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RELAXED EXTRAGRADIENT METHOD FOR FINDING A COMMON ELEMENT OF SYSTEMS OF VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

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Abstract. In this paper, we investigate the problem of finding a common element of the solution set of a general system of variational inequalities, the solution set of a convex feasibility problem and the fixed point set of a strict pseudocontraction in a real Hilbert space. Based on the well-known extragradient method, viscosity approximation method and Mann iterative method, we propose and analyze a new relaxed extragradient method for computing a common element. Under very mild assumptions, we obtain a strong convergence theorem for three sequences generated by the proposed method. Our proposed method is quite general and flexible and includes the iterative methods considered in the earlier and recent literature as special cases. Our results represent the modification, supplement, extension and improvement of some corresponding results in the references.

1. Introduction

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. For a given nonlinear mapping $A : C \to H$, consider the following classical variational inequality of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

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The set of solutions of problem (1.1) is denoted by $VI(A, C)$. Variational inequality theory has emerged as an important tool in the investigation of a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. For finding an element of $\mathrm{Fix}(S)\cap VI(A, C)$ under the assumption that a set $C \subset H$ is nonempty, closed and convex, a mapping $S : C \to C$ is nonexpansive and a mapping $A : C \to H$ is α-inverse strongly monotone, Takahashi and Toyoda [15] introduced the following iterative algorithm:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n A x_n), \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. It was proven in [15] that if $\mathrm{Fix}(S) \cap VI(A, C) \neq \emptyset$ then the sequence $\{x_n\}$ converges weakly to some $z \in \mathrm{Fix}(S) \cap VI(A, C)$. Recently, Nadezhkina and Takahashi [31] and Zeng and Yao [40] proposed some so-called extragradient method motivated by the idea of Korpelevich [32] for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality. Further, these iterative methods were extended in [4] to develop a general iterative method for finding a element of $\mathrm{Fix}(S) \cap VI(A, C)$.

Let $B_1, B_2 : C \to H$ be two mappings. Consider the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.2}$$

which is called a general system of variational inequalities, where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants. It was introduced and considered by Ceng, Wang and Yao [3]. In particular, if $B_1 = B_2 = A$, then problem (1.2) reduces to the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 A y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 A x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.3}$$

which was defined by Verma [20] (see also [33]) and it is called a new system of variational inequalities. Further, if $x^* = y^*$ additionally, then problem (1.3) reduces to the classical variational inequality (1.1). Recently, Ceng, Wang and Yao [3] transformed problem (1.2) into a fixed point problem in the following way:

Lemma 1.1. [3]. *For given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of problem (1.2) if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)], \quad \forall x \in C,$$

where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

In particular, if the mapping $B_j : C \to H$ is β_j-inverse strongly monotone for $j = 1, 2$, then the mapping G is nonexpansive provided $\mu_j \in (0, 2\beta_j)$ for $j = 1, 2$.

Utilizing Lemma 1.1, they proposed and analyzed a relaxed extragradient method for solving problem (1.2). Throughout this paper, the set of fixed points of the mapping G is denoted by $GSVI(B_1, B_2, C)$. Based on the relaxed extragradient method and viscosity approximation method, Yao, Liou and Kang [21] introduced and studied an iterative algorithm for finding a common solution of problem (1.2) and the fixed-point problem of a strictly pseudo-contractive mapping $S : C \to C$. It is worth pointing out that in their main result (that is, [21, Theorem 3.2]), the boundedness restriction imposed on C is much stronger.

Recently, many authors studied the following convex feasibility problem (for short, CFP):

$$\text{finding an } \bar{x} \in \bigcap_{i=1}^{m} K_i, \tag{1.4}$$

where $m \geq 1$ is an integer and each K_i is a nonempty closed convex subset of H. There is a considerable investigation on the CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [5,10], computer tomography [14] and radiation therapy treatment planning [6]. In this paper, we shall consider the case when K_i is the solution set of the variational inequality (1.1). Furthermore, it is worth pointing out that related iterative methods for solving fixed point problems, variational inequalities, equilibrium problems and optimization problems can be found in [1, 3, 4, 6-9, 11-13, 15-31, 33, 35-40].

In 2007, Yao and Yao [17] introduced and considered a relaxed extragradient algorithm for finding an element of $\mathrm{Fix}(S) \cap VI(A, C)$ and derived a strong convergence result which improves Iiduk and Takahashi's theorem [7].

Theorem 1.1. (see [17]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $S : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap VI(A, C) \neq \emptyset$. *Suppose that $\{x_n\}, \{y_n\}$ are given by*

$$\begin{cases} x_1 = u \in C \text{ chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda_n A y_n), \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$ and

(a) $\alpha_n + \beta_n + \gamma_n = 1, \ \forall n \geq 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(d) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$,
then $\{x_n\}$ *converges strongly to* $P_{\mathrm{Fix}(S)\cap VI(A,C)}u$.

In 2008, Ceng, Wang and Yao [3] further considered the problem of finding a common element of the solution set of the general system (1.2) of variational inequalities and the fixed point set of a nonexpansive mapping by the following iterative algorithm:

$$\tag{1.6} \begin{cases} x_1 = u \in C \text{ chosen arbitrarily,} \\ y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \quad \forall n \geq 1, \end{cases}$$

where $A, B : C \to H$ are two inverse-strongly monotone mappings and $S : C \to C$ is a nonexpansive mapping. They also obtained a strong convergence theorem of algorithm (1.6).

Very recently, Cho and Kang [19] studied the convex feasibility problem (1.4) (where $K_i = VI(A_i, C)$ for $i = 1, 2, ..., m$) by considering a finite family of inverse-strongly monotone mappings $\{A_i\}_{i=1}^m : C \to H$ and a strict pseudocontraction, and established a strong convergence theorem which extends the corresponding results in [3, 7, 11, 17].

Theorem 1.2. (see [19]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $A_i : C \to H$ *be a* $\widehat{\alpha}_i$*-inverse-strongly monotone mapping for each* $1 \leq i \leq m$, *where* m *is some positive integer. Let* $S : C \to C$ *be a* κ*-strict pseudocontraction with a fixed point. Assume that* $\Omega := \cap_{i=1}^m VI(A_i, C) \cap \mathrm{Fix}(S) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated in the following iterative algorithm:*

$$\tag{1.7} \begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n u + (1-\alpha_n) \sum_{i=1}^{m} [\eta_n^i P_C(x_n - \lambda_i A_i x_n)], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)[\gamma_n y_n + (1-\gamma_n) S y_n], \quad \forall n \geq 1, \end{cases}$$

where $\{\gamma_n\}$ *is a sequence in* $[\kappa, \gamma)$, *where* γ *is some constant in* $(\kappa, 1)$, u *is a fixed point in* C, $\lambda_1, \lambda_2, ..., \lambda_m$ *are real numbers such that* $\lambda_i \in (0, 2\widehat{\alpha}_i)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\eta_n^i\} \subset (0, 1)$ *for* $i = 1, 2, ..., m$. *Assume that the above control sequences satisfy the following restrictions:*

(a) $\sum_{i=1}^m \eta_n^i = 1, \ \forall n \geq 1$;

(b) $\lim_{n\to\infty} \eta_n^i = \eta^i \in (0, 1)$ *for* $i = 1, 2, ..., m$;

(c) $\lim_{n\to\infty}\alpha_n = 0,\ \sum_{n=1}^{\infty}\alpha_n = \infty$;

(d) $\beta_n \le a < 1$, *where* a *is a constant in* $(0,1)$;

(e) $\lim_{n\to\infty}(\gamma_{n+1}-\gamma_n) = 0$ *and* $\gamma_n \le b < 1$, *where* b *is a constant in* $(0,1)$.
Then the sequence $\{x_n\}$ *defined by algorithm (1.7) converges strongly to* $\bar{x} = P_{\Omega}u$.

