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NOTE ON LOCAL INTEGRATED C-COSINE FUNCTIONS AND ABSTRACT CAUCHY PROBLEMS

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Abstract. Let α be a nonnegative number, and $C : X \to X$ a bounded linear operator on a Banach space X. In this paper, we shall deduce some basic properties of a nondegenerate local α-times integrated C-cosine function on X and some generation theorems of local α-times integrated C-cosine functions on X with or without the nondegeneracy, which can be applied to obtain some equivalence relations between the generation of a nondegenerate local α-times integrated C-cosine function on X with generator A and the unique existence of solutions of the abstract Cauchy problem:

$$\text{ACP}(A, f, x, y) \qquad \begin{cases} u''(t) = Au(t) + f(t) & \text{for } t \in (0, T_0), \\ u(0) = x, u'(0) = y, \end{cases}$$

just as the case of α-times integrated C-cosine function when $C : X \to X$ is injective and $A : \mathrm{D}(A) \subset X \to X$ a closed linear operator in X such that $CA \subset AC$. Here $0 < T_0 \leq \infty$, $x, y \in X$, and f is an X-valued function defined on a subset of $\mathbb{R}$ containing $(0, T_0)$.

1. Introduction

Let X be a Banach space over $\mathbb{F}$(=$\mathbb{R}$ or $\mathbb{C}$) with norm $\|\cdot\|$, and let $\mathrm{L}(X)$ denote the set of all bounded linear operators from X into itself. For each $0 < T_0 \leq \infty$, we consider the following abstract Cauchy problem:

$$\text{(1.1)} \qquad \text{ACP}(A, f, x, y) \qquad \begin{cases} u''(t) = Au(t) + f(t) & \text{for } t \in (0, T_0), \\ u(0) = x, u'(0) = y, \end{cases}$$

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where $x, y \in X$ are given, $A : \mathrm{D}(A) \subset X \to X$ is a closed linear operator, and f is an X-valued function defined on a subset of $\mathbb{R}$ containing $(0, T_0)$. A function u is called a strong solution of ACP(A, f, x, y), if $u \in \mathrm{C}^2((0, T_0),\ X) \cap \mathrm{C}^1([0, T_0), X) \cap \mathrm{C}((0, T_0), [\mathrm{D}(A)])$, and satisfies ACP$(A, f, x, y)$. Here $[\mathrm{D}(A)]$ denotes the Banach space $\mathrm{D}(A)$ equipped with the graph norm $|x|_A = \|x\| + \|Ax\|$ for $x \in \mathrm{D}(A)$. For each $C \in \mathrm{L}(X)$ and $\alpha > 0$, a family $C(\cdot)(= \{C(t) \,|\, 0 \leq t < T_0\})$ in $\mathrm{L}(X)$ is called a local α-times integrated C-cosine function on X if it is strongly continuous, $C(\cdot)C = CC(\cdot)$, and satisfies

$$\begin{aligned} 2C(t)C(s)x =& \frac{1}{\Gamma(\alpha)}[(\int_0^{t+s} - \int_0^t - \int_0^s)(t+s-r)^{\alpha-1}C(r)Cxdr \\ &+ \int_{|t-s|}^t (s-t+r)^{\alpha-1}C(r)Cxdr \\ &+ \int_{|t-s|}^s (t-s+r)^{\alpha-1}C(r)Cxdr \\ &+ \int_0^{|t-s|} (|t-s|+r)^{\alpha-1}C(r)Cxdr] \end{aligned} \tag{1.2}$$

for all $0 \leq t, s, t+s < T_0$ and $x \in X$ (see [12, 13]); or called a local (0-times integrated) C-cosine function on X if it is strongly continuous, $C(\cdot)C = CC(\cdot)$, and satisfies

$$\begin{aligned} &2C(t)C(s)x \\ =&C(t+s)Cx + C(|t-s|)Cx \quad \text{for all } 0 \leq t, s, t+s < T_0 \text{ and } x \in X, \end{aligned} \tag{1.3}$$

(see [4, 6, 18, 20]), where $\Gamma(\cdot)$ denotes the Gamma function. Moreover, we say that $C(\cdot)$ is nondegenerate, if $x = 0$ whenever $C(t)x = 0$ for all $0 \leq t < T_0$. In this case, its (integral) generator $A : \mathrm{D}(A) \subset X \to X$ is a closed linear operator in X defined by

$$\mathrm{D}(A) = \{x \in X |, \text{ there exists a } y_x \in X \text{ such that } C(\cdot)x - j_\alpha(\cdot)Cx = \widetilde{S}(\cdot)y_x \text{ on } [0, T_0)\}$$

and $Ax = y_x$ for all $x \in \mathrm{D}(A)$. Here $j_\alpha(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$, $S(s)z = \int_0^s C(r)zdr$, and $\widetilde{S}(t)z = \int_0^t S(s)zds$. In general, a local α-times integrated (resp.,0-times integrated) C-cosine function on X is called an α-times integrated C-cosine function (resp., (0-times integrated) C-cosine function) on X if $T_0 = \infty$ (see [7, 10, 11, 15, 17, 23-25] (resp., [9, 22])); or called a local α-times integrated cosine function on X if $C = I$, the identity operator on X (see [14, 20]), and a local α-times integrated cosine function on X is also called an α-times integrated cosine function on X if $T_0 = \infty$ (see [2, 26]); or called a cosine function on X if $\alpha = 0$ (see [1, 3, 5, 8, 19]). Moreover, a local α-times integrated cosine function on X is not necessarily extendable to an α-times integrated cosine function on X except for $\alpha = 0$ (see [5]), the nondegeneracy of a local α-times integrated C-cosine function on X does not imply the injectivity of C except for $T_0 = \infty$ (see [11]), and the injectivity of C does not imply the nondegeneracy of a local α-times integrated C-cosine function on X except for $\alpha = 0$ (see [18]). Some basic properites of a nondegenerate α-times integrated C-cosine function on X have been established by many authors when $\alpha = 0$ (see [9, 22]), $\alpha \in \mathbb{N}$ (see [7, 15, 17, 23-25), and $\alpha > 0$ is arbitrary (see [11]), which can be applied to deduce some equivalence relations between the generation of a nondegenerate α-times integrated C-cosine function on X with generator A and the unique existence of strong or weak solutions of the abstract Cauchy problem ACP(A, f, x, y) with $T_0 = \infty$ (see [7, 10, 11, 24]). The purpose of this paper is to investigate the following basic properties of a nondegenerate local α-times integrated C-cosine function on X when C is injective:

(1.4) $C(0) = C$ on X if $\alpha = 0$, and $C(0) = 0$ (, the zero operator) on X if $\alpha > 0$;

$$C^{-1}AC = A; \tag{1.5}$$

$$\begin{aligned} &\widetilde{S}(t)x \in D(A) \quad \text{and } A\widetilde{S}(t)x \\ &= C(t)x - j_\alpha(t)Cx \quad \text{for all } x \in X \quad \text{and } 0 \le t < T_0; \end{aligned} \tag{1.6}$$

$$\begin{aligned} &C(t)x \in D(A) \quad \text{and } AC(t)x \\ &= C(t)Ax \quad \text{for all } x \in D(A) \quad \text{and } 0 \le t < T_0; \end{aligned} \tag{1.7}$$

$$C(t)C(s) = C(s)C(t) \quad \text{for all } 0 \le t, s, t+s < T_0; \tag{1.8}$$

and then deduce some equivalence relations between the generation of a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on X with generator A and the unique existence of strong solutions of ACP(A, f, x, y), just as some results in [12,13] concerning the unique existence of strong and weak solutions of ACP(A, f, x, y). To do these, we shall first prove an important lemma which shows that a strongly continuous family $C(\cdot)(= \{C(t) \,|\, 0 \le t < T_0\})$ in $\mathrm{L}(X)$ is a local α-times integrated C-cosine function on X (with closed subgenerator A) is equivalent to $\widetilde{S}(\cdot)$ is a local $(\alpha + 2)$-times integrated C-cosine function on X (with closed subgenerator A), and then show that a strongly continuous family $C(\cdot)(= \{C(t) \,|\, 0 \le t < T_0\})$ in $\mathrm{L}(X)$ which commutes with C on X is a local α-times integrated C-cosine function on X is equivalent to $\widetilde{S}(t)[C(s) - j_\alpha(s)C]$=$[C(t) - j_\alpha(t)C]\widetilde{S}(s)$ for all $0 \le t, s, t+s < T_0$. We also show that $j_\beta * C(\cdot)$ is a local $(\alpha + \beta + 1)$-times integrated C-cosine function on X (with closed subgenerator A) if $C(\cdot)$ is a local α-times integrated C-cosine function on X (with closed subgenerator A) and $\beta > -1$, which can be applied to show that its " only if " part is also true when β is a nonnegative integer, where $f * C(t)x = \int_0^t f(t-s)C(s)xds$ for all $x \in X$ and $f \in \mathrm{L}^1_{loc}([0, T_0), \mathbb{F})$. In order, we show that the generator of a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on X is the unique subgenerator of $C(\cdot)$ which contains all subgenerators of $C(\cdot)$ and each subgenerator of $C(\cdot)$ is closable and its closure is also a subgenerator of $C(\cdot)$ when $C(\cdot)$ has a subgenerator. In particular, which is also so when C is injective. This can be applied to show that $CA \subset AC$ and $C(\cdot)$ is a nondegenerate local α-times integrated C-cosine function on X with generator $C^{-1}AC$ when C is injective and $C(\cdot)$ is a strongly continuous family in $\mathrm{L}(X)$ with closed subgenerator A. In this case, $C^{-1}\overline{A_0}C$ is the generator of $C(\cdot)$ for each subgenerator A_0 of $C(\cdot)$. Some illustrative examples concerning these theorems are also presented in the final part of this paper.

2. Basic Properties for Local α-Times Integrated C-Cosine Functions

We first deduce an important lemma which can be applied to obtain an equivalence relation between the generation of a local α-times integrated C-cosine function $C(\cdot)$ on X and the equality of

$$\widetilde{S}(t)[C(s) - j_\alpha(s)C] = [C(t) - j_\alpha(t)C]\widetilde{S}(s) \tag{2.1}$$

for all $0 \le t, s, t+s < T_0$, just as a result in [16] for the case of local α-times integrated C-semigroup when $C(\cdot)$ is a strongly continuous family in $\mathrm{L}(X)$ commuting with C on X.

