TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 2, pp. 517-528, April 2013
DOI: 10.11650/tjm.17.2013.2122
This paper is available online at http://journal.taiwanmathsoc.org.tw

A GENERAL L_p-VERSION OF PETTY'S AFFINE PROJECTION INEQUALITY

Wang Weidong and Feng Yibin

Abstract. About a decade ago Lutwak, Yang, and Zhang introduced the notion of L_p-projection body. More recently, Wang and Leng established an L_p-version of Petty's affine projection inequality. At the same time Ludwig discovered a family of general L_p-projection bodies and Haberl and Schuster established Petty's projection inequality for general L_p-projection bodies. In this paper we establish a general L_p-version of Petty's affine projection inequality for general L_p-projection bodies. Moreover, we obtain an analogous inequality for L_p-geominimal surface area.

1. INTRODUCTION

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, we write $\mathcal{K}_o^n$. Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Let S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ and $V(K)$ the n-dimensional volume of the body K. For the standard unit ball B in $\mathbb{R}^n$ let $\omega_n = V(B)$.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \longrightarrow (-\infty, +\infty)$, is defined by (see [7])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \qquad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

The classical projection bodies were introduced by Minkowski at the turn of the previous century. For each $K \in \mathcal{K}^n$, the projection body, ΠK, of K is an origin-symmetric convex body whose support function is defined by (see [7])

$$h_{\Pi K}(u) = \frac{1}{2} \int_{S^{n-1}} | u \cdot v | \, dS(K, v),$$

Received June 9, 2012, accepted August 23, 2012.
Communicated by Sun-Yung Alice Chang.
2010 *Mathematics Subject Classification*: 52A40, 52A20.
Key words and phrases: General L_p-projection body, L_p-Petty affine projection inequality, L_p-Affine surface area, L_p-Geominimal surface area.
Research is supported in part by the Natural Science Foundation of China (Grant No. 10671117) and Science Foundation of China Three Gorges University.

for all $u \in S^{n-1}$. Here $S(K, \cdot)$ denotes the classical surface area measure of K. Projection bodies are a very important object for the study of projections in the Brunn-Minkowski theory. During the past four decades, a number of important results regarding classical projection bodies were obtained (see e.g. [1-4, 12, 13, 18, 20, 27, 28, 42] or the book [7]).

The notion of L_p-projection body was introduced by Lutwak, Yang, and Zhang (see [23]). For each $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-projection body, $\Pi_p K$, of K is defined as the origin-symmetric convex body whose support function is given by

$$h^p_{\Pi_p K}(u) = \alpha_{n,p} \int_{S^{n-1}} | u \cdot v |^p \, dS_p(K, v). \tag{1.1}$$

for all $u \in S^{n-1}$. Here

$$\alpha_{n,p} = \frac{1}{n\omega_n c_{n-2,p}} \tag{1.2}$$

with $c_{n,p} = \omega_{n+p}/\omega_2\omega_n\omega_{p-1}$, and $S_p(K, \cdot)$ is a positive Borel measure on S^{n-1}, called the L_p-surface area measure of $K \in \mathcal{K}_o^n$ (see [21]). It turns out that the measure $S_p(K, \cdot)$ is absolutely continuous with respect to the classical surface area measure $S(K, \cdot)$ of K, and has Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h(K, \cdot)^{1-p}. \tag{1.3}$$

The unusual normalization of definition (1.1) is chosen so that for the unit ball B, we have $\Pi_p B = B$. In particular, for $p = 1$, the convex body $\Pi_1 K$ is the classical projection body ΠK of K under the normalization of definition (1.1).

L_p-projection bodies extended the classical projection bodies from the Brunn-Minkowski theory to the L_p-Brunn-Minkowski theory. L_p-projection bodies have been investigated intensively in recent years, see [8, 11, 14, 26, 30, 34-37, 39].