Motivated and inspired by the recent research work going on in this field, we consider the problem of finding an element of $\Omega := \cap_{i=1}^{m} VI(A_i, C) \cap \mathrm{Fix}(S) \cap GSVI(B_1, B_2, C)$ where $A_i : C \to H$ is $\widehat{\alpha}_i$-inverse strongly monotone for $i = 1, 2, ..., m$, $B_j : C \to H$ is $\widehat{\beta}_j$-inverse strongly monotone for $j = 1, 2$ and $S : C \to C$ is a κ-strict pseudocontraction. Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, \frac{1}{2})$. Based on the well-known extragradient method, viscosity approximation method and Mann iterative method, we introduce a new relaxed extragradient algorithm for finding an element in Ω, that is,

$$
(1.8)\qquad \begin{cases} x_1 \in C, \text{ choosen arbitrarily,} \\ z_n = P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\ y_n = \alpha_n Q x_n + (1-\alpha_n)\sum_{i=1}^{m}[\eta_n^i P_C(z_n - \lambda_i A_i z_n)], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)[\gamma_n y_n + (1-\gamma_n) S y_n], \quad \forall n \ge 1, \end{cases}
$$

where $\{\gamma_n\} \subset [\kappa, \gamma)$ for some $\gamma \in (\kappa, 1)$, $\mu_j \in (0, 2\widehat{\beta}_j)$ for $j = 1, 2$, $\lambda_i \in (0, 2\widehat{\alpha}_i)$ and $\{\alpha_n\}, \{\beta_n\}, \{\eta_n^i\} \subset (0,1)$ for $i = 1, 2, ..., m$. It is proven that under very mild conditions three sequences $\{x_n\}, \{y_n\}, \{z_n\}$ generated by algorithm (1.8) converge strongly to the same point $\bar{x} = P_{\Omega} Q \bar{x}$. Furthermore, $(\bar{x}, \bar{y})$ is a solution of the general system (1.2) of variational inequalities, where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$. Our result represents the modification, supplement, extension and improvement of the above Theorems 1.1 and 1.2 in the following aspects.

(i) our problem of finding an element of $\cap_{i=1}^{m} VI(A_i, C) \cap \mathrm{Fix}(S) \cap GSVI(B_1, B_2, C)$ is more general and more complex than the problem of finding an element of $\mathrm{Fix}(S) \cap VI(A, C)$ in the above Theorem 1.1.

(ii) our problem of finding an element of $\cap_{i=1}^{m} VI(A_i, C) \cap \mathrm{Fix}(S) \cap GSVI(B_1, B_2, C)$ is also more general and more complex than the problem of finding an element of $\cap_{i=1}^{m} VI(A_i, C) \cap \mathrm{Fix}(S)$ in the above Theorem 1.2.

(iii) our algorithm (1.8) is very different from algorithm (1.5) in the above Theorem 1.1 and also very different from algorithm (1.7) in the above Theorem 1.2 because algorithm (1.8) is closely related to the viscosity approximation method with the ρ-contraction $Q : C \to C$ and involves the Picard successive iteration for the general system (1.2) of variational inequalities.

(iv) the techniques of proving strong convergence in our result are very different from those in the above theorems 1.1 and 1.2 because our techniques depend on the norm inequality in Lemma 2.2 and the inverse-strong monotonicity of mappings $A_i, B_j : C \to H$ for $i = 1, 2, ..., m$ and $j = 1, 2$, the demiclosedness principle for strict pseudocontractions, and the transformation of the general system (1.2) of variational inequalities into the fixed-point problem of the nonexpansive self-mapping $G : C \to C$ (see the above Lemma 1.1).

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and C be a nonempty closed convex subset of H. We write $\to$ to indicate that the sequence $\{x_n\}$ converges strongly to x and $\rightharpoonup$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall x \in C.$$

P_C is called the metric projection of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C; that is, there holds the following relation

$$\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Consequently, P_C is nonexpansive and monotone. It is also known that P_C is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \tag{2.1}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.2}$$

for all $x \in H, y \in C$; see [34] for more details. Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality, this implies that

$$x \in VI(A, C) \quad \Leftrightarrow \quad x = P_C(x - \lambda A x) \ \forall \lambda > 0. \tag{2.3}$$

Recall that a mapping $S : C \to C$ is called a strict pseudocontraction if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.4}$$

In this case, we also say that S is a k-strict pseudocontraction. A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.5}$$

It is obvious that any α-inverse strongly monotone mapping is Lipschitz continuous. Meantime, observe that (2.4) is equivalent to

$$\langle Sx - Sy, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

It is easy to see that if S is a k-strictly pseudocontractive mapping, then $I - S$ is $\frac{1-k}{2}$-inverse strongly monotone and hence $\frac{2}{1-k}$-Lipschitz continuous. Thus, S is Lipschitz continuous with constant $\frac{1+k}{1-k}$. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S. It is clear that the class of strict pseudocontractions strictly includes the one of nonexpansive mappings which are mappings $S : C \to C$ such that $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$.

In order to prove our main result in the next section, we need the following lemmas and propositions.

Lemma 2.1. (see Bruck [2]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i : 1 \leq i \leq m\}$ be a sequence of nonexpansive mappings on C. Suppose $\cap_{i=1}^m \mathrm{Fix}(T_i)$ is nonempty. Let $\{\lambda_i\}$ be a sequence of positive numbers with $\sum_{i=1}^m \lambda_i = 1$. Then a mapping S on C defined by*

$$Sx = \sum_{i=1}^m \lambda_i T_i x, \quad \forall x \in C$$

is well defined, nonexpansive and $\mathrm{Fix}(S) = \cap_{i=1}^m \mathrm{Fix}(T_i)$ holds.

The following lemma is an immediate consequence of an inner product.

Lemma 2.2. *In a real Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Recall that $S : C \to C$ is called a quasi-strict pseudocontraction if the fixed point set of S, $\mathrm{Fix}(S)$, is nonempty and if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - p\|^2 \leq \|x - p\|^2 + k\|x - Sx\|^2 \quad \text{for all } x \in C \text{ and } p \in \mathrm{Fix}(S). \tag{2.6}$$

We also say that S is a k-quasi-strict pseudocontraction if condition (2.6) holds.

Proposition 2.1. (see [8, Proposition 2.1]). *Assume C is a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a self-mapping on C.*

(i) *If S is a k-strict pseudocontraction, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1+k}{1-k}\|x-y\|, \quad \forall x, y \in C. \tag{2.7}$$

(ii) *If S is a k-strict pseudocontraction, then the mapping $I - S$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-S)x_n \to 0$, then $(I-S)\tilde{x} = 0$, i.e., $\tilde{x} \in \mathrm{Fix}(S)$.*

(iii) *If S is a k-quasi-strict pseudocontraction, then the fixed point set $\mathrm{Fix}(S)$ of S is closed and convex so that the projection $P_{\mathrm{Fix}(S)}$ is well defined.*

The following lemma was proved by Suzuki [13].