Lemma 2.1. *Let $C(\cdot)$ be a strongly continuous family in $L(X)$. Then $C(\cdot)$ is a local α-times integrated C-cosine function on X if and only if $\widetilde{S}(\cdot)$ is a local $(\alpha+2)$-times integrated C-cosine function on X.*

Proof. We consider only the case $\alpha > 0$, for the case $\alpha = 0$ can be treated similarly. In this case, we shall first show that

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{\Gamma(\alpha+2)}[(\int_0^{t+s} - \int_0^t - \int_0^s)(t+s-r)^{\alpha+1}\widetilde{S}(r)Cxdr \\
&+ \int_{|t-s|}^t (s-t+r)^{\alpha+1}\widetilde{S}(r)Cxdr \\
&+ \int_{|t-s|}^s (t-s+r)^{\alpha+1}\widetilde{S}(r)Cxdr + \int_0^{|t-s|}(|t-s|+r)^{\alpha+1}\widetilde{S}(r)Cxdr] \\
=&\frac{1}{\Gamma(\alpha+1)}[(\int_0^{t+s} - \int_0^t - \int_0^s)(t+s-r)^{\alpha}\widetilde{S}(r)Cxdr \\
&+ sgn(s-t)\int_{|t-s|}^t (s-t+r)^{\alpha}\widetilde{S}(r)Cxdr \\
&+ sgn(t-s)\int_{|t-s|}^s (t-s+r)^{\alpha}\widetilde{S}(r)Cxdr + \int_0^{|t-s|}(|t-s|+r)^{\alpha}\widetilde{S}(r)Cxdr]
\end{aligned} \tag{2.2}$$

and

$$
\begin{aligned}
&\frac{d^2}{dt^2}\frac{1}{\Gamma(\alpha+2)}[(\int_0^{t+s}-\int_0^t-\int_0^s)(t+s-r)^{\alpha+1}\widetilde{S}(r)Cxdr\\
&+\int_{|t-s|}^t(s-t+r)^{\alpha+1}\widetilde{S}(r)Cxdr+\int_{|t-s|}^s(t-s+r)^{\alpha+1}\widetilde{S}(r)Cxdr\\
&+\int_0^{|t-s|}(|t-s|+r)^{\alpha+1}\widetilde{S}(r)Cxdr]+2j_\alpha(s)\widetilde{S}(t)Cx\\
=&\frac{1}{\Gamma(\alpha)}[(\int_0^{t+s}-\int_0^t-\int_0^s)(t+s-r)^{\alpha-1}\widetilde{S}(r)Cxdr\\
&+\int_{|t-s|}^t(s-t+r)^{\alpha-1}\widetilde{S}(r)Cxdr+\int_{|t-s|}^s(t-s+r)^{\alpha-1}\widetilde{S}(r)Cxdr\\
&+\int_0^{|t-s|}(|t-s|+r)^{\alpha-1}\widetilde{S}(r)Cxdr]
\end{aligned}
\tag{2.3}
$$

for all $x\in$ X and $0\le t,s,t+s<T_0$. Indeed, for $0\le s\le t<T_0$ with $t+s<T_0$, we have

$$
\begin{aligned}
&\frac{d}{dt}[\frac{1}{\Gamma(\alpha+2)}(\int_0^{t+s}-\int_0^t-\int_0^s)(t+s-r)^{\alpha+1}\widetilde{S}(r)Cxdr\\
&+\frac{1}{\Gamma(\alpha+2)}\int_{t-s}^t(s-t+r)^{\alpha+1}\widetilde{S}(r)Cxdr+\frac{1}{\Gamma(\alpha+2)}\int_0^s(t-s+r)^{\alpha+1}\widetilde{S}(r)Cxdr]\\
=&[\frac{1}{\Gamma(\alpha+1)}(\int_0^{t+s}-\int_0^t-\int_0^s)(t+s-r)^{\alpha}\widetilde{S}(r)Cxdr-j_{\alpha+1}(s)\tilde{S}(t)Cx]\\
&+[j_{\alpha+1}(s)\widetilde{S}(t)Cx-\frac{1}{\Gamma(\alpha+1)}\int_{t-s}^t(s-t+r)^{\alpha}\tilde{S}(r)Cxdr]\\
&+\frac{1}{\Gamma(\alpha+1)}\int_0^s(t-s+r)^{\alpha}\tilde{S}(r)Cxdr\\
=&\frac{1}{\Gamma(\alpha+1)}[(\int_0^{t+s}-\int_0^t-\int_0^s)(t+s-r)^{\alpha}\widetilde{S}(r)Cxdr\\
&+sgn(s-t)\int_{|t-s|}^t(s-t+r)^{\alpha}\widetilde{S}(r)Cxdr+sgn(t-s)\int_{|t-s|}^s(t-s+r)^{\alpha}\widetilde{S}(r)Cxdr\\
&+\int_0^{|t-s|}(|t-s|+r)^{\alpha}\widetilde{S}(r)Cxdr]
\end{aligned}
$$

and

$$\begin{aligned}
&\frac{d}{dt}[\frac{1}{\Gamma(\alpha+1)}(\int_0^{t+s}-\int_0^t-\int_0^s)(t+s-r)^{\alpha}\widetilde{S}(r)Cxdr\\
&-\frac{1}{\Gamma(\alpha+1)}\int_{t-s}^t(s-t+r)^{\alpha}\tilde{S}(r)Cxdr\\
&+\frac{1}{\Gamma(\alpha+1)}\int_0^s(t-s+r)^{\alpha}\tilde{S}(r)Cxdr]+2j_\alpha(s)\widetilde{S}(t)Cx\\
=&\frac{1}{\Gamma(\alpha)}(\int_0^{t+s}-\int_0^t-\int_0^s)(t+s-r)^{\alpha-1}\widetilde{S}(r)Cxdr-2j_\alpha(s)\widetilde{S}(t)Cx\\
&+\frac{1}{\Gamma(\alpha)}\int_{t-s}^t(s-t+r)^{\alpha-1}\widetilde{S}(r)Cxdr\\
&+\frac{1}{\Gamma(\alpha)}\int_0^s(t-s+r)^{\alpha-1}\widetilde{S}(r)Cxdr+2j_\alpha(s)\widetilde{S}(t)Cx\\
=&\frac{1}{\Gamma(\alpha)}(\int_0^{t+s}-\int_0^t-\int_0^s)(t+s-r)^{\alpha-1}\widetilde{S}(r)Cxdr\\
&+\frac{1}{\Gamma(\alpha)}\int_{t-s}^t(s-t+r)^{\alpha-1}\widetilde{S}(r)Cxdr+\frac{1}{\Gamma(\alpha)}\int_0^s(t-s+r)^{\alpha-1}\widetilde{S}(r)Cxdr\\
=&\frac{1}{\Gamma(\alpha)}[(\int_0^{t+s}-\int_0^t-\int_0^s)(t+s-r)^{\alpha-1}\widetilde{S}(r)Cxdr+\int_{|t-s|}^t(s-t+r)^{\alpha-1}\widetilde{S}(r)Cxdr\\
&+\int_{|t-s|}^s(t-s+r)^{\alpha-1}\widetilde{S}(r)Cxdr+\int_0^{|t-s|}(|t-s|+r)^{\alpha-1}\widetilde{S}(r)Cxdr].
\end{aligned}$$

That is, (2.2) and (2.3) both hold for all $0\le s\le t<T_0$ with $t+s<T_0$. Similarly, we can show that (2.2) and (2.3) both also hold when $0\le t\le s<T_0$ with $t+s<T_0$. Clearly, the right-hand side of (2.3) is symmetric in t,s with $0\le t,s,t+s<T_0$. It follows that

$$\begin{aligned}
&\frac{d^2}{ds^2}\frac{1}{\Gamma(\alpha+2)}[(\int_0^{t+s}-\int_0^t-\int_0^s)(t+s-r)^{\alpha+1}\widetilde{S}(r)Cxdr\\
&+\int_{|t-s|}^t(s-t+r)^{\alpha+1}\widetilde{S}(r)Cxdr+\int_{|t-s|}^s(t-s+r)^{\alpha+1}\widetilde{S}(r)Cxdr\\
&+\int_0^{|t-s|}(|t-s|+r)^{\alpha+1}\widetilde{S}(r)Cxdr]+2j_\alpha(t)\widetilde{S}(s)Cx\\
=&\frac{1}{\Gamma(\alpha)}[(\int_0^{t+s}-\int_0^t-\int_0^s)(t+s-r)^{\alpha-1}\widetilde{S}(r)Cxdr\\
&+\int_{|t-s|}^t(s-t+r)^{\alpha-1}\widetilde{S}(r)Cxdr+\int_{|t-s|}^s(t-s+r)^{\alpha-1}\widetilde{S}(r)Cxdr\\
&+\int_0^{|t-s|}(|t-s|+r)^{\alpha-1}\widetilde{S}(r)Cxdr]
\end{aligned}\tag{2.4}$$

for all $x \in$ X and $0 \leq t, s, t+s < T_0$. Using integration by parts twice, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}[(\int_0^{t+s} - \int_0^t - \int_0^s)(t+s-r)^{\alpha-1}\widetilde{S}(r)Cxdr \\
&+ \int_{|t-s|}^t (s-t+r)^{\alpha-1}\widetilde{S}(r)Cxdr + \int_{|t-s|}^s (t-s+r)^{\alpha-1}\widetilde{S}(r)Cxdr \\
&+ \int_0^{|t-s|} (|t-s|+r)^{\alpha-1}\widetilde{S}(r)Cxdr] \\
=&\frac{1}{\Gamma(\alpha+2)}[(\int_0^{t+s} - \int_0^t - \int_0^s)(t+s-r)^{\alpha+1}C(r)Cxdr \\
&+ \int_{|t-s|}^t (s-t+r)^{\alpha+1}C(r)Cxdr + \int_{|t-s|}^s (t-s+r)^{\alpha+1}C(r)Cxdr \\
&+ \int_0^{|t-s|} (|t-s|+r)^{\alpha+1}C(r)Cxdr]
\end{aligned} \tag{2.5}$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Now if $\widetilde{S}(\cdot)$ is a local $(\alpha+2)$-times integrated C-cosine function on X. By (2.4) and (2.5), we have