For $p \geq 1$, Ludwig discovered in [14] a new notion of asymmetric L_p-projection bodies, $\Pi_p^+ K$, of $K \in \mathcal{K}_o^n$, defined by

$$h^p_{\Pi_p^+ K}(u) = 2\alpha_{n,p} \int_{S^{n-1}} (u \cdot v)_+^p dS_p(K, v), \tag{1.4}$$

where $(u \cdot v)_+ = \max\{u \cdot v, 0\}$. From (1.2) and (1.4), we see that $\Pi_p^+ B = B$. In [8] Haberl and Schuster also defined

$$\Pi_p^- K = \Pi_p^+(-K).$$

Moreover, the authors of [14, 8] introduced a function $\varphi_\tau : \mathbb{R} \longrightarrow [0, +\infty)$, given by

$$\varphi_\tau(t) = |t| + \tau t,$$

for $\tau \in [-1,1]$, and for $K \in \mathcal{K}_o^n$, $p \geq 1$, they defined $\Pi_p^\tau K \in \mathcal{K}_o^n$ via its support function by

$$h_{\Pi_p^\tau K}^p(u) = \alpha_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p dS_p(K,v), \tag{1.5}$$

where

$$\alpha_{n,p}(\tau) = \frac{2\alpha_{n,p}}{(1+\tau)^p + (1-\tau)^p}.$$

The normalization is chosen such that $\Pi_p^\tau B = B$ for every $\tau \in [-1,1]$. The family of convex bodies $\Pi_p^\tau K$ is called the general L_p-projection bodies of K. Obviously, if $\tau = 0$ then $\Pi_p^\tau K = \Pi_p K$.

For the general L_p-projection bodies, Haberl and Schuster (see [8]) proved a general version of the L_p-Petty projection inequality:

Theorem 1.A. *If $K \in \mathcal{K}_o^n$, $p \geq 1$, then for every $\tau \in [-1,1]$,*

$$V(K)^{(n-p)/p} V(\Pi_p^{\tau,*} K) \leq \omega_n^{n/p}, \tag{1.6}$$

with equality if and only if K is an ellipsoid centered at the origin.

Here, $\Pi_p^{\tau,*} K$ denotes the polar of the general L_p-projection body $\Pi_p^\tau K$. If $\tau = 0$, then inequality (1.6) is just the L_p-Petty projection inequality which was established by Lutwak, Yang and Zhang (see [23]). If $\tau = 0$ and $p = 1$, then inequality (1.6) becomes the classical Petty projection inequality (see [28]) under the normalization of definition (1.1).

The classical Petty projection inequality and its L_p-extension have become a major focus in different areas. For example, the family of L_p-Petty projection inequalities has been used to establish a number of affine analytic inequalities, see [5, 9, 10, 24, 25, 31, 32].

Associated with the L_p-projection bodies, Wang and Leng established in [37] an L_p-version of Petty's affine projection inequality:

Theorem 1.B. *If $K \in \mathcal{F}_o^n$ and $p \geq 1$, then*

$$\Omega_p(K) \leq n\omega_n^{\frac{n}{n+p}} V(\Pi_p K)^{\frac{p}{n+p}}, \tag{1.7}$$

with equality if and only if K is an ellipsoid centered at the origin.

Here $\mathcal{F}_o^n$ denotes the set of convex bodies in $\mathcal{K}_o^n$ with positive continuous curvature function, and $\Omega_p(K)$ denotes the L_p-affine surface area of K (see Section 2).

Note that for $p = 1$, inequality (1.7) is just Petty's affine projection inequality (see [28]) under the normalization of definition (1.1).

In this paper, we continue to investigate the family of general L_p-projection bodies. First, we extend inequality (1.7), to obtain the following general L_p-version of Petty's affine projection inequality.

Theorem 1.1. *If $K \in \mathcal{F}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, then*

$$\Omega_p(K) \leq n\omega_n^{\frac{n}{n+p}} V(\Pi_p^\tau K)^{\frac{p}{n+p}}, \tag{1.8}$$

with equality if and only if K is an ellipsoid centered at the origin.

Note that if $\tau = 0$, then inequality (1.8) is just inequality (1.7).

We also establish a general version of the L_p-geominimal surface area inequality.

Theorem 1.2. *If $K \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, then*

$$G_p(K) \leq n\omega_n^{\frac{n-p}{n}} V(\Pi_p^\tau K)^{\frac{p}{n}}, \tag{1.9}$$

with equality if and only if $\Pi_p^\tau K$ is an ellipsoid centered at the origin.

Here, $G_p(K)$ denotes the L_p-geominimal surface area of $K \in \mathcal{K}_o^n$ (see Section 2).