Lemma 2.3. (see [13]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.4. (see [16]). *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1-\delta_n)a_n + \delta_n \sigma_n, \quad \forall n \geq 0,$$

where $\{\delta_n\}, \{\sigma_n\}$ are sequences of real numbers such that

(i) *$\{\delta_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty} \delta_n = \infty$, or equivalently,*

$$\prod_{n=0}^{\infty}(1-\delta_n) := \lim_{n\to\infty} \prod_{j=0}^{n}(1-\delta_j) = 0;$$

(ii) *$\limsup_{n\to\infty} \sigma_n \leq 0$, or*

(ii') *$\sum_{n=0}^{\infty} \delta_n \sigma_n$ is convergent.*
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. (see Zhou [18]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a κ-strict pseudocontraction with a fixed point. Define $S_a : C \to C$ by $S_a x = ax + (1-a)Sx$ for each $x \in C$. Then, as $a \in [\kappa, 1)$, S_a is nonexpansive such that $\mathrm{Fix}(S_a) = \mathrm{Fix}(S)$.*

3. Strong Convergence Theorem

We are now in a position to state and prove our main result.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Given an integer $m \geq 1$. Let $A_i : C \to H$ be $\widehat{\alpha}_i$-inverse strongly monotone for*

$i = 1, 2, ..., m$, and $B_j : C \to H$ be $\widehat{\beta}_j$-inverse strongly monotone for $j = 1, 2$. Let $S : C \to C$ be a κ-strict pseudocontraction such that $\Omega := \cap_{i=1}^{m} VI(A_i, C) \cap \mathrm{Fix}(S) \cap GSVI(B_1, B_2, C) \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, \frac{1}{2})$. For given $x_1 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ be generated iteratively by

$$
(3.1) \qquad \begin{cases} z_n = P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\ y_n = \alpha_n Q x_n + (1 - \alpha_n) \sum_{i=1}^{m} [\eta_n^i P_C(z_n - \lambda_i A_i z_n)], \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)[\gamma_n y_n + (1 - \gamma_n) S y_n], \quad \forall n \geq 1, \end{cases}
$$

where $\{\gamma_n\} \subset [\kappa, \gamma)$ for some $\gamma \in (\kappa, 1)$, $\mu_j \in (0, 2\widehat{\beta}_j)$ for $j = 1, 2$, $\lambda_i \in (0, 2\widehat{\alpha}_i)$ and $\{\alpha_n\}, \{\beta_n\}, \{\eta_n^i\} \subset (0, 1)$ for $i = 1, 2, ..., m$, such that

- (i) $\sum_{i=1}^{m} \eta_n^i = 1, \ \forall n \geq 1$;
- (ii) $\lim_{n\to\infty} \eta_n^i = \eta^i \in (0, 1)$ *for* $i = 1, 2, ..., m$;
- (iii) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$;
- (iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
- (v) $\lim_{n\to\infty}(\gamma_{n+1} - \gamma_n) = 0$ *and* $\limsup_{n\to\infty} \gamma_n < 1$.
  Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to the same point $\bar{x} = P_\Omega Q \bar{x}$. Furthermore, $(\bar{x}, \bar{y})$ is a solution of the general system (1.2) of variational inequalities, where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

Proof. First, let us show that the mapping $I - \lambda_i A_i$ is nonexpansive for $i = 1, 2, ..., m$. Indeed, for all $x, y \in C$, we have

$$
\begin{aligned}
& \|(I - \lambda_i A_i)x - (I - \lambda_i A_i)y\|^2 \\
&= \|(x - y) - \lambda_i(A_i x - A_i y)\|^2 \\
&= \|x - y\|^2 - 2\lambda_i \langle A_i x - A_i y, x - y\rangle + \lambda_i^2 \|A_i x - A_i y\|^2 \\
&\leq \|x - y\|^2 - 2\lambda_i \widehat{\alpha}_i \|A_i x - A_i y\|^2 + \lambda_i^2 \|A_i x - A_i y\|^2 \\
&= \|x - y\|^2 - \lambda_i(2\widehat{\alpha}_i - \lambda_i)\|A_i x - A_i y\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}
$$

This shows that $I - \lambda_i A_i$ is nonexpansive for $i = 1, 2, ..., m$.

We divide the rest of the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, let $x^* \in \Omega$. Then $Sx^* = x^*, \ x^* = P_C(x^* - \lambda_i A_i x^*)$ for $i = 1, 2, ..., m$, and

$$
x^* = P_C[P_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 P_C(x^* - \mu_2 B_2 x^*)].
$$

For simplicity, we write

$$y^* = P_C(x^* - \mu_2 B_2 x^*),\ \tilde{x}_n = P_C(x_n - \mu_2 B_2 x_n)\ v_n^i = P_C(z_n - \lambda_i A_i z_n) \text{ and } u_n = \sum_{i=1}^{m} \eta_n^i v_n^i$$

for each $n \geq 1$. Then $y_n = \alpha_n Q x_n + (1-\alpha_n) u_n$ for each $n \geq 1$. Since $B_j : C \to H$ is $\widehat{\beta}_j$-inverse strongly monotone for $j = 1, 2$ and $0 < \mu_j < 2\widehat{\beta}_j$ for $j = 1, 2$, we know that for all $n \geq 1$

$$\begin{aligned}
&\|z_n - x^*\|^2 \\
&= \|P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] - x^*\|^2 \\
&= \|P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] \\
&\quad - P_C[P_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 P_C(x^* - \mu_2 B_2 x^*)]\|^2 \\
&\leq \|[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] \\
&\quad - [P_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 P_C(x^* - \mu_2 B_2 x^*)]\|^2 \\
&= \|[P_C(x_n - \mu_2 B_2 x_n) - P_C(x^* - \mu_2 B_2 x^*)] \\
&\quad - \mu_1 [B_1 P_C(x_n - \mu_2 B_2 x_n) - B_1 P_C(x^* - \mu_2 B_2 x^*)]\|^2 \\
&\leq \|P_C(x_n - \mu_2 B_2 x_n) - P_C(x^* - \mu_2 B_2 x^*)\|^2 \\
&\quad - \mu_1 (2\widehat{\beta}_1 - \mu_1) \|B_1 P_C(x_n - \mu_2 B_2 x_n) - B_1 P_C(x^* - \mu_2 B_2 x^*)\|^2 \\
&\leq \|(x_n - \mu_2 B_2 x_n) - (x^* - \mu_2 B_2 x^*)\|^2 - \mu_1 (2\widehat{\beta}_1 - \mu_1) \|B_1 \tilde{x}_n - B_1 y^*\|^2 \\
&= \|(x_n - x^*) - \mu_2 (B_2 x_n - B_2 x^*)\|^2 - \mu_1 (2\widehat{\beta}_1 - \mu_1) \|B_1 \tilde{x}_n - B_1 y^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \mu_2 (2\widehat{\beta}_2 - \mu_2) \|B_2 x_n - B_2 x^*\|^2 - \mu_1 (2\widehat{\beta}_1 - \mu_1) \|B_1 \tilde{x}_n - B_1 y^*\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned} \tag{3.2}$$