$$\begin{aligned}
&2\widetilde{S}(t)C(s)x = 2\frac{d^2}{ds^2}\widetilde{S}(t)\widetilde{S}(s)x \\
=&\frac{1}{\Gamma(\alpha+2)}[(\int_0^{t+s} - \int_0^t - \int_0^s)(t+s-r)^{\alpha+1}C(r)Cxdr \\
&+ \int_{|t-s|}^t (s-t+r)^{\alpha+1}C(r)Cxdr + \int_{|t-s|}^s (t-s+r)^{\alpha+1}C(r)Cxdr \\
&+ \int_0^{|t-s|} (|t-s|+r)^{\alpha+1}C(r)Cxdr]
\end{aligned}$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$, so that

$$\begin{aligned}
&2C(t)C(s)x = 2\frac{d^2}{dt^2}\widetilde{S}(t)C(s)x \\
=&\frac{1}{\Gamma(\alpha)}[(\int_0^{t+s} - \int_0^t - \int_0^s)(t+s-r)^{\alpha-1}C(r)Cxdr \\
&+ \int_{|t-s|}^t (s-t+r)^{\alpha-1}C(r)Cxdr \\
&+ \int_{|t-s|}^s (t-s+r)^{\alpha-1}C(r)Cxdr \\
&+ \int_0^{|t-s|} (|t-s|+r)^{\alpha-1}C(r)Cxdr]
\end{aligned} \tag{2.6}$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Hence $C(\cdot)$ is a local α-times integrated C-cosine function on X. Conversely, if $C(\cdot)$ is a local α-times integrated C-cosine function on X. We shall first apply Fubini's theorem for double integrals twice to obtain

$$\begin{aligned} & 2C(t)\widetilde{S}(s)x \\ =& \frac{1}{\Gamma(\alpha+2)}[(\int_0^{t+s} - \int_0^t - \int_0^s)(t+s-r)^{\alpha+1}C(r)Cxdr \\ & + \int_{|t-s|}^t (s-t+r)^{\alpha+1}C(r)Cxdr + \int_{|t-s|}^s (t-s+r)^{\alpha+1}C(r)Cxdr \\ & + \int_0^{|t-s|}(|t-s|+r)^{\alpha+1}C(r)Cxdr] + 2j_\alpha(t)\widetilde{S}(s)Cx \end{aligned} \tag{2.7}$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Indeed, if $x \in X$ is given, then for $0 \leq t, s, t+s < T_0$ with $t \geq s$, we have

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)}\int_0^\tau \int_t^{t+s} (t+s-r)^{\alpha-1}C(r)Cxdrds \\ =& \frac{1}{\Gamma(\alpha)}\int_t^{t+\tau} \int_{r-t}^\tau (t+s-r)^{\alpha-1}C(r)Cxdsdr \\ =& \frac{1}{\Gamma(\alpha+1)}\int_t^{t+\tau} (t+\tau-r)^\alpha C(r)Cxdsdr, \end{aligned} \tag{2.8}$$

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)}\int_0^\tau \int_0^s (t+s-r)^{\alpha-1}C(r)Cxdrds \\ =& \frac{1}{\Gamma(\alpha)}\int_0^\tau \int_r^\tau (t+s-r)^{\alpha-1}C(r)Cxdsdr \\ =& \frac{1}{\Gamma(\alpha+1)}\int_0^\tau (t+\tau-r)^\alpha C(r)Cxdr - j_\alpha(t)S(\tau)Cx, \end{aligned} \tag{2.9}$$

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)}\int_0^\tau \int_{t-s}^t (s-t+r)^{\alpha-1}C(r)Cxdrds \\ =& \frac{1}{\Gamma(\alpha)}\int_{t-\tau}^t \int_{t-r}^\tau (s-t+r)^{\alpha-1}C(r)Cxdsdr \\ =& \frac{1}{\Gamma(\alpha+1)}\int_{t-\tau}^t (\tau-t+r)^\alpha C(r)Cxdr, \end{aligned} \tag{2.10}$$

and

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^\tau\int_0^s (t-s+r)^{\alpha-1}C(r)Cxdrds\\
&=\frac{1}{\Gamma(\alpha)}\int_0^\tau\int_r^\tau (t-s+r)^{\alpha-1}C(r)Cxdsdrs\\
&=j_\alpha(t)S(\tau)Cx-\frac{1}{\Gamma(\alpha+1)}\int_0^\tau (t-\tau+r)^\alpha C(r)Cxdr.
\end{aligned}\tag{2.11}
$$

We observe from (2.8)-(2.11) that we also have

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_0^s\int_t^{t+\tau} (t+\tau-r)^\alpha C(r)Cxdrd\tau\\
&=\frac{1}{\Gamma(\alpha+2)}\int_t^{t+s} (t+s-r)^{\alpha+1}C(r)Cxdr,
\end{aligned}\tag{2.12}
$$

$$
\begin{aligned}
&\int_0^s[\frac{1}{\Gamma(\alpha+1)}\int_0^\tau (t+\tau-r)^\alpha C(r)Cxdr-j_\alpha(t)S(\tau)Cx]d\tau\\
&=\left[\frac{1}{\Gamma(\alpha+2)}\int_0^s (t+s-r)^{\alpha+1}C(r)Cxdr-j_{\alpha+1}(t)S(s)Cx\right]-j_\alpha(t)\widetilde{S}(s)Cx,
\end{aligned}\tag{2.13}
$$

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_0^s\int_{t-\tau}^{t} (\tau-t+r)^\alpha C(r)Cxdrd\tau\\
&=\frac{1}{\Gamma(\alpha+2)}\int_{t-s}^{t} (s-t+r)^{\alpha+1}C(r)Cxdr,
\end{aligned}\tag{2.14}
$$

and

$$
\begin{aligned}
&\int_0^s[j_\alpha(t)S(\tau)Cx-\frac{1}{\Gamma(\alpha+1)}\int_0^\tau (t-\tau+r)^\alpha C(r)Cxdr]d\tau\\
&=j_\alpha(t)\widetilde{S}(s)Cx+\left[\frac{1}{\Gamma(\alpha+2)}\int_0^s (t-s+r)^{\alpha+1}C(r)Cxdr-j_{\alpha+1}(t)S(s)Cx\right].
\end{aligned}\tag{2.15}
$$

Combining (2.12)-(2.15), we obtain (2.7) for all $0 \le t, s, t+s < T_0$ with $t \ge s$. Similarly, we can show that (2.7) also holds when $0 \le t, s, t+s < T_0$ with $s \ge t$. By (2.3), (2.5) and (2.7), we have

$$
\begin{aligned}
&2C(t)\widetilde{S}(s)x\\
&=\frac{d^2}{dt^2}\frac{1}{\Gamma(\alpha+2)}[(\int_0^{t+s}-\int_0^t-\int_0^s)(t+s-r)^{\alpha+1}\widetilde{S}(r)Cxdr\\
&\quad+\int_{|t-s|}^t (s-t+r)^{\alpha+1}\widetilde{S}(r)Cxdr+\int_{|t-s|}^s (t-s+r)^{\alpha+1}\widetilde{S}(r)Cxdr\\
&\quad+\int_0^{|t-s|}(|t-s|+r)^{\alpha+1}\widetilde{S}(r)Cxdr]
\end{aligned}
$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Combining this and (2.2) with $t = 0$, we conclude that $\widetilde{S}(\cdot)$ is a local $(\alpha+2)$-times integrated C-cosine function on X. ■

Theorem 2.2. *Let $C(\cdot)$ be a strongly continuous family in $L(X)$ which commutes with C on X. Then $C(\cdot)$ is a local α-times integrated C-cosine function on X if and only if $\widetilde{S}(t)[\,C(s) - j_\alpha(s)C\,]$=$[\,C(t) - j_\alpha(t)C\,]\widetilde{S}(s)$ for all $0 \leq t, s, t+s < T_0$.*

Proof. Indeed, if $C(\cdot)$ is a local α-times integrated C-cosine function on X. By (2.3) and (2.4), we have $2C(t)\widetilde{S}(s)x + 2j_\alpha(s)\widetilde{S}(t)Cx = 2\widetilde{S}(t)C(s)x + 2j_\alpha(t)\widetilde{S}(s)Cx$ for all $x \in X$ and $0 \leq t, s, t+s < T_0$ or equivalently, $\widetilde{S}(t)[C(s) - j_\alpha(s)C]$=$[C(t) - j_\alpha(t)C]\widetilde{S}(s)$ for all $0 \leq t, s, t+s < T_0$. Conversely, if (2.1) holds for all $0 \leq t, s, t+s < T_0$. We may assume that $\alpha > 0$, then $\widetilde{S}(t)C(s)x - C(t)\widetilde{S}(s)x = j_\alpha(s)\widetilde{S}(t)Cx - j_\alpha(t)\widetilde{S}(s)Cx$ for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Fix $x \in X$ and $0 \leq t, s, t+s < T_0$ with $t \geq s$, we have

$$\begin{aligned}(2.16)\qquad &\widetilde{S}(t+s-r)C(r)x - C(t+s-r)\widetilde{S}(r)x\\ &= j_\alpha(r)\widetilde{S}(t+s-r)Cx - j_\alpha(t+s-r)\widetilde{S}(r)Cx\end{aligned}$$

for all $0 \leq r \leq t$, and

$$\begin{aligned}(2.17)\qquad &\widetilde{S}(s-t+r)C(r)x - C(s-t+r)\widetilde{S}(r)x\\ &= j_\alpha(r)\widetilde{S}(s-t+r)Cx - j_\alpha(s-t+r)\widetilde{S}(r)Cx\end{aligned}$$

for all $t-s \leq r \leq t$. Using integration by parts to left-hand sides of the integrations of (2.16)-(2.17) and change of variables to right-hand sides of the integrations of (2.16)-(2.17), we obtain

$$\begin{aligned}(2.18)\qquad &S(t)\widetilde{S}(s)x + \widetilde{S}(t)S(s)x\\ &= \left(\int_0^{t+s} - \int_0^t - \int_0^s\right) j_\alpha(t+s-r)\widetilde{S}(r)Cxdr\end{aligned}$$

and

$$\begin{aligned}(2.19)\qquad &S(t)\widetilde{S}(s)x - \widetilde{S}(t)S(s)x\\ &= \int_0^s j_\alpha(t-s+r)\widetilde{S}(r)Cxdr - \int_{t-s}^t j_\alpha(s-t+r)\widetilde{S}(r)Cxdr,\end{aligned}$$

so that

$$\begin{aligned}&2\widetilde{S}(t)S(s)x\\ &= \left(\int_0^{t+s} \int_0^t - \int_0^s\right) j_\alpha(t+s-r)\widetilde{S}(r)Cxdr\\ &\quad + \int_{t-s}^t j_\alpha(s-t+r)\widetilde{S}(r)Cxdr - \int_0^s j_\alpha(t-s+r)\widetilde{S}(r)Cxdr.\end{aligned}$$

