pass through Γ_R. Therefore, when $\varepsilon \to 0$, the generalized entropy solution of (1.1)-(1.3) can be expressed as

$$u(x,t) = \begin{cases} u_L, & x < 0, \ t > 0, \\ (f')^{-1}(\dfrac{x}{t}), & 0 \le x \le f'(u_R)t, \ t > 0, \\ u_R, & f'(u_R)t < x, \ t > 0. \end{cases} \tag{5.1}$$

(B) If $(a_R - a_L)g(u) < 0$, $u_L > u_R > 0$ and $0 < \delta < \dfrac{a_L - a_R}{f'(u_L)} \max\limits_{[u_R,u_L]} g(u)$.

In this case, perturbed Riemann solution $u^\varepsilon(x,t)$ consists of a shock. But when δ is sufficiently small and $|(a_R - a_L)g(u)|$ is sufficiently large for $u \in [u_R, u_L]$, the time for the existence of shock is only of order $O(1)\varepsilon$. It implies that the generalized entropy solution $u(x,t)$ of (1.1)-(1.3) is a rarefaction wave which can be expressed as

$$
(5.2) \qquad u(x,t) = \begin{cases} u_L, & x < 0,\ t > 0, \\ u^*, & 0 \le x \le f'(u^*)t,\ t > 0, \\ (f')^{-1}(\frac{x}{t}), & f'(u^*)t < x \le f'(u_R)t,\ t > 0, \\ u_R, & f'(u_R)t < x,\ t > 0, \end{cases}
$$

where $0 \le u^* < u_R < u_L$, and u^* is in (2.25).

(C) If $(a_R - a_L)g(u) > 0$, $u_R > u_L > 0$ and $\delta < \dfrac{a_R - a_L}{f'(u_L)} \max\limits_{[u_L, u_R]} g(u)$.

In this case, the value of $u^\varepsilon(x,t)$ is increasing along each characteristic curve in Ω_ε. It follows that, if δ is sufficiently small and $(a_R - a_L)g(u)$ is sufficiently large for $u \in [u_L, u_R]$, then $u^\varepsilon(x,t)|_{x=\varepsilon}$ and $f'(u^\varepsilon(x,t))|_{x=\varepsilon}$ are increasing functions of t. It implies that there exists $t^* > 0$ such that the shock occurs when $t \ge t^*$. By letting $\varepsilon \to 0$ and the fact that $t^* = O(1)\varepsilon$, we obtain the generalized entropy solution

$$
(5.3) \qquad u(x,t) = \begin{cases} u_L, & x < 0,\ t > 0, \\ u^*, & 0 \le x \le \dfrac{f(u_R) - f(u^*)}{u_R - u^*}t,\ t > 0, \\ u_R, & \dfrac{f(u_R) - f(u^*)}{u_R - u^*}t < x,\ t > 0, \end{cases}
$$

where u^* is in (2.25).

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