Now, observe that

$$\|v_n^i - x^*\| = \|P_C(I - \lambda_i A_i) z_n - P_C(I - \lambda_i A_i) x^*\| \leq \|z_n - x^*\| \leq \|x_n - x^*\| \tag{3.3}$$

for $i = 1, 2, ..., m$. Define a mapping $S_{\gamma_n} : C \to C$ by

$$S_{\gamma_n} x = \gamma_n x + (1 - \gamma_n) S x, \quad \forall x \in C.$$

From Lemma 2.5, it is known that $\text{Fix}(S) = \text{Fix}(S_{\gamma_n})$ for each $n \geq 1$. It follows from (3.1) that

$$
\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\le \beta_n\|x_n - x^*\| + (1-\beta_n)\|S_{\gamma_n}y_n - x^*\| \\
&\le \beta_n\|x_n - x^*\| + (1-\beta_n)\|\alpha_n Qx_n + (1-\alpha_n)\sum_{i=1}^{m}\eta_n^i v_n^i - x^*\| \\
&\le \beta_n\|x_n - x^*\| + (1-\beta_n)[\alpha_n\|Qx_n - x^*\| + (1-\alpha_n)\|\sum_{i=1}^{m}\eta_n^i v_n^i - x^*\|] \\
&\le [1-\alpha_n(1-\beta_n)]\|x_n - x^*\| + \alpha_n(1-\beta_n)\|Qx_n - x^*\| \\
&\le [1-\alpha_n(1-\beta_n)]\|x_n - x^*\| + \alpha_n(1-\beta_n)(\rho\|x_n - x^*\| + \|Qx^* - x^*\|) \\
&= [1-\alpha_n(1-\beta_n)(1-\rho)]\|x_n - x^*\| + \alpha_n(1-\beta_n)\|Qx^* - x^*\| \\
&\le \max\{\|x_n - x^*\|, \frac{\|Qx^* - x^*\|}{1-\rho}\}.
\end{aligned}
$$

By induction, we can obtain

$$\|x_{n+1} - x^*\| \le \max\{\|x_1 - x^*\|, \frac{\|Qx^* - x^*\|}{1-\rho}\}.$$

This shows that the sequence $\{x_n\}$ is bounded.

Step 2. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.
Indeed, note that

$$
\begin{aligned}
&\|z_{n+1} - z_n\|^2 \\
&= \|P_C[P_C(x_{n+1} - \mu_2 B_2 x_{n+1}) - \mu_1 B_1 P_C(x_{n+1} - \mu_2 B_2 x_{n+1})] \\
&\quad - P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)]\|^2 \\
&\le \|[P_C(x_{n+1} - \mu_2 B_2 x_{n+1}) - \mu_1 B_1 P_C(x_{n+1} - \mu_2 B_2 x_{n+1})] \\
&\quad - [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)]\|^2 \\
&= \|[P_C(x_{n+1} - \mu_2 B_2 x_{n+1}) - P_C(x_n - \mu_2 B_2 x_n)] \\
&\quad - \mu_1[B_1 P_C(x_{n+1} - \mu_2 B_2 x_{n+1}) - B_1 P_C(x_n - \mu_2 B_2 x_n)]\|^2 \\
&\le \|P_C(x_{n+1} - \mu_2 B_2 x_{n+1}) - P_C(x_n - \mu_2 B_2 x_n)\|^2 \\
&\quad - \mu_1(2\widehat{\beta}_1 - \mu_1)\|B_1 P_C(x_{n+1} - \mu_2 B_2 x_{n+1}) - B_1 P_C(x_n - \mu_2 B_2 x_n)\|^2 \\
&\le \|P_C(x_{n+1} - \mu_2 B_2 x_{n+1}) - P_C(x_n - \mu_2 B_2 x_n)\|^2 \\
&\le \|(x_{n+1} - \mu_2 B_2 x_{n+1}) - (x_n - \mu_2 B_2 x_n)\|^2 \\
&= \|(x_{n+1} - x_n) - \mu_2(B_2 x_{n+1} - B_2 x_n)\|^2 \\
&\le \|x_{n+1} - x_n\|^2 - \mu_2(2\widehat{\beta}_2 - \mu_2)\|B_2 x_{n+1} - B_2 x_n\|^2 \\
&\le \|x_{n+1} - x_n\|^2,
\end{aligned}
$$

and hence

$$\|v_{n+1}^i - v_n^i\| = \|P_C(I-\lambda_i A_i)z_{n+1} - P_C(I-\lambda_i A_i)z_n\| \le \|z_{n+1} - z_n\| \le \|x_{n+1} - x_n\| \tag{3.4}$$

for $i = 1, 2, ..., m$. On the other hand, we have

$$y_{n+1} - y_n = (\alpha_{n+1} - \alpha_n)(Qx_{n+1} - u_n) + (1 - \alpha_{n+1})(u_{n+1} - u_n) + \alpha_n(Qx_{n+1} - Qx_n).$$

It follows from (3.4) that

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\leq |\alpha_{n+1} - \alpha_n|\|Qx_{n+1} - u_n\| + (1 - \alpha_{n+1})\|u_{n+1} - u_n\| + \alpha_n\|Qx_{n+1} - Qx_n\| \\
&= |\alpha_{n+1} - \alpha_n|\|Qx_{n+1} - u_n\| + (1 - \alpha_{n+1})\|\sum_{i=1}^{m} \eta_{n+1}^i v_{n+1}^i - \sum_{i=1}^{m} \eta_n^i v_n^i\| \\
&\quad + \alpha_n\|Qx_{n+1} - Qx_n\| \\
&\leq |\alpha_{n+1} - \alpha_n|\|Qx_{n+1} - u_n\| + (1 - \alpha_{n+1})\sum_{i=1}^{m} \eta_{n+1}^i\|v_{n+1}^i - v_n^i\| \qquad (3.5) \\
&\quad + (1 - \alpha_{n+1})\sum_{i=1}^{m} |\eta_{n+1}^i - \eta_n^i|\|v_n^i\| + \alpha_n\|Qx_{n+1} - Qx_n\| \\
&\leq |\alpha_{n+1} - \alpha_n|\|Qx_{n+1} - u_n\| + \|x_{n+1} - x_n\| + M\sum_{i=1}^{m} |\eta_{n+1}^i - \eta_n^i| \\
&\quad + \alpha_n\|Qx_{n+1} - Qx_n\|,
\end{aligned}$$

where M is an appropriate constant such that

$$M = \max\{\sup\{\|P_C(I - \lambda_i A_i)z_n\| : n \geq 1\} : 1 \leq i \leq m\}.$$

Note that

$$\|S_{\gamma_{n+1}}y_{n+1} - S_{\gamma_n}y_n\| \leq |\gamma_{n+1} - \gamma_n|\|y_n - Sy_n\| + \|y_{n+1} - y_n\|,$$

which together with (3.5) yields that

$$\begin{aligned}
&\|S_{\gamma_{n+1}}y_{n+1} - S_{\gamma_n}y_n\| - \|x_{n+1} - x_n\| \\
&\leq |\gamma_{n+1} - \gamma_n|\|y_n - Sy_n\| + \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\
&\leq |\gamma_{n+1} - \gamma_n|\|y_n - Sy_n\| + |\alpha_{n+1} - \alpha_n|\|Qx_{n+1} - u_n\| + M\sum_{i=1}^{m} |\eta_{n+1}^i - \eta_n^i| \\
&\quad + \alpha_n\|Qx_{n+1} - Qx_n\|.
\end{aligned}$$