Hence

$$
\begin{aligned}
&2\widetilde{S}(t)C(s)x \\
=&\left(\int_0^{t+s}-\int_0^t-\int_0^s\right) j_{\alpha-1}(t+s-r)\widetilde{S}(r)Cxdr \\
&+\int_{t-s}^t j_{\alpha-1}(s-t+r)\widetilde{S}(r)Cxdr+\int_0^s j_{\alpha-1}(t-s+r)\widetilde{S}(r)Cxdr \\
&-2j_\alpha(t)\widetilde{S}(s)Cx,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&2\widetilde{S}(t)C(s)x+2j_\alpha(t)\widetilde{S}(s)Cx \\
=&\frac{1}{\Gamma(\alpha)}\Bigg[\left(\int_0^{t+s}-\int_0^t-\int_0^s\right)(t+s-r)^{\alpha-1}\widetilde{S}(r)Cxdr \\
&+\int_{|t-s|}^t (s-t+r)^{\alpha-1}\widetilde{S}(r)Cxdr+\int_{|t-s|}^s (t-s+r)^{\alpha-1}\widetilde{S}(r)Cxdr \\
&+\int_0^{|t-s|}(|t-s|+r)^{\alpha-1}\widetilde{S}(r)Cxdr\Bigg].
\end{aligned}
\tag{2.20}
$$

Similarly, we can show that (2.20) also holds when $x \in X$ and $0 \leq t, s, t+s < T_0$ with $s \geq t$. Combining this with (2.4), we have

$$
\begin{aligned}
&2\widetilde{S}(t)C(s)x \\
=&\frac{d^2}{ds^2}\Bigg[\frac{1}{\Gamma(\alpha+2)}\left(\int_0^{t+s}-\int_0^t-\int_0^s\right)(t+s-r)^{\alpha+1}\widetilde{S}(r)Cxdr \\
&+\frac{1}{\Gamma(\alpha+2)}\int_{|t-s|}^t (s-t+r)^{\alpha+1}\widetilde{S}(r)Cxdr \\
&+\frac{1}{\Gamma(\alpha+2)}\int_{|t-s|}^s (t-s+r)^{\alpha+1}\widetilde{S}(r)Cxdr \\
&+\frac{1}{\Gamma(\alpha+2)}\int_0^{|t-s|}(|t-s|+r)^{\alpha+1}\widetilde{S}(r)Cxdr\Bigg].
\end{aligned}
$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Consequently, $\widetilde{S}(\cdot)$ is a local $(\alpha+2)$-times integrated C-cosine function on X. Similarly, we can show that the conclusion of this theorem is also true when $\alpha = 0$. ■

Proposition 2.3. *Let $C(\cdot)$ be a local α-times integrated C-cosine function on X and $\beta > -1$. Then $j_\beta * C(\cdot)$ is a local $(\alpha+\beta+1)$-times integrated C-cosine function on X. Moreover, $C(\cdot)$ is a local α-times integrated C-cosine function on X if it is a* *strongly continuous family in $L(X)$ such that $S(\cdot)$ is a local $(\alpha+1)$-times integrated C-cosine function on X.*

Proof. We set $C_\beta(\cdot) = j_\beta * C(\cdot)$ and $\widetilde{S}_\beta(\cdot) = j_1 * C_\beta(\cdot)$. Then $C_\beta(\cdot)C = CC_\beta(\cdot)$ and $\widetilde{S}_\beta(\cdot)C = C\widetilde{S}_\beta(\cdot)$, so that for $x \in X$ and $0 \le t < T_0$, we have

$$\begin{aligned}
&[C_\beta(t) - j_{\alpha+\beta+1}(t)C]\widetilde{S}_\beta(\cdot)x \\
&= [j_\beta * C(t) - j_\beta * j_\alpha(t)C]j_\beta * \widetilde{S}(\cdot)x \\
&= j_\beta * ([j_\beta * C(t) - j_\beta * j_\alpha(t)C]\widetilde{S}(\cdot)x) \\
&= j_\beta * (\int_0^t j_\beta(t-s)[C(s) - j_\alpha(s)C]\widetilde{S}(\cdot)xds) \\
&= j_\beta * (\int_0^t j_\beta(t-s)\widetilde{S}(s)[C(\cdot) - j_\alpha(\cdot)C]xds) \\
&= \int_0^t j_\beta(t-s)\widetilde{S}(s)j_\beta * [C(\cdot) - j_\alpha(\cdot)C]xds \\
&= j_\beta * \widetilde{S}(t)j_\beta * [C(\cdot) - j_\alpha(\cdot)C]x \\
&= \widetilde{S}_\beta(t)[C_\beta(\cdot) - j_{\alpha+\beta+1}(\cdot)C]x.
\end{aligned}$$

on $[0, s]$ for all $0 < s < T_0$ with $t + s < T_0$. Hence $C_\beta(\cdot)$ is a local $(\alpha+\beta+1)$-times integrated C-cosine function on X, which together with Lemma 2.1 implies that $C(\cdot)$ is a local α-times integrated C-cosine function on X if it is a strongly continuous family in L(X) such that $S(\cdot)$ is a local $(\alpha+1)$-times integrated C-cosine function on X. ■

Lemma 2.4. *Let $C(\cdot)$ be a local α-times integrated C-cosine function on X. Assume that $CC(\cdot)x = 0$ on $[0, t_0)$ for some $x \in X$ and $0 < t_0 < T_0$. Then $CC(\cdot)x = 0$ on $[0, T_0)$. In particular, $C(t)x = 0$ for all $0 \le t < T_0$ if the injectivity of C is added.*

Proof. Indeed, if $0 \le t < T_0$ is given, then $t + s < T_0$ for some $0 < s < t_0$. By hypothesis, we have $\widetilde{S}(s)C(t)x$=$C(t)\widetilde{S}(s)x = 0$ and $\widetilde{S}(s)j_\alpha(t)Cx$=$j_\alpha(t)C\widetilde{S}(s)x = 0$. By (1.2) and (1.3), we also have $C(s)\widetilde{S}(t)x$=$\widetilde{S}(t)C(s)x = 0$. By Theorem 2.2, we have $\widetilde{S}(s)[C(t) - j_\alpha(t)C]x$ =$[C(s) - j_\alpha(s)C]\widetilde{S}(t)x$, so that $j_\alpha(s)\widetilde{S}(t)Cx = j_\alpha(s)C\widetilde{S}(t)x = 0$. Hence $\widetilde{S}(t)Cx = 0$. Since $0 \le t < T_0$ is arbitrary, we conclude that $CC(t)x = C(t)Cx = 0$ for all $0 \le t < T_0$. In particular, $C(t)x = 0$ for all $0 \le t < T_0$ if the injectivity of C is added. ■

Proposition 2.5. *Let $C(\cdot)$ be a nondegenerate local α-times integrated C-cosine function on X. Assume that C is injective. Then (1.4)-(1.7) hold .*

Proof. It is easy to see from (1.2)(resp.,(1.3)), the nondegeneracy of $C(\cdot)$ and the injectivity of C that (1.4) holds. Just as in the proof of [11, Prop. 1.5], we can show that (1.5) also holds. Next, to show that (1.6) holds. Indeed, if $0 \leq t_0 < T_0$ is fixed. Then for each $x \in X$ and $0 \leq s < T_0$, we set $y = \widetilde{S}(t_0)x$. By Theorem 2.2, we have

$$\begin{aligned}
&\widetilde{S}(r)[C(s) - j_\alpha(s)C]y \\
&= [C(r) - j_\alpha(r)C]\widetilde{S}(s)y \\
&= \widetilde{S}(s)[C(r) - j_\alpha(r)C]y \\
&= \widetilde{S}(s)([C(r) - j_\alpha(r)C]\widetilde{S}(t_0)x) \\
&= \widetilde{S}(s)(\widetilde{S}(r)[C(t_0) - j_\alpha(t_0)C]x) \\
&= [\widetilde{S}(s)\widetilde{S}(r)][C(t_0) - j_\alpha(t_0)C]x \\
&= \widetilde{S}(r)\widetilde{S}(s)[C(t_0) - j_\alpha(t_0)C]x
\end{aligned}$$

for all $0 \leq r < T_0$ with $r + s, r + t < T_0$. Clearly, $\widetilde{S}(\cdot)$ is also nondegenerate. It follows from Lemma 2.4 that we have $[C(s) - j_\alpha(s)C]y = \widetilde{S}(s)[C(t_0) - j_\alpha(t_0)C]x$. Since $0 \leq s < T_0$ is arbitrary, we conclude that (1.6) holds. Now if $x \in D(A)$ is given. By (1.6) and the definition of $D(A)$, we have $A\widetilde{S}(t)x = C(t)x - j_\alpha(t)Cx = \widetilde{S}(t)Ax$ for all $0 \leq t < T_0$. By the closedness of A, we also have $\frac{d^2}{dt^2}\widetilde{S}(t)x \in D(A)$ and $AC(t)x = A\frac{d^2}{dt^2}\widetilde{S}(t)x = \frac{d^2}{dt^2}A\widetilde{S}(t)x = \frac{d^2}{dt^2}\widetilde{S}(t)Ax = C(t)Ax$ for all $0 \leq t < T_0$. ■

Just as in the proof of [11, Lemma 1.6], the next lemma is also attained.