From Theorem 1.2 and a combination of the definitions of L_p-affine surface area and L_p-geominimal surface area, we obtain a further extension of Theorem 1.1:

Corollary 1.1. *If $K \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, then*

$$\Omega_p(K) \leq n\omega_n^{\frac{n}{n+p}} V(\Pi_p^\tau K)^{\frac{p}{n+p}}, \tag{1.10}$$

with equality if and only if $\Pi_p^\tau K$ is an ellipsoid centered at the origin.

Let $\mathcal{K}_c^n$ denote the set of convex bodies whose centroid is at the origin. If $K \in \mathcal{K}_c^n$, then the equality conditions of inequality (1.9) may be improved as follows:

Theorem 1.3. *If $K \in \mathcal{K}_c^n$, $p \geq 1$, $\tau \in [-1,1]$, then*

$$G_p(K) \leq n\omega_n^{\frac{n-p}{n}} V(\Pi_p^\tau K)^{\frac{p}{n}}, \tag{1.11}$$

with equality if and only if K is an ellipsoid centered at the origin.

Similarly, we obtain the following generalization of Theorem 1.3:

Corollary 1.2. *If $K \in \mathcal{K}_c^n$, $p \geq 1$, $\tau \in [-1,1]$, then*

$$\Omega_p(K) \leq n\omega_n^{\frac{n}{n+p}} V(\Pi_p^\tau K)^{\frac{p}{n+p}},$$

with equality if and only if K is an ellipsoid centered at the origin.

2. Basic Notions

2.1. Radial Function and Polar Bodies

If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \backslash \{0\} \longrightarrow [0,+\infty)$, is defined by (see [7])

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \qquad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $E \subseteq \mathbb{R}^n$ is nonempty, the polar set of E, E^*, is defined by (see [7])

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in E\}.$$

For $K \in \mathcal{K}_c^n$ and its polar body, the well-known Blaschke-Santaló inequality can be stated as follows (see [29]):

Theorem 2.A. *If $K \in \mathcal{K}_c^n$, then*

$$V(K)V(K^*) \leq \omega_n^2, \tag{2.1}$$

with equality if and only if K is an ellipsoid.

Note that an extension of the Blaschke-Santaló inequality (2.1) to star bodies whose centroid is at the origin was obtained by Lutwak (see[19]).

2.2. L_p-Mixed Volume

For $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the Firey L_p-combination, $\lambda \cdot K +_p \mu \cdot L \in \mathcal{K}_o^n$, of K and L is defined by (see [6])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K,\cdot)^p + \mu h(L,\cdot)^p,$$

where " $\cdot$ " in $\lambda \cdot K$ denotes the Firey scalar multiplication.

Associated with the Firey L_p-combination of convex bodies, Lutwak ([21]) introduced for $K, L \in \mathcal{K}_o^n$, $\varepsilon > 0$ and $p \geq 1$, the L_p-mixed volume, $V_p(K,L)$, of K and L, defined by

$$\frac{n}{p} V_p(K,L) = \lim_{\varepsilon \longrightarrow 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Corresponding to each $K \in \mathcal{K}_o^n$, Lutwak ([21]) proved that there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1} such that

$$V_p(K,L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(v) dS_p(K,v) \tag{2.2}$$

for each $L \in \mathcal{K}_o^n$. The measure $S_p(K,\cdot)$ is called the L_p-surface area measure of K.

From formula (2.2) and (1.3), it follows immediately that for each $K \in \mathcal{K}_o^n$,

$$V_p(K,K) = \frac{1}{n}\int_{S^{n-1}} h_K(v)dS(K,v) = V(K). \tag{2.3}$$

2.3. L_p-Affine Surface Area and L_p-Geominimal Surface Area

A convex body $K \in \mathcal{K}_o^n$ is said to have an L_p-curvature function (see [22]) $f_p(K,\cdot) : S^{n-1} \longrightarrow \mathbb{R}$, if its L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$f_p(K,\cdot) = \frac{dS_p(K,\cdot)}{dS}. \tag{2.4}$$

Using definition (2.4) of the L_p-curvature function, Lutwak [22] defined the notion of L_p-curvature image as follows: For each $K \in \mathcal{F}_o^n$ and real $p \geq 1$, define $\Lambda_p K \in \mathcal{S}_o^n$, the L_p-curvature image of K, by

$$\rho(\Lambda_p K,\cdot)^{n+p} = \frac{V(\Lambda_p K)}{\omega_n} f_p(K,\cdot).$$

The notion of L_p-affine surface area was introduced by Lutwak (see [22]). For each $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-affine surface area, $\Omega_p(K)$, of K is defined by

$$n^{-\frac{p}{n}}\Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K,Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\}. \tag{2.5}$$

Moreover, Lutwak in [22] proved that if $K \in \mathcal{F}_o^n$ and $p \geq 1$, then the L_p-affine surface area of K has the integral representation

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}} dS(u). \tag{2.6}$$

L_p-affine surface area is very important in the L_p-Brunn-Minkowski theory, see [15-17, 33, 40, 41].