It follows from conditions (ii), (iii) and (v) that

$$\limsup_{n\to\infty}(\|S_{\gamma_{n+1}}y_{n+1} - S_{\gamma_n}y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

From Lemma 2.3 it follows that

$$\lim_{n\to\infty} \|S_{\gamma_n}y_n - x_n\| = 0. \qquad (3.6)$$

Since $x_{n+1} - x_n = (1 - \beta_n)(S_{\gamma_n} y_n - x_n)$, we deduce from (3.6) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

Step 3. $\lim_{n\to\infty} \|B_1\tilde{x}_n - B_1 y^*\| = \lim_{n\to\infty} \|B_2 x_n - B_2 x^*\| = \lim_{n\to\infty} \|A_i z_n - A_i x^*\| = 0$ for $i = 1, 2, ..., m$. Indeed, utilizing Lemma 2.5 we get from (3.2)

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|S_{\gamma_n} y_n - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|\alpha_n Q x_n + (1-\alpha_n)u_n - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|Qx_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\beta_n)\|\sum_{i=1}^m \eta_n^i v_n^i - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|Qx_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\beta_n)\sum_{i=1}^m \eta_n^i\|v_n^i - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|Qx_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\beta_n)\sum_{i=1}^m \eta_n^i\|z_n - x^* - \lambda_i(A_i z_n - A_i x^*)\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|Qx_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\beta_n)\sum_{i=1}^m \eta_n^i(\|z_n - x^*\|^2 - \lambda_i(2\widehat{\alpha}_i - \lambda_i)\|A_i z_n - A_i x^*\|^2) \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|Qx_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\beta_n)(\|z_n - x^*\|^2 - \sum_{i=1}^m \eta_n^i\lambda_i(2\widehat{\alpha}_i - \lambda_i)\|A_i z_n - A_i x^*\|^2) \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|Qx_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\beta_n)[\|x_n - x^*\|^2 - \mu_2(2\widehat{\beta}_2 - \mu_2)\|B_2 x_n - B_2 x^*\|^2 \\
&\quad -\mu_1(2\widehat{\beta}_1 - \mu_1)\|B_1\tilde{x}_n - B_1 y^*\|^2 - \sum_{i=1}^m \eta_n^i\lambda_i(2\widehat{\alpha}_i - \lambda_i)\|A_i z_n - A_i x^*\|^2] \\
&\le \|x_n - x^*\|^2 + \alpha_n\|Qx_n \\
&\quad -x^*\|^2 - (1-\alpha_n)(1-\beta_n)[\mu_1(2\widehat{\beta}_1 - \mu_1)\|B_1\tilde{x}_n - B_1 y^*\|^2 \\
&\quad +\mu_2(2\widehat{\beta}_2 - \mu_2)\|B_2 x_n - B_2 x^*\|^2 + \sum_{i=1}^m \eta_n^i\lambda_i(2\widehat{\alpha}_i - \lambda_i)\|A_i z_n - A_i x^*\|^2],
\end{aligned} \tag{3.8}$$

and hence

$$
\begin{aligned}
&(1-\alpha_n)(1-\beta_n)[\mu_1(2\widehat{\beta}_1-\mu_1)\|B_1\tilde{x}_n-B_1y^*\|^2+\mu_2(2\widehat{\beta}_2-\mu_2)\|B_2x_n-B_2x^*\|^2\\
&+\sum_{i=1}^m \eta_n^i\lambda_i(2\widehat{\alpha}_i-\lambda_i)\|A_iz_n-A_ix^*\|^2]\\
\le\ &\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n\|Qx_n-x^*\|^2\\
\le\ &(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|+\alpha_n\|Qx_n-x^*\|^2.
\end{aligned}
$$

Utilizing conditions (iii) and (iv), we see from (3.7) that

$$
\lim_{n\to\infty}\|B_1\tilde{x}_n-B_1y^*\|=\lim_{n\to\infty}\|B_2x_n-B_2x^*\|=\lim_{n\to\infty}\|A_iz_n-A_ix^*\|=0 \tag{3.9}
$$

for $i=1,2,...,m$.

Step 4. $\lim_{n\to\infty}\|Sx_n-x_n\|=0$.
Indeed, observe that

$$
\begin{aligned}
\|v_n^i-x^*\|^2 &= \|P_C(I-\lambda_iA_i)z_n-P_C(I-\lambda_iA_i)x^*\|^2\\
&\le \langle (I-\lambda_iA_i)z_n-(I-\lambda_iA_i)x^*, v_n^i-x^*\rangle\\
&= \frac{1}{2}(\|(I-\lambda_iA_i)z_n-(I-\lambda_iA_i)x^*\|^2+\|v_n^i-x^*\|^2\\
&\quad -\|(I-\lambda_iA_i)z_n-(I-\lambda_iA_i)x^*-(v_n^i-x^*)\|^2)\\
&\le \frac{1}{2}(\|z_n-x^*\|^2+\|v_n^i-x^*\|^2-\|z_n-v_n^i-\lambda_i(A_iz_n-A_ix^*)\|^2)\\
&= \frac{1}{2}(\|z_n-x^*\|^2+\|v_n^i-x^*\|^2-\|z_n-v_n^i\|^2\\
&\quad +2\lambda_i\langle A_iz_n-A_ix^*, z_n-v_n^i\rangle-\lambda_i^2\|A_iz_n-A_ix^*\|^2)
\end{aligned}
$$

for $i=1,2,...,m$. Hence it follows that

$$
\begin{aligned}
\|v_n^i-x^*\|^2 &\le \|z_n-x^*\|^2-\|z_n-v_n^i\|^2\\
&\qquad +2\lambda_i\langle A_iz_n-A_ix^*, z_n-v_n^i\rangle-\lambda_i^2\|A_iz_n-A_ix^*\|^2\\
&\le \|z_n-x^*\|^2-\|z_n-v_n^i\|^2+\Gamma^i\|A_iz_n-A_ix^*\|
\end{aligned} \tag{3.10}
$$

for $i=1,2,...,m$, where Γ^i is an appropriate constant such that

$$
\Gamma^i=\sup\{2\lambda_i\|z_n-v_n^i\|: n\ge 1\}
$$

for $i=1,2,...,m$. In the meantime, we have

$$
\|u_n-z_n\|^2\le\sum_{i=1}^m\eta_n^i\|v_n^i-z_n\|^2,
$$

which together with (3.10) yields that

$$
\begin{aligned}
&\sum_{i=1}^m \eta_n^i \|v_n^i - x^*\|^2 \\
&\le \|z_n - x^*\|^2 - \sum_{i=1}^m \eta_n^i \|z_n - v_n^i\|^2 + \sum_{i=1}^m \eta_n^i \Gamma^i \|A_i z_n - A_i x^*\| \\
&\le \|z_n - x^*\|^2 - \|u_n - z_n\|^2 + \sum_{i=1}^m \Gamma^i \|A_i z_n - A_i x^*\|.
\end{aligned}
\tag{3.11}
$$