Lemma 2.6. *Let $C(\cdot)$ be a nondegenerate local α-times integrated C-cosine function on X with generator A. Assume that C is injective, and $u \in C([0, t_0), X)$ satisfies $u(\cdot) = Aj_1 * u(\cdot)$ on $[0, t_0)$ for some $0 < t_0 < T_0$. Then $u \equiv 0$ on $[0, t_0)$.*

Proposition 2.7. *Let $C(\cdot)$ be a nondegenerate local α-times integrated C-cosine function on X with generator A. Assume that C is injective. Then (1.8) holds.*

Proof. To show that $C(t)C(s)x = C(s)C(t)x$ for all $x \in X$ and $0 \leq t, s < T_0$, we need only to show that $\widetilde{S}(t)\widetilde{S}(s)x = \widetilde{S}(s)\widetilde{S}(t)x$ for all $x \in X$ and $0 \leq t, s < T_0$. Indeed, if $x \in X$ and $0 \leq s < T_0$ are given. By (1.7) and the closedness of A, we have

$$\begin{aligned}
&\widetilde{S}(\cdot)\widetilde{S}(s)x - Aj_1 * \widetilde{S}(\cdot)\widetilde{S}(s)x \\
&= j_{\alpha+2}(\cdot)C\widetilde{S}(s)x \\
&= \widetilde{S}(s)j_{\alpha+2}(\cdot)Cx \\
&= \widetilde{S}(s)[\widetilde{S}(\cdot)x - Aj_1 * \widetilde{S}(\cdot)x] \\
&= \widetilde{S}(s)\widetilde{S}(\cdot)x - \widetilde{S}(s)Aj_1 * \widetilde{S}(\cdot)x \\
&= \widetilde{S}(s)\widetilde{S}(\cdot)x - Aj_1 * \widetilde{S}(s)\widetilde{S}(\cdot)x
\end{aligned}$$

on $[0, T_0)$, and so $[\widetilde{S}(\cdot)\widetilde{S}(s)x - \widetilde{S}(s)\widetilde{S}(\cdot)x] = Aj_1 * [\widetilde{S}(\cdot)\widetilde{S}(s)x - \widetilde{S}(s)\widetilde{S}(\cdot)x]$ on $[0, T_0)$. Hence $\widetilde{S}(\cdot)\widetilde{S}(s)x = \widetilde{S}(s)\widetilde{S}(\cdot)x$ on $[0, T_0)$, which implies that $\widetilde{S}(t)\widetilde{S}(s)x = \widetilde{S}(s)\widetilde{S}(t)x$ for all $0 \le t, s < T_0$. Consequently, (1.8) holds. ■

Definition 2.8. Let $C(\cdot)$ be a strongly continuous family in $L(X)$. A linear operator A in X is called a subgenerator of $C(\cdot)$ if

$$C(t)x - j_\alpha(t)Cx = \int_0^t \int_0^s C(r)Axdrds \tag{2.21}$$

for all $x \in D(A)$ and $0 \le t < T_0$, and

$$\int_0^t \int_0^s C(r)xdrds \in D(A) \quad \text{and } A\int_0^t \int_0^s C(r)xdrds = C(t)x - j_\alpha(t)Cx \tag{2.22}$$

for all $x \in X$ and $0 \le t < T_0$. A subgenerator A of $C(\cdot)$ is called the maximal subgenerator of $C(\cdot)$ if it is an extension of each subgenerator of $C(\cdot)$ to $D(A)$.

Theorem 2.9. *Let $C(\cdot)$ be a strongly continuous family in $L(X)$ which commutes with C on X. Assume that $C(\cdot)$ has a subgenerator. Then $C(\cdot)$ is a local α-times integrated C-cosine function on X. Moreover, $C(\cdot)$ is nondegenerate if the injectivity of C is added.*

Proof. Indeed, if A is a subgenerator of $C(\cdot)$. By (2.22), we have

$$[C(t)x - j_\alpha(t)C]\widetilde{S}(\cdot)x = \widetilde{S}(t)A\widetilde{S}(\cdot)x = \widetilde{S}(t)[C(\cdot)x - j_\alpha(\cdot)C]x$$

on $[0, T_0)$ for all $x \in X$ and $0 \le t < T_0$. Applying Theorem 2.2, we get that $C(\cdot)$ is a local α-times integrated C-cosine function on X. Now if the injectivity of C is added, and $C(\cdot)x = 0$ on $[0, T_0)$ for some $x \in X$. By (2.22), we have $j_\alpha(\cdot)Cx = 0$ on $[0, T_0)$, and so $Cx = 0$. Hence $x = 0$, which implies that $C(\cdot)$ is nondegenerate. ■

Corollary 2.10. *Let $C(\cdot)$ be a local α-times integrated C-cosine function on X. Assume that C is injective. Then $C(\cdot)$ is nondegenerate if and only if it has a subgenerator.*

Theorem 2.11. *Let $C(\cdot)$ be a local α-times integrated C-cosine function on X which has a subgenerator. Assume that $A : D(A) \subset X \to X$ defined by*

$$D(A) = \{x \in X |\ \text{there exists a unique } y_x \in X \text{ such that } C(\cdot)x - j_\alpha(\cdot)Cx = \widetilde{S}(\cdot)y_x \text{ on } [0, T_0)\}$$

and $Ax = y_x$ for all $x \in D(A)$, is a closed linear operator in X. Then A is the maximal subgenerator of $C(\cdot)$. Moreover, each subgenerator of $C(\cdot)$ is closable and its closure is also a subgenerator of $C(\cdot)$.

Proof. Indeed, if A_0 is a subgenerator of $C(\cdot)$. Clearly, $A_0 \subset A$. It is easy to see from Zorn's lemma that $C(\cdot)$ has a subgenerator B which is an extension of A_0, but does not have a proper extension that is still a subgenerator of $C(\cdot)$, which together with the definition of A implies that B is the maximal subgenerator of $C(\cdot)$. To show that $A = B$ or equivalently, $A \subset B$, we shall first show that B is closable. Indeed, if $x_k \in D(B)$, $x_k \to 0$, and $Bx_k \to y$ in X. Then $x_k \in D(A)$ and $Ax_k = Bx_k \to y$. By the closedness of A, we have $y = 0$. In order to show that $B = \overline{B}$ (the closure of B) or equivalently, $\overline{B}$ is a subgenerator of $C(\cdot)$. Indeed, if $x \in D(\overline{B})$ is given, then $x_k \to x$ and $Bx_k \to \overline{B}x$ in X for sequence $\{x_k\}_{k=1}^{\infty}$ in $D(B)$. By (2.21), we have $C(t)x_k - j_\alpha(t)Cx_k = \int_0^t \int_0^s C(r)Bx_k drds$ for all $k \in \mathbb{N}$ and $0 \le t < T_0$. Letting $k \to \infty$, we get $C(t)x - j_\alpha(t)Cx = \int_0^t \int_0^s C(r)\overline{B}xdrds$ for all $0 \le t < T_0$. Since $B \subset \overline{B} \subset A$, we also have $C(t)z - j_\alpha(t)Cz = B\int_0^t \int_0^s C(r)zdrds = \overline{B}\int_0^t \int_0^s C(r)zdrds$ for all $z \in X$ and $0 \le t < T_0$. Consequently, the closure of B is a subgenerator of $C(\cdot)$. Similarly, we can show that A is also a subgenerator of $C(\cdot)$ and each subgenerator of $C(\cdot)$ is closable and its closure is also a subgenerator of $C(\cdot)$. In particular, $A = B$. ■

Corollary 2.12. *Let $C(\cdot)$ be a nondegenerate local α-times integrated C-cosine function on X with generator A. Assume that $C(\cdot)$ has a subgenerator. Then A is the maximal subgenerator of $C(\cdot)$. Moreover, each subgenerator of $C(\cdot)$ is closable and its closure is also a subgenerator of $C(\cdot)$.*

Corollary 2.13. *Let $C(\cdot)$ be a nondegenerate local α-times integrated C-cosine function on X with generator A. Assume that C is injective. Then A is the maximal subgenerator of $C(\cdot)$. Moreover, each subgenerator of $C(\cdot)$ is closable and its closure is also a subgenerator of $C(\cdot)$.*

Proof. This follows from (2.21), (2.22) and the definition of A. ■

Theorem 2.14. *Let A be a closed subgenerator of a strongly continuous family $C(\cdot)$ in $L(X)$. Assume that C is injective. Then $CA \subset AC$, and $C(\cdot)$ is a nondegenerate local α-times integrated C-cosine function on X with generator $C^{-1}AC$. In particular, $C^{-1}\overline{A_0}C$ is the generator of $C(\cdot)$ for each subgenerator A_0 of $C(\cdot)$.*

Proof. We first show that $CA \subset AC$. Indeed, if $x \in D(A)$ is given, then $j_{\alpha+2}(t)Cx = \widetilde{S}(t)x - j_1 * \widetilde{S}(t)Ax \in D(A)$ and

$$\begin{aligned} Aj_{\alpha+2}(t)Cx &= A\widetilde{S}(t)x - Aj_1 * \widetilde{S}(t)Ax \\ &= A\widetilde{S}(t)x - [\widetilde{S}(t)Ax - j_{\alpha+2}(t)CAx] \\ &= j_{\alpha+2}(t)CAx \end{aligned}$$

for all $0 \le t < T_0$, so that $CAx = ACx$. Hence $CA \subset AC$. To show that $C(\cdot)$ is a nondegenerate local α-times integrated C-cosine function on X. By Theorem 2.9, we remain only to show that $CC(\cdot) = C(\cdot)C$ or equivalently, $C\widetilde{S}(\cdot) = \widetilde{S}(\cdot)C$. Just as in the proof of Proposition 2.7, we have $[\widetilde{S}(\cdot)Cx - C\widetilde{S}(\cdot)x] = Aj_1 * [\widetilde{S}(\cdot)Cx - C\widetilde{S}(\cdot)x]$ on $[0, T_0)$. By a parallel argument of [11, Lemma 1.6], we also have $\widetilde{S}(\cdot)Cx = C\widetilde{S}(\cdot)x$ on $[0, T_0)$. Now if B denotes the generator of $C(\cdot)$. By Corollary 2.13, we have $A \subset B$. By (1.5), we also have $C^{-1}AC \subset C^{-1}BC = B$. Conversely, if $x \in D(B)$ is given, then $j_{\alpha+2}(t)Cx = \widetilde{S}(t)x - j_1 * \widetilde{S}(t)Bx \in D(A)$ for all $0 \le t < T_0$, so that $Cx \in D(A)$ and

$$\begin{aligned} Aj_{\alpha+2}(\cdot)Cx &= A\widetilde{S}(\cdot)x - Aj_1 * \widetilde{S}(\cdot)Bx \\ &= A\widetilde{S}(\cdot)x - [\widetilde{S}(\cdot)Bx - j_{\alpha+2}(\cdot)CBx] \\ &= A\widetilde{S}(\cdot)x - [B\widetilde{S}(\cdot)x - j_{\alpha+2}(\cdot)CBx] \\ &= j_{\alpha+2}(\cdot)CBx \end{aligned}$$

on $[0, T_0)$. Hence $ACx = CBx \in R(C)$, which implies that $x \in D(C^{-1}AC)$ and $C^{-1}ACx = Bx$. Consequently, $B \subset C^{-1}AC$. ■

Remark 2.15. Let $C(\cdot)$ be a strongly continuous family in $\mathrm{L}(X)$. Then $C(\cdot)$ is a local α-times integrated C-cosine function on X with closed subgenerator A if and only if $S(\cdot)$ is a local $(\alpha + 1)$-times integrated C-cosine function on X with closed subgenerator A.