In [22] Lutwak also introduced the notion of L_p-geominimal surface area. For each $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-geominimal surface area, $G_p(K)$, of K is be defined by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf\{nV_p(K,Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}. \tag{2.7}$$

For the study of L_p-geominimal surface area, apart from [22], also see [43, 38].

From definitions (2.5) and (2.7), the following fact can be obtained (see [22]):

Theorem 2.B. *If $K \in \mathcal{K}_o^n$, $p \geq 1$, then*

$$\Omega_p(K)^{n+p} \leq (n\omega_n)^p G_p(K)^n. \tag{2.8}$$

3. The General L_p-Version of Petty's Affine Projection Inequality

In this section, we give the proofs of Theorem 1.1-1.3. The proof of Theorem 1.1 requires the following lemmas.

Lemma 3.1. *If $K,\ L \in \mathcal{K}_o^n,\ p \geq 1,\ \tau \in [-1,1]$, then*

$$V_p(K, \Pi_p^\tau L) = V_p(L, \Pi_p^\tau K). \tag{3.1}$$

Proof. From (1.5) and (2.2), we easily obtain

$$\begin{aligned} V_p(L, \Pi_p^\tau K) &= \frac{1}{n}\int_{S^{n-1}} h^p_{\Pi_p^\tau K}(u)dS_p(L,u) \\ &= \frac{1}{n}\int_{S^{n-1}} \alpha_{n,p}(\tau)\int_{S^{n-1}} \varphi_\tau(u\cdot v)^p dS_p(K,v)dS_p(L,u) \\ &= \frac{1}{n}\int_{S^{n-1}} h^p_{\Pi_p^\tau L}(v)dS_p(K,v) \\ &= V_p(K, \Pi_p^\tau L). \end{aligned}$$ ∎

Using (2.6), Wang and Leng in [37] proved the following result.

Lemma 3.2. *If $K \in \mathcal{F}_o^n,\ L \in \mathcal{K}_o^n$ and $p \geq 1$, then*

$$\Omega_p(K) \leq nV_p(K,L)^{\frac{n}{n+p}} V(L^*)^{\frac{p}{n+p}}, \tag{3.2}$$

with equality if and only if $\Lambda_p K$ and L^* are dilates.

Note that for $K, L \in \mathcal{K}_o^n$, inequality (3.2) can immediately be deduced from definition (2.5), however without the equality conditions.

Lemma 3.3. *If $K \in \mathcal{F}_o^n,\ L \in \mathcal{K}_o^n$ and $p \geq 1$, then*

$$\Omega_p(K)V(L)^{\frac{n-p}{n+p}} \leq n\omega_n^{\frac{n}{n+p}} V_p(K, \Pi_p^\tau L)^{\frac{n}{n+p}}, \tag{3.3}$$

with equality if and only if K and L are dilates of the same ellipsoid centered at the origin.

Proof. Taking $\Pi_p^\tau L$ for L in inequality (3.2), we obtain

$$\Omega_p(K) \leq nV_p(K, \Pi_p^\tau L)^{\frac{n}{n+p}} V(\Pi_p^{\tau,*} L)^{\frac{p}{n+p}}, \tag{3.4}$$

with equality if and only if $\Lambda_p K$ and $\Pi_p^{\tau,*} L$ are dilates.

Using the general L_p-Petty projection inequality (1.6), we have

$$\begin{aligned}\Omega_p(K)V(L)^{\frac{n-p}{n+p}} &\leq nV_p(K,\Pi_p^\tau L)^{\frac{n}{n+p}}[V(L)^{n-p}V(\Pi_p^{\tau,*}L)^p]^{\frac{1}{n+p}}\\ &\leq n\omega_n^{\frac{n}{n+p}}V_p(K,\Pi_p^\tau L)^{\frac{n}{n+p}}.\end{aligned}$$

Equality holds in inequality (1