On the other hand, observe that

$$
\begin{aligned}
&\|\tilde{x}_n - y^*\|^2 = \|P_C(x_n - \mu_2 B_2 x_n) - P_C(x^* - \mu_2 B_2 x^*)\|^2 \\
&\le \langle (x_n - \mu_2 B_2 x_n) - (x^* - \mu_2 B_2 x^*), \tilde{x}_n - y^* \rangle \\
&= \frac{1}{2}[\|x_n - x^* - \mu_2(B_2 x_n - B_2 x^*)\|^2 + \|\tilde{x}_n - y^*\|^2 \\
&\quad -\|(x_n - x^*) - \mu_2(B_2 x_n - B_2 x^*) - (\tilde{x}_n - y^*)\|^2] \\
&\le \frac{1}{2}[\|x_n - x^*\|^2 + \|\tilde{x}_n - y^*\|^2 - \|(x_n - x^*) - \mu_2(B_2 x_n - B_2 x^*) - (\tilde{x}_n - y^*)\|^2] \\
&= \frac{1}{2}[\|x_n - x^*\|^2 + \|\tilde{x}_n - y^*\|^2 - \|x_n - \tilde{x}_n - (x^* - y^*)\|^2 \\
&\quad +2\mu_2 \langle x_n - \tilde{x}_n - (x^* - y^*), B_2 x_n - B_2 x^* \rangle - \mu_2^2 \|B_2 x_n - B_2 x^*\|^2] \\
&\le \frac{1}{2}[\|x_n - x^*\|^2 + \|\tilde{x}_n - y^*\|^2 - \|x_n - \tilde{x}_n - (x^* - y^*)\|^2 \\
&\quad +2\mu_2 \|x_n - \tilde{x}_n - (x^* - y^*)\| \|B_2 x_n - B_2 x^*\|],
\end{aligned}
$$

that is,

$$
\begin{aligned}
\|\tilde{x}_n - y^*\|^2 \le \|x_n - x^*\|^2 &- \|x_n - \tilde{x}_n - (x^* - y^*)\|^2 \\
&+2\mu_2 \|x_n - \tilde{x}_n - (x^* - y^*)\| \|B_2 x_n - B_2 x^*\|.
\end{aligned}
\tag{3.12}
$$

Further, similarly to the above argument, we derive

$$
\begin{aligned}
&\|z_n - x^*\|^2 = \|P_C(\tilde{x}_n - \mu_1 B_1 \tilde{x}_n) - P_C(y^* - \mu_1 B_1 y^*)\|^2 \\
&\le \langle (\tilde{x}_n - \mu_1 B_1 \tilde{x}_n) - (y^* - \mu_1 B_1 y^*), z_n - x^* \rangle \\
&= \frac{1}{2}[\|\tilde{x}_n - y^* - \mu_1(B_1 \tilde{x}_n - B_1 y^*)\|^2 + \|z_n - x^*\|^2 \\
&\quad -\|(\tilde{x}_n - y^*) - \mu_1(B_1 \tilde{x}_n - B_1 y^*) - (z_n - x^*)\|^2] \\
&\le \frac{1}{2}[\|\tilde{x}_n - y^*\|^2 + \|z_n - x^*\|^2 - \|(\tilde{x}_n - z_n) - \mu_1(B_1 \tilde{x}_n - B_1 y^*) + (x^* - y^*)\|^2] \\
&= \frac{1}{2}[\|\tilde{x}_n - y^*\|^2 + \|z_n - x^*\|^2 - \|\tilde{x}_n - z_n + (x^* - y^*)\|^2 \\
&\quad +2\mu_1 \langle \tilde{x}_n - z_n + (x^* - y^*), B_1 \tilde{x}_n - B_1 y^* \rangle - \mu_1^2 \|B_1 \tilde{x}_n - B_1 y^*\|^2]
\end{aligned}
$$

$$\leq \frac{1}{2}[\|\tilde{x}_n - y^*\|^2 + \|z_n - x^*\|^2 - \|\tilde{x}_n - z_n + (x^* - y^*)\|^2 + 2\mu_1\|\tilde{x}_n - z_n + (x^* - y^*)\|\|B_1\tilde{x}_n - B_1y^*\|],$$

that is,

$$\|z_n - x^*\|^2 \leq \|\tilde{x}_n - y^*\|^2 - \|\tilde{x}_n - z_n + (x^* - y^*)\|^2 + 2\mu_1\|\tilde{x}_n - z_n + (x^* - y^*)\|\|B_1\tilde{x}_n - B_1y^*\|. \tag{3.13}$$

Combining (3.12) with (3.13), we have

$$\begin{aligned} &\|z_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \|x_n - \tilde{x}_n - (x^* - y^*)\|^2 - \|\tilde{x}_n - z_n + (x^* - y^*)\|^2 \\ &\quad +2\mu_1\|\tilde{x}_n - z_n + (x^* - y^*)\|\|B_1\tilde{x}_n - B_1y^*\| \\ &\quad +2\mu_2\|x_n - \tilde{x}_n - (x^* - y^*)\|\|B_2x_n - B_2x^*\|. \end{aligned} \tag{3.14}$$

In terms of (3.8), (3.11) and (3.14) we have

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|Qx_n - x^*\|^2 \\ &\quad +(1-\alpha_n)(1-\beta_n)\sum_{i=1}^{m}\eta_n^i\|v_n^i - x^*\|^2 \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|Qx_n - x^*\|^2 \\ &\quad +(1-\alpha_n)(1-\beta_n)[\|z_n - x^*\|^2 - \|u_n - z_n\|^2 + \sum_{i=1}^{m}\Gamma^i\|A_iz_n - A_ix^*\|] \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|Qx_n - x^*\|^2 \\ &\quad +(1-\alpha_n)(1-\beta_n)[\|x_n - x^*\|^2 \\ &\quad -\|x_n - \tilde{x}_n - (x^* - y^*)\|^2 - \|\tilde{x}_n - z_n + (x^* - y^*)\|^2 \\ &\quad +2\mu_1\|\tilde{x}_n - z_n + (x^* - y^*)\|\|B_1\tilde{x}_n - B_1y^*\| \\ &\quad +2\mu_2\|x_n - \tilde{x}_n - (x^* - y^*)\|\|B_2x_n - B_2x^*\| \\ &\quad -\|u_n - z_n\|^2 + \sum_{i=1}^{m}\Gamma^i\|A_iz_n - A_ix^*\|] \\ &\leq \|x_n - x^*\|^2 + \alpha_n\|Qx_n - x^*\|^2 + \sum_{i=1}^{m}\Gamma^i\|A_iz_n - A_ix^*\| \\ &\quad +2\mu_1\|\tilde{x}_n - z_n + (x^* - y^*)\|\|B_1\tilde{x}_n - B_1y^*\| \\ &\quad +2\mu_2\|x_n - \tilde{x}_n - (x^* - y^*)\|\|B_2x_n - B_2x^*\| \\ &\quad -(1-\alpha_n)(1-\beta_n)[\|x_n - \tilde{x}_n - (x^* - y^*)\|^2 \\ &\quad +\|\tilde{x}_n - z_n + (x^* - y^*)\|^2 + \|u_n - z_n\|^2]. \end{aligned}$$