Remark 2.16. A strongly continuous family in $\mathrm{L}(X)$ may not have a subgenerator; a local α-times integrated C-cosine function on X is degenerate when it has a subgenerator but does not have a maximal subgenerator; and a closed linear operator in X generates at most one nondegenerate local α-times integrated C-cosine function on X when C is injective.

3. Abstract Cauchy Problems

In the following, we always assume that $\alpha > 0$, $C \in \mathrm{L}(X)$ is injective, and A a closed linear operator in X such that $CA \subset AC$. We first note some basic properties concerning the strong solutions of ACP(A, f, x, y), just as results in [11] when A is the generator of a nondegenerate α-times integrated C-cosine function on X.

Proposition 3.1. *Let A be a closed subgenerator of a nondegenerate local $(\alpha+1)$-times integrated C-cosine function $C(\cdot)$ on X. Then for each $x \in D(A)$ $C(\cdot)x$ is the unique solution of ACP($j_{\alpha-1}(\cdot)Cx, 0, 0)$ in* $C([0, T_0), [D(A)])$.

Proposition 3.2. *Let A be a closed subgenerator of a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on X and $C^1 = \{x \in X \,|\, C(\cdot)x$ is continuously differentiable on $(0, T_0)\}$. Then*

(i) *$S(t)C^1 \subset D(A)$ for all $0 < t < T_0$;*

(ii) *for each $x \in C^1$ $S(\cdot)x$ is the unique solution of ACP($j_{\alpha-1}(\cdot)Cx, 0, 0$);*

(iii) *for each $x \in D(A)$ $S(\cdot)x$ is the unique solution of ACP($j_{\alpha-1}(\cdot)Cx, 0, 0$) in $C^1([0, T_0), [D(A)])$.*

Proposition 3.3. *Let A be the generator of a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on X and $x \in X$. Assume that $C(t)x \in R(C)$ for all $0 \leq t < T_0$, and $C^{-1}C(\cdot)x$ is continuously differentiable on $(0, T_0)$. Then $C^{-1}S(t)x \in D(A)$ for all $0 < t < T_0$, and $C^{-1}S(\cdot)x$ is the unique solution of ACP($j_{\alpha-1}(\cdot)x, 0, 0$).*

Applying Theorem 2.14, we can investigate an important result concerning the relation between the generation of a nondegenerate local α-times integrated C-cosine function on X with generator A and the unique existence of strong solutions of ACP(A, f, x, y), which has been established by another method in [11] when $T_0 = \infty$ or in [9] when $\alpha = 0$ and $T_0 = \infty$.

Theorem 3.4. *The following statements are equivalent :*

(i) *A is a subgenerator of a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on X;*

(ii) *for each $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$ the problem ACP($j_\alpha(\cdot)Cx + j_\alpha * Cg(\cdot), 0, 0$) has a unique solution in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$;*

(iii) *for each $x \in X$ the problem ACP($j_\alpha(\cdot)Cx, 0, 0$) has a unique solution in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$;*

(iv) *for each $x \in X$ the integral equation $v(\cdot) = Aj_1 * v(\cdot) + j_\alpha(\cdot)Cx$ has a unique solution $v(\cdot; x)$ in $C([0, T_0), X)$.*

In this case, $\widetilde{S}(\cdot)x + \widetilde{S} * g(\cdot)$ is the unique solution of ACP($j_\alpha(\cdot)Cx + j_\alpha * Cg(\cdot), 0, 0$) and $v(\cdot; x) = C(\cdot)x$.

Proof. We first show that "(i)⇒(ii)" holds. Indeed, if $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$ are given. We set $u(\cdot) = \widetilde{S}(\cdot)x + \widetilde{S} * g(\cdot)$, then $u \in C^2([0.T_0), X) \cap C([0, T_0), [D(A)])$, $u(0) = u'(0) = 0$, and

$$
\begin{aligned}
Au(t) &= A\widetilde{S}(t)x + A\int_0^t \widetilde{S}(t-s)g(s)ds \\
&= C(t)x - j_\alpha(t)Cx + \int_0^t [C(t-s) - j_\alpha(t-s)C]g(s)ds \\
&= C(t)x + \int_0^t C(t-s)g(s)ds - [j_\alpha(t)Cx + j_\alpha * Cg(t)] \\
&= u''(t) - [j_\alpha(t)Cx + j_\alpha * Cg(t)]
\end{aligned}
$$

for all $0 \leq t < T_0$. Hence u is a solution of ACP$(j_\alpha(\cdot)Cx + j_\alpha * Cg(\cdot), 0, 0)$ in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$. The uniqueness of solutions for ACP$(j_\alpha(\cdot)Cx + j_\alpha * Cg(\cdot), 0, 0)$ follows directly from the uniqueness of solutions for ACP$(0, 0, 0)$. Clearly, "(ii)⇒(iii)" holds, and (iii) and (iv) both are equivalent. We remain only to show that "(iv)⇒(i)" holds. Indeed, if $C(t) : X \to X$ is defined by $C(t)x = v(\cdot; x)$ for all $x \in X$ and $0 \leq t < T_0$. Clearly, $C(\cdot)$ is strongly continuous, and satisfies (2.22). Combining the uniqueness of solutions for the integral equation $v(\cdot)$=$Aj_1 * v(\cdot) + j_\alpha(\cdot)Cx$ with the assumption $CA \subset AC$, we have $v(\cdot; Cx) = Cv(\cdot; x)$ for each $x \in X$, which implies that $C(t)$ for $0 \leq t < T_0$ are linear, and commute with C. Now let $\{t_k\}_{k=1}^{\infty}$ be an increasing sequence in $(0, T_0)$ such that $t_k \to T_0$, and $C([0, T_0), X)$ a Frechet space with the quasi-norm $|\cdot|$ defined by $|v| = \sum_{k=1}^{\infty} \frac{\|v\|_k}{2^k(1+\|v\|_k)}$ for $v \in C([0, T_0), X)$. Here $\|v\|_k = \max_{t\in[0,t_k]} \|v(t)\|$ for all $k \in \mathbb{N}$. To show that $C(\cdot)$ is a family in $L(X)$, we need only to the linear map $\eta : X \to C([0, T_0), X)$ defined by $\eta(x) = v(\cdot; x)$ for $x \in X$, is continuous or equivalently, $\eta : X \to C([0, T_0), X)$ is a closed linear operator. Indeed, if $\{x_k\}_{k=1}^{\infty}$ is a sequence in X such that $x_k \to x$ in X and $\eta(x_k) \to v$ in $C([0, T_0), X)$, then $v(\cdot; x_k)$=$Aj_1 * v(\cdot; x_k) + j_\alpha(\cdot)Cx_k$ on $[0, T_0)$. Combining the closedness of A with the uniform convergence of $\{\eta(x_k)\}_{k=1}^{\infty}$ on $[0, t_k]$, we have $v(\cdot)$=$Aj_1 * v(\cdot) + j_\alpha(\cdot)Cx$ on $[0, T_0)$. By the uniqueness of solutions for integral equations, we have $v(\cdot)$=$v(\cdot; x)$=$\eta(x)$. Consequently, $\eta : X \to C([0, T_0), X)$ is a closed linear operator. To show that A is a subgenerator of $C(\cdot)$, we remain only to show that $\widetilde{S}(t)A \subset A\widetilde{S}(t)$ for all $0 \leq t < T_0$. Indeed, if $x \in D(A)$ is given, then $\widetilde{S}(t)x - j_{\alpha+2}(t)Cx$=$Aj_1 * \widetilde{S}(t)x$=$j_1 * A\widetilde{S}(t)x$ for all $0 \leq t < T_0$, and so

$$\begin{aligned}
&\widetilde{S}(t)Ax - Aj_1 * \widetilde{S}(t)Ax \\
&= j_{\alpha+2}(t)CAx \\
&= Aj_{\alpha+2}(t)Cx \\
&= A\widetilde{S}(t)x - Aj_1 * \widetilde{S}(t)Ax
\end{aligned}$$

for all $0 \leq t < T_0$. Hence $Aj_1 * [\widetilde{S}(\cdot)Ax - A\widetilde{S}(\cdot)x]$=$\widetilde{S}(\cdot)Ax - A\widetilde{S}(\cdot)x$ on $[0, T_0)$. By the uniqueness of solutions of ACP$(0, 0, 0)$, we have $\widetilde{S}(\cdot)Ax$=$A\widetilde{S}(\cdot)x$ on $[0, T_0)$. Applying Theorem 2.11, we get that $C(\cdot)$ is a nondegenerate local α-times integrated C-cosine function on X with subgenerator A. ■

By slightly modifying the proof of [11, Theorem 2.4], we can apply Theorem 3.4 to obtain the next result.

Theorem 3.5. *Assume that $R(C) \subset R(\lambda - A)$ for some $\lambda \in \mathbb{F}$, and ACP$(j_{\alpha-1}(\cdot)x, 0, 0)$ has a unique solution in $C([0, T_0), [D(A)])$ for each $x \in D(A)$ with $(\lambda - A)x \in R(C)$. Then A is a subgenerator of a nondegenerate local $(\alpha + 1)$-times integrated C-cosine function on X.*

Proof. Clearly, it suffices to show that the integral equation

$$v(\cdot) = A\int_0^{\cdot}\int_0^s v(r)drds + j_{\alpha+1}(\cdot)Cx \tag{3.1}$$

has a (unique) solution $v(\cdot;x)$ in $\mathrm{C}([0,T_0),X)$ for each $x \in X$. Indeed, if $x \in X$ is given, then there exists a $y_x \in \mathrm{D}(A)$ such that $(\lambda - A)y_x = Cx$. By hypothesis, $\mathrm{ACP}(j_{\alpha-1}(\cdot)y_x, 0, 0)$ has a unique solution $u(\cdot;y_x)$ in $\mathrm{C}([0,T_0),[\mathrm{D}(A)])$. By the closedness of A and the continuity of $Au(\cdot)$, we have $\int_0^t\int_0^s u(r;y_x)drds \in \mathrm{D}(A)$ and

$$A\int_0^t\int_0^s u(r;y_x)drds = \int_0^t\int_0^s Au(r;y_x)drds = u(t;y_x) - j_{\alpha+1}(t)y_x \in \mathrm{D}(A)$$

for all $0 \le t < T_0$, so that

$$\begin{aligned}(\lambda - A)u(t;y_x) &= (\lambda - A)[A\int_0^t\int_0^s u(r;y_x)drds + j_{\alpha+1}(t)y_x] \\ &= A\int_0^t\int_0^s (\lambda - A)u(r;y_x)drds + j_{\alpha+1}(t)Cx\end{aligned} \tag{3.2}$$

for all $0 \le t < T_0$. Hence $v(\cdot;x) = (\lambda - A)u(\cdot;y_x)$ is a solution of (3.1) in $\mathrm{C}([0,T_0),X)$. ■

Combining Theorem 3.4 with Theorem 3.5, the next theorem is also attained.