This in turn implies that

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)[\|x_n-\tilde{x}_n-(x^*-y^*)\|^2\\
&+\|\tilde{x}_n-z_n+(x^*-y^*)\|^2+\|u_n-z_n\|^2]\\
\le\ &\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n\|Qx_n-x^*\|^2+\sum_{i=1}^{m}\Gamma^i\|A_iz_n-A_ix^*\|\\
&+2\mu_1\|\tilde{x}_n-z_n+(x^*-y^*)\|\|B_1\tilde{x}_n-B_1y^*\|\\
&+2\mu_2\|x_n-\tilde{x}_n-(x^*-y^*)\|\|B_2x_n-B_2x^*\|\\
\le\ &(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|\\
&+\alpha_n\|Qx_n-x^*\|^2+\sum_{i=1}^{m}\Gamma^i\|A_iz_n-A_ix^*\|\\
&+2\mu_1\|\tilde{x}_n-z_n+(x^*-y^*)\|\|B_1\tilde{x}_n-B_1y^*\|\\
&+2\mu_2\|x_n-\tilde{x}_n-(x^*-y^*)\|\|B_2x_n-B_2x^*\|.
\end{aligned}$$

From (3.7), (3.9) and conditions (iii) and (iv), we derive

$$\lim_{n\to\infty}\|x_n-\tilde{x}_n-(x^*-y^*)\|=\lim_{n\to\infty}\|\tilde{x}_n-z_n+(x^*-y^*)\|=\lim_{n\to\infty}\|u_n-z_n\|=0.$$

Thus, it is easy to see that

$$\lim_{n\to\infty}\|x_n-z_n\|=\lim_{n\to\infty}\|u_n-z_n\|=0. \tag{3.15}$$

Note that

$$\begin{aligned}
\|S_{\gamma_n}x_n-x_n\|&\le\|S_{\gamma_n}x_n-S_{\gamma_n}y_n\|+\|S_{\gamma_n}y_n-x_n\|\\
&\le\|x_n-y_n\|+\|S_{\gamma_n}y_n-x_n\|\\
&\le\|x_n-z_n\|+\|z_n-y_n\|+\|S_{\gamma_n}y_n-x_n\|\\
&\le\|x_n-z_n\|+\|z_n-u_n\|+\|u_n-y_n\|+\|S_{\gamma_n}y_n-x_n\|\\
&=\|x_n-z_n\|+\|u_n-z_n\|+\alpha_n\|Qx_n-u_n\|+\|S_{\gamma_n}y_n-x_n\|.
\end{aligned}$$

So, it follows from (3.6), (3.15) and condition (iii) that

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|S_{\gamma_n}x_n-x_n\|=0. \tag{3.16}$$

Since $\|S_{\gamma_n}x_n-x_n\|=(1-\gamma_n)\|Sx_n-x_n\|$, this together with condition (v) implies that

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0.$$

Step 5. $\limsup_{n\to\infty}\langle Q\bar{x}-\bar{x},x_n-\bar{x}\rangle\le 0$.

Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle = \lim_{i\to\infty}\langle Q\bar{x}-\bar{x}, x_{n_i}-\bar{x}\rangle. \tag{3.17}$$

Also, since H is reflexive and $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $\hat{x}\in\omega_w(x_n)$. Without loss of generality, we may assume that $x_{n_i}\rightharpoonup\hat{x}$. Define a mapping $F: C\to C$ by

$$Fx=\sum_{i=1}^{m}\eta^i P_C(I-\lambda_i A_i)x, \quad \forall x\in C,$$

where $\eta^i=\lim_{n\to\infty}\eta_n^i$. By Lemma 2.1 we deduce that F is nonexpansive such that

$$\text{Fix}(F)=\bigcap_{i=1}^{m}\text{Fix}(P_C(I-\lambda_i A_i))=\bigcap_{i=1}^{m} VI(C,A_i).$$

Note that

$$\begin{aligned}\|z_n-Fz_n\| &\le \|z_n-u_n\|+\|u_n-Fz_n\|\\ &= \|u_n-z_n\|+\|\sum_{i=1}^{m}\eta_n^i P_C(I-\lambda_i A_i)z_n-\sum_{i=1}^{m}\eta^i P_C(I-\lambda_i A_i)z_n\|\\ &\le \|u_n-z_n\|+\sum_{i=1}^{m}|\eta_n^i-\eta^i|\|P_C(I-\lambda_i A_i)z_n\|\\ &\le \|u_n-z_n\|+M\sum_{i=1}^{m}|\eta_n^i-\eta^i|,\end{aligned}$$

where $M=\max\{\sup\{\|P_C(I-\lambda_i A_i)z_n\|: n\ge 1\}: 1\le i\le m\}$. From condition (ii) and (3.15) we get

$$\lim_{n\to\infty}\|z_n-Fz_n\|=0. \tag{3.18}$$

In the meantime, from (3.15) and $x_{n_i}\rightharpoonup\hat{x}$ we have $z_{n_i}\rightharpoonup\hat{x}$. By Proposition 2.1 (ii) we obtain that $\hat{x}\in\text{Fix}(F)$. Moreover, in view of (3.16) and Proposition 2.1 (ii) we also have $\hat{x}\in\text{Fix}(S)$. This immediately shows that $\hat{x}\in\cap_{i=1}^m VI(A_i,C)\cap\text{Fix}(S)$.

Next, let us show that $\hat{x}\in GSVI(B_1,B_2,C)$. As a matter of fact, observe that

$$\begin{aligned}\|x_n-G(x_n)\| &= \|x_n-P_C[P_C(x_n-\mu_2B_2x_n)-\mu_1B_1P_C(x_n-\mu_2B_2x_n)]\|\\ &= \|x_n-z_n\|\to 0 \quad (n\to\infty)\end{aligned}$$

due to (3.15), where $G: C\to C$ is defined as that in Lemma 1.1. According to Proposition 2.1 (ii) we conclude from $x_{n_i}\rightharpoonup\hat{x}$ that $\hat{x}\in GSVI(B_1,B_2,C)$. Therefore,

$$\hat{x}\in\bigcap_{i=1}^{m} VI(A_i,C)\cap\text{Fix}(S)\cap GSVI(B_1,B_2,C)=\Omega.$$

Consequently, from (2.1) and (3.17) it follows that

$$\limsup_{n\to\infty}\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle = \lim_{i\to\infty}\langle Q\bar{x}-\bar{x}, x_{n_i}-\bar{x}\rangle = \langle Q\bar{x}-\bar{x}, \hat{x}-\bar{x}\rangle \le 0.$$

Step 6. $\lim_{n\to\infty}\|x_n-\bar{x}\| = 0$.

Indeed, utilizing Lemma 2.2 we get from (3.1) and (3.2)

$$\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2\\
&= \|\beta_n(x_n-\bar{x})+(1-\beta_n)(S_{\gamma_n}y_n-\bar{x})\|^2\\
&\le \beta_n\|x_n-\bar{x}\|^2+(1-\beta_n)\|y_n-\bar{x}\|^2\\
&\le \beta_n\|x_n-\bar{x}\|^2+(1-\beta_n)[2\alpha_n\langle Qx_n-\bar{x}, y_n-\bar{x}\rangle+(1-\alpha_n)^2\|u_n-\bar{x}\|^2]\\
&\le \beta_n\|x_n-\bar{x}\|^2+(1-\beta_n)[2\alpha_n\langle Qx_n-\bar{x}, y_n-\bar{x}\rangle+(1-\alpha_n)\|z_n-\bar{x}\|^2]\\
&\le \beta_n\|x_n-\bar{x}\|^2+(1-\beta_n)[2\alpha_n\langle Qx_n-\bar{x}, y_n-\bar{x}\rangle+(1-\alpha_n)\|x_n-\bar{x}\|^2]\\
&= [1-\alpha_n(1-\beta_n)]\|x_n-\bar{x}\|^2+2\alpha_n(1-\beta_n)\langle Qx_n-\bar{x}, y_n-\bar{x}\rangle.
\end{aligned}\tag{3.19}$$