Theorem 3.6. *Assume that $R(C) \subset R(\lambda - A)$ for some $\lambda \in \mathbb{F}$, and ACP($j_{\alpha-1}(\cdot)$ $x, 0, 0$) has a unique solution in $C^1([0,T_0),[D(A)])$ for each $x \in D(A)$ with $(\lambda - A)x \in R(C)$. Then A is a subgenerator of a nondegenerate local α-times integrated C-cosine function on X.*

Proof. Indeed, if $x \in X$ is given, and $u(\cdot;y_x)$ and $v(\cdot;x)$ both are given as in the proof of Theorem 3.5. By hypothesis, $v(\cdot;x)$ is continuously differentiable on $[0,T_0)$ and $v'(t;x) = (\lambda - A)u'(t;y_x)$ for all $0 \le t < T_0$. By (3.2), we also have $v'(t;x) = A\int_0^t v(r;x)dr + j_\alpha(t)Cx$ for all $0 \le t < T_0$. In particular, $v'(0;x) = 0$, and so $v'(\cdot;x){=}Aj_1 * v'(\cdot;x) + j_\alpha(\cdot)Cx$ on $[0,T_0)$. Hence $v'(\cdot;x)$ is a (unique) solution of the integral equation $v(\cdot){=}Aj_1 * v(\cdot) + j_\alpha(\cdot)Cx$ in $\mathrm{C}([0,T_0),X)$. ■

Since $C^{-1}AC = A$ and $R((\lambda - A)^{-1}C) = C(\mathrm{D}(A))$ if $\rho(A) \neq \emptyset$ (see [21]), we can apply Proposition 3.1, Theorem 3.5 and Theorem 3.6 to obtain the next two corollaries.

Corollary 3.7. *Let $A : D(A) \to X$ be a closed linear operator with nonempty resolvent set. Then A is the generator of a nondegenerate local $(\alpha{+}1)$-times integrated*

C-cosine function on X if and only if for each $x \in D(A)$ ACP($j_{\alpha-1}(\cdot)Cx, 0, 0)$ has a unique solution in $C([0, T_0), [D(A)])$.

Corollary 3.8. *Let $A : D(A) \to X$ be a closed linear operator with nonempty resolvent set. Then A is the generator of a nondegenerate local α-times integrated C-cosine function on X if and only if for each $x \in D(A)$ ACP($j_{\alpha-1}(\cdot)Cx, 0, 0)$ has a unique solution in* $C^1([0, T_0), [D(A)])$.

Just as in [11, Theorems 2.9 and 2.10], we can apply Theorem 3.4 to obtain the next two theorems.

Theorem 3.9. *Let $A : D(A) \to X$ be a densely defined closed linear operator. Then the following are equivalent :*

- (i) *A is a subgenerator of a nondegenerate local $(\alpha+1)$-times integrated C-cosine function $S(\cdot)$ on X;*
- (ii) *for each $x \in D(A)$ ACP($j_{\alpha-1}(\cdot)Cx, 0, 0)$ has a unique solution $u(\cdot; Cx)$ in $C([0, T_0), [D(A)])$ which depends continuously on x. That is, if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(D(A), \|\cdot\|)$, then $\{u(\cdot; Cx_n)\}_{n=1}^{\infty}$ converges uniformly on compact subsets of* $[0, T_0)$.

Proof. (i)⇒(ii). It is easy to see from the definition of a subgenerator of $S(\cdot)$ that $S(\cdot)x$ is the unique solution of ACP($j_{\alpha-1}(\cdot)Cx, 0, 0)$ in C($[0, T_0), [D(A)])$ which depends continuously on $x \in D(A)$. (ii)⇒(i). In view of Theorem 3.4, we need only to show that for each $x \in X$ (3.1) has a unique solution $v(\cdot; x)$ in C($[0, T_0), X)$. Indeed, if $x \in X$ is given. By the denseness of D(A), we have $x_m \to x$ in X for some sequence $\{x_m\}_{m=1}^{\infty}$ in $D(A)$. We set $u(\cdot; Cx_m)$ to denote the unique solution of ACP($j_{\alpha-1}(\cdot)Cx_m, 0, 0)$ in C($[0, T_0), [D(A)])$. By hypothesis, we have $u(\cdot; Cx_m) \to u(\cdot)$ uniformly on compact subsets of $[0, T_0)$ for some $u \in$ C($[0, T_0), X)$, so that $\int_0^{\cdot}\int_0^s u(r; Cx_m)drds \to \int_0^{\cdot}\int_0^s u(r)drds$ uniformly on compact subsets of $[0, T_0)$. Since $Au(\cdot; Cx_m) = u''(\cdot; Cx_m) - j_{\alpha-1}(\cdot)Cx_m$ on $(0, T_0)$, we have

$$
\begin{aligned}
&A\int_0^{\cdot}\int_0^s u(r; Cx_m)drds \\
(3.3)\qquad &= \int_0^{\cdot}\int_0^s Au(r; Cx_m)drds = u(\cdot; Cx_m) - j_{\alpha+1}(\cdot)Cx_m
\end{aligned}
$$

on $[0, T_0)$ for all $m \in \mathbb{N}$. Clearly, the right-hand side of the last equality of (3.3) converges uniformly to $u(\cdot) - j_{\alpha+1}(\cdot)Cx$ on compact subsets of $[0, T_0)$. It follows from the closedness of A that $\int_0^t\int_0^s u(r)drds \in$ D(A) for all $0 \le t < T_0$ and $A\int_0^{\cdot}\int_0^s u(r)drds$=$u(\cdot) - j_{\alpha+1}(\cdot)Cx$ on $[0, T_0)$, which implies that $u(\cdot)$ is a (unique) solution of (3.1) in C($[0, T_0), X)$. ■

Theorem 3.10. *Let $A : D(A) \to X$ be a densely defined (closed) linear operator. Then the following are equivalent :*

(i) *A is a subgenerator of a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on X;*

(ii) *for each $x \in D(A)$ ACP($j_{\alpha-1}(\cdot)Cx, 0, 0$) has a unique solution $u(\cdot; Cx)$ in $C^1([0, T_0), [D(A)])$ which depends continuously differentiable on x. That is, if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(D(A), \|\cdot\|)$, then $\{u(\cdot; Cx_n)\}_{n=1}^{\infty}$ and $\{u'(\cdot; Cx_n)\}_{n=1}^{\infty}$ both converge uniformly on compact subsets of $[0, T_0)$.*

Proof. (i)⇒(ii). For each $0 \leq t < T_0$ and $x \in X$, we set $S(t)x = \int_0^t C(r)xdr$. Then $S(\cdot)x$ is the unique solution of ACP($j_{\alpha-1}(\cdot)Cx, 0, 0$) in $C^1([0, T_0), [\mathrm{D}(A)])$. Now if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathrm{D}(A), \|\cdot\|)$. We set $u(\cdot; Cx_n) = S(\cdot)x_n$ for $n \in \mathbb{N}$, then $\{u(\cdot; Cx_n)\}_{n=1}^{\infty}$ and $\{u'(\cdot; Cx_n)\}_{n=1}^{\infty}$ both converge uniformly on compact subsets of $[0, T_0)$. (ii)⇒(i). For each $x \in X$ and $0 \leq t < T_0$, we define $u(t) = \lim_{n\to\infty} u(t; Cx_n)$ whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in $D(A)$ which converges to x in X. By hypothesis, $u(\cdot; Cx_m) \to u(\cdot)$ and $u'(\cdot; Cx_m) \to u'(\cdot)$ uniformly on compact subsets of $[0, T_0)$ for some $u \in \mathrm{C}^1([0, T_0), X)$. Just as in the proof of Theorem 3.9, we also have

$$(3.4) \quad A\int_0^t \int_0^s u'(r; Cx_m)drds = A\int_0^t u(r; Cx_m)drds = u'(\cdot; Cx_m) - j_\alpha(\cdot)Cx_m$$

on $[0, T_0)$ for all $m \in \mathbb{N}$. Similarly, we also have $A\int_0^{\cdot}\int_0^s u'(r)drds = u'(\cdot) - j_\alpha(\cdot)Cx$ on $[0, T_0)$, which implies that $u'(\cdot)$ is a solution of the integral equation $v(\cdot) = Aj_1 * v(\cdot) + j_\alpha(\cdot)Cx$ in $\mathrm{C}([0, T_0), X)$. The uniqueness of solutions for the integral equation $v(\cdot) = Aj_1 * v(\cdot) + j_\alpha(\cdot)Cx$ in $\mathrm{C}([0, T_0), X)$ follows from the uniqueness of solutions for the integral equation (3.1) in $\mathrm{C}([0, T_0), X)$. ■

We end this paper with several illustrative examples.

Example 1. Let $X = C_b(\mathbb{R})$, and $\mathrm{C}(t)$ for $t \geq 0$ be bounded linear operators on X defined by $\mathrm{C}(t)f(x) = \frac{1}{2}[f(x+t) + f(x-t)]$ for all $x \in \mathbb{R}$. Then for each $\beta > -1$, $j_\beta * C(\cdot)$ is a $(\beta + 1)$-times integrated cosine function on X with generator $\frac{d^2}{dx^2}$, but $\mathrm{C}(\cdot)$ is not a cosine function on X.