Note that

$$\begin{aligned}
&\langle Qx_n-\bar{x}, y_n-\bar{x}\rangle\\
&= \langle Qx_n-\bar{x}, x_n-\bar{x}\rangle+\langle Qx_n-\bar{x}, y_n-x_n\rangle\\
&= \langle Qx_n-Q\bar{x}, x_n-\bar{x}\rangle+\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle+\langle Qx_n-\bar{x}, y_n-x_n\rangle\\
&\le \rho\|x_n-\bar{x}\|^2+\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle+\|Qx_n-\bar{x}\|\|y_n-x_n\|.
\end{aligned}\tag{3.20}$$

Thus, combining (19) with (20), we have

$$\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2\\
&\le [1-\alpha_n(1-\beta_n)]\|x_n-\bar{x}\|^2+2\alpha_n(1-\beta_n)[\rho\|x_n-\bar{x}\|^2\\
&\quad+\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle+\|Qx_n-\bar{x}\|\|y_n-x_n\|]\\
&= [1-\alpha_n(1-\beta_n)(1-2\rho)]\|x_n-\bar{x}\|^2\\
&\quad+2\alpha_n(1-\beta_n)[\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle+\|Qx_n-\bar{x}\|\|y_n-x_n\|]\\
&= [1-\alpha_n(1-\beta_n)(1-2\rho)]\|x_n-\bar{x}\|^2\\
&\quad+(1-2\rho)\alpha_n(1-\beta_n)\frac{2[\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle+\|Qx_n-\bar{x}\|\|y_n-x_n\|]}{1-2\rho}.
\end{aligned}$$

Note that $\liminf_{n\to\infty}(1-\beta_n)(1-2\rho) > 0$. It follows that $\sum_{n=1}^{\infty}\alpha_n(1-\beta_n)(1-2\rho) = \infty$. It is clear that

$$\limsup_{n\to\infty}\frac{2[\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle+\|Qx_n-\bar{x}\|\|y_n-x_n\|]}{1-2\rho} \le 0$$

because $\limsup_{n\to\infty}\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle \le 0$ and $\lim_{n\to\infty}\|x_n-y_n\| = 0$ (due to (3.16)). Therefore, all conditions of Lemma 2.3 are satisfied. Consequently, we immediately conclude that $\|x_n-\bar{x}\| \to 0$ as $n\to\infty$. This completes the proof. ■

If $Q \equiv u$ a constant in C and S is nonexpansive, then Theorem 3.1 is reduced to the following

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Given an integer $m \geq 1$. Let $A_i : C \to H$ be $\widehat{\alpha}_i$-inverse strongly monotone for $i = 1, 2, ..., m$, and $B_j : C \to H$ be $\widehat{\beta}_j$-inverse strongly monotone for $j = 1, 2$. Let $S : C \to C$ be a nonexpansive mapping such that $\Omega := \cap_{i=1}^m VI(A_i, C) \cap \mathrm{Fix}(S) \cap GSVI(B_1, B_2, C) \neq \emptyset$. For fixed $u \in C$ and given $x_1 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ be generated iteratively by*

$$(3.21) \qquad \begin{cases} z_n = P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\ y_n = \alpha_n u + (1 - \alpha_n) \sum_{i=1}^m [\eta_n^i P_C(z_n - \lambda_i A_i z_n)], \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S y_n, \quad \forall n \geq 1, \end{cases}$$

where $\mu_j \in (0, 2\widehat{\beta}_j)$ for $j = 1, 2$, $\lambda_i \in (0, 2\widehat{\alpha}_i)$ and $\{\alpha_n\}, \{\beta_n\}, \{\eta_n^i\} \subset (0, 1)$ for $i = 1, 2, ..., m$, such that

(i) $\sum_{i=1}^m \eta_n^i = 1, \ \forall n \geq 1$;

(ii) $\lim_{n\to\infty} \eta_n^i = \eta^i \in (0, 1)$ *for* $i = 1, 2, ..., m$;

(iii) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^\infty \alpha_n = \infty$;

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to the same point $\bar{x} = P_\Omega Q \bar{x}$. Furthermore, $(\bar{x}, \bar{y})$ is a solution of the general system (1.2) of variational inequalities, where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

Proof. Put $Q \equiv u$ and $\kappa = \gamma_n = 0$ for each $n \geq 1$. Then, by Theorem 3.1 we obtain the desired result. ■

Finally, we consider the common fixed point problem of a finite family of strict pseudocontractions.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Given an integer $m \geq 1$. Let $T_i : C \to C$ be a k_i-strict pseudocontraction for $i = 1, 2, ..., m$, and $B_j : C \to H$ be $\widehat{\beta}_j$-inverse strongly monotone for $j = 1, 2$. Let $S : C \to C$ be a κ-strict pseudocontraction such that $\Omega := \cap_{i=1}^m \mathrm{Fix}(T_i) \cap \mathrm{Fix}(S) \cap GSVI(B_1, B_2, C) \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, \frac{1}{2})$. For given $x_1 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ be generated iteratively by*

$$(3.22) \qquad \begin{cases} z_n = P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\ y_n = \alpha_n Q x_n + (1 - \alpha_n) \sum_{i=1}^m \eta_n^i ((1 - \lambda_i) z_n + \lambda_i T_i z_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)[\gamma_n y_n + (1 - \gamma_n) S y_n], \quad \forall n \geq 1, \end{cases}$$

where $\{\gamma_n\} \subset [\kappa, \gamma)$ *for some* $\gamma \in (\kappa, 1)$, $\mu_j \in (0, 2\widehat{\beta}_j)$ *for* $j = 1, 2$, $\lambda_i \in (0, 1 - k_i)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\eta_n^i\} \subset (0, 1)$ *for* $i = 1, 2, ..., m$, *such that*

(*i*) $\sum_{i=1}^{m} \eta_n^i = 1, \ \forall n \geq 1$;

(*ii*) $\lim_{n\to\infty} \eta_n^i = \eta^i \in (0, 1)$ *for* $i = 1, 2, ..., m$;

(*iii*) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(*iv*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(*v*) $\lim_{n\to\infty}(\gamma_{n+1} - \gamma_n) = 0$ *and* $\limsup_{n\to\infty} \gamma_n < 1$.

Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ *converge strongly to the same point* $\bar{x} = P_\Omega Q\bar{x}$. *Furthermore,* $(\bar{x}, \bar{y})$ *is a solution of the general system (1.2) of variational inequalities, where* $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

Proof. In Theorem 3.1, put $A_i = I - T_i$ for $i = 1, 2, ..., m$. Then it is easy to see that A_i is $\widehat{\alpha}_i$-inverse strongly monotone with $\widehat{\alpha}_i = \frac{1-k_i}{2}$ for $i = 1, 2, ..., m$ and that $\mathrm{Fix}(T_i) = VI(A_i, C)$ for $i = 1, 2, ..., m$. Note that $\lambda_i \in (0, 2\widehat{\alpha}_i) = (0, 1 - k_i)$ and

$$P_C(z_n - \lambda_i A_i z_n) = (1 - \lambda_i)z_n + \lambda_i T_i z_n$$

for $i = 1, 2, ..., m$. Thus, by Theorem 3.1 we obtain the desired result. This completes the proof. ∎

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