Example 2. Let k be a fixed nonnegative integer, and let $C(t)$ for $t \geq 0$ and C be bounded linear operators on c_0 (the family of all convergent sequences in $\mathbb{F}$ with limit 0) defined by $C(t)x = \{x_n(n-k)e^{-n}\int_0^t j_{\alpha-1}(t-s)\cosh nsds\}_{n=1}^{\infty}$ and $Cx = \{x_n(n-k)e^{-n}\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in c_0$, then $\{C(t)|0 \leq t < 1\}$ is a local α-times integrated C-cosine function on c_0 which is degenerate except for $k = 0$ and generator A defined by $Ax = \{n^2x_n\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in c_0$

with $\{n^2 x_n\}_{n=1}^{\infty} \in c_0$, and for each $r > 1$ $\{C(t)|0 \leq t < r\}$ is not a local α-times integrated C-cosine function on c_0. Now if $k \in \mathbb{N}$, then $A_a : c_0 \to c_0$ for $a \in \mathbb{F}$ defined by $A_a x = \{n^2 y_n\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in c_0$ with $\{n^2 x_n\}_{n=1}^{\infty} \in c_0$, are subgenerators of $\{C(t)|0 \leq t < 1\}$ which do not have proper extensions that are still subgenerators of $\{C(t)|0 \leq t < 1\}$. Here $y_n = ak^2 x_k$ if $n = k$, and $y_n = n^2 x_n$ otherwise. Consequently, $\{C(t)|0 \leq t < 1\}$ does not have a maximal subgenerator.

Example 3. Let C $\in$ L(X) be fixed, and let C$(\cdot)$ be an α-times integrated C-cosine function on X defined by $C(t) = j_\alpha(t)C$ for $t \geq 0$. Then C$(\cdot)$ is nondegenerate with generator 0 (the zero operator on X) if and only if C is injective. Now if $D(\cdot)$ is a nondegenerate local α-times integrated D-cosine function on a Banach space Y over $\mathbb{F}$. Then $\widetilde{C}(\cdot)$ defined by $\widetilde{C}(t)(x, y) = (C(t)x, D(t)y)$ for all $0 \leq t < T_0$ and $(x, y) \in X \times Y$, is a local α-times integrated (C,D)-cosine function on the product Banach space $X \times Y$. Here $(C, D) : X \times Y \to X \times Y$ is defined by $(C, D)(x, y) = (Cx, Dy)$ for all $(x, y) \in X \times Y$. In this case, $\widetilde{C}(\cdot)$ is nondegenerate with generator $(0, D)$ defined by $(0, D)(x, y) = (0, Dy)$ for all $x \in X$ and $y \in D$ if and only if C is injective. Next if X is the direct sum of X_1 and X_2 for some nonzero subspaces X_1 and X_2 of X, $C : X \to X$ is the projection of X to a nonzero subspace of X_1, and $A : X \to X$ is the projection of X to a nonzero subspace of X_2, then $A : X \to X$ and the zero operator on X are subgenerators of $C(\cdot)$ which do not have common proper extensions that are still subgenerators of $\{C(t)|0 \leq t < 1\}$. In particular, $C(\cdot)$ does not have a maximal subgenerator. Similarly, we can show that $(0, D)$ and (A, D) are subgenerators of the degenerate local α-times integrated (C, D)-cosine function $\widetilde{C}(\cdot)$ on $X \times Y$ which do not have common proper extensions that are still subgenerators of $\widetilde{C}(\cdot)$. In particular, $\widetilde{C}(\cdot)$ does not have a maximal subgenerator.

Example 4. Let $X = C_b(\mathbb{R})$(or $L^\infty(\mathbb{R})$), and A be the maximal differential operator in X defined by $Au = \sum\limits_{j=0}^{k} a_j D^j u$ on $\mathbb{R}$ for all $u \in D(A)$, then $UC_b(\mathbb{R})$ (or $C_0(\mathbb{R})$) $= \overline{D(A)}$. Here $a_0, a_1, \cdots, a_k \in \mathbb{C}$ and $D^j u(x) = u^{(j)}(x)$ for all $x \in \mathbb{R}$. It is shown in [2, Theorem 6.7] that A generates an exponentially bounded, norm continuous 1-times integrated cosine function C$(\cdot)$ on X which is defined by $(C(t)f)(x) = \frac{1}{\sqrt{2\pi}}(\widetilde{\phi_t} * f)(x)$ for all $f \in X$ and $t \geq 0$ if the real-valued polynomial $p(x) = \sum\limits_{j=0}^{k} a_j (ix)^j$ satisfies $\sup\limits_{x \in \mathbb{R}} p(x) < \infty$. Here $\widetilde{\phi_t}$ denotes the inverse Fourier transform of ϕ_t with $\phi_t(x) = \int_0^t cosh(\sqrt{p(x)}s)ds$. Applying Theorem 3.4, we get that for each $f \in X$ and continuous function g on $[0, T_0) \times \mathbb{R}$ with $\int_0^t \sup\limits_{x \in \mathbb{R}} |g(s, x)|ds < \infty$ for all $0 \leq t < T_0$, the function u on $[0, T_0) \times \mathbb{R}$ defined by $u(t, x) = \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} (t - s)\widetilde{\phi_s}(x - y)f(y)dyds + \frac{1}{\sqrt{2\pi}} \int_0^t \int_0^{t-r} \int_{-\infty}^{\infty} (t - r - s)\widetilde{\phi_s}(x - y)g(s, y)dydsdr$ for all

$0 \le t < T_0$ and $x \in \mathbb{R}$, is the unique solution of

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} \\ = \displaystyle\sum_{j=0}^{k} a_j (\frac{\partial}{\partial x})^j u(t,x) + t f(x) + \int_0^t (t-s) g(s,x) ds \text{ for } t \in (0,T_0) \text{ and a.e. } x \in \mathbb{R}, \\ u(0,x) = 0 \text{ and } \dfrac{\partial u}{\partial t}(0,x) = 0 \quad \text{for a.e. } x \in \mathbb{R} \end{cases}$$

in $\mathrm{C}^2([0,T_0),\mathrm{X}) \cap \mathrm{C}([0,T_0),[\mathrm{D}(\mathrm{A})])$.

References

1. W. Arendt, C. J. K. Batty, H. Hieber and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, Vol. 96, Birkhauser Verlag, Basel-Boston-Berlin, 2001.

2. W. Arendt and H. Kellermann, *Integrated Solutions of Volterra Integrodifferential Equations and Applications*, Pitman Res. Notes Math., Vol. 190, Longman, Harlow, 1989, pp. 21-51.

3. H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, in North-Holland Math. Stud., Vol. 108, North-Holland, Amsterdam, 1985.

4. M. C. Gao, Local C-Semigroups and C-Cosine Functions, *Acta Math. Sci.*, **19** (1999), 201-213.

5. J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford, 1985.

6. F. Huang and T. Huang, Local C-Cosine Family Theory and Application, *Chin. Ann. Math.*, **16** (1995), 213-232.

7. Z. Huang and S. Wang, Strongly Continuous Integrated C-Cosine Operator Function and the Application to the Second Order Abstract Cauchy Problem, in: *Functional Analysis in China*, Kluwer Academic Publishes, Holland, 1996, pp. 330-350.

8. S. Kanda, Cosine Families and Weak Solution of Second Order Differential Equations, *Proc. Japan Acad. Ser. A. Math. Sci.*, **54** (1978), 119-123.

9. C.-C. Kuo and S.-Y. Shaw, C-Cosine Functions and the Abstract Cauchy Problem I, II, *J. Math. Anal. Appl.*, **210** (1997), 632-646, 647-666.

10. C.-C. Kuo, On exponentially bounded α-Times Integrated C-Cosine Functions, *Yokohama Math. J.*, **52** (2005), 59-72.

11. C.-C. Kuo, On α-Times Integrated C-Cosine Functions and Abstract Cauchy Problem I, *J. Math. Anal. Appl.*, **313** (2006), 142-162.

12. C.-C. Kuo, On Existence and Approximation of Solutions of Second Order Abstract Cauchy Problem, *Taiwanese J. Math.*, **14(3B)** (2010), 1093-1109.

13. C.-C. Kuo, On Local Integrated C-Cosine Function and Weak Solution of Second Order Abstract Cauchy Problem, *Taiwanese J. Math.*, **14(5)** (2010), 2027-2042.

14. C.-C. Kuo, On Perturbation of local integrated cosine functions, *Taiwanese J. Math.*, **16(5)** (2012), 1613-1628.

15. Y.-C. Li and S.-Y. Shaw, On Generators of Integrated C-Semigroups and C-Cosine Functions, *Semigroup Forum*, **47** (1993), 29-35.

16. Y.-C. Li and S.-Y. Shaw, *On Local α-Times Integrated C-Semigroups*, Abstract and Applied Anal., 2007, Article ID34890, 18 pages.

17. S.-Y. Shaw and Y.-C. Li, On N-Times Integrated C-Cosine Functions, in: *Evolution Equation*, Dekker, New York, 1994, pp. 393-406.

18. S.-Y. Shaw and Y.-C. Li, Characterization and generator of local C-osine and C-sine Functions, *Inter. J. Evolution Equations*, **1(4)** (2005), 373-401.

19. Sova, Cosine Operator Functions, *Rozprawy Mat.*, **49** (1966), 1-47.

20. T. Takenaka and S. Piskarev, Local C-Cosine Families and N-Times Integrated Local Cosine Families, *Taiwanese J. Math.*, **8** (2004), 515-546.

21. N. Takenaka and I. Miyadera, C-semigroups and the Abstract Cauchy Problem, *J. Math. Anal. Appl.*, **170** (1992), 196-206.

22. H.-Y. Wang, C-Cosine Operator Functions and the Second Order Abstract Cauchy Problem, *Northeast Math. J.*, **11(1)** (1995), 1-10.

23. S.-W. Wang and Z. Huang, Strongly Continuous Integrated C-Cosine Operator Functions, *Studia Math.*, **126** (1997), 273-289.

24. T.-J. Xiao and J. Liang, *The Cauchy Problem for Higher-Order Abstract Differential Equations*, Lectures Notes in Math., 1701, Springer, 1998.

25. R. Zhao and Z. Huang, Properties of Subgenerators of Integrated C-Cosine Operator Functions, *Northeast Math. J.*, **14(3)** (1998), 281-290.

26. Q. Zheng, Coercive Differential Operators and Fractionally Integrated Cosine Functions, *Taiwanese J. Math.*, **6** (2002), 59-65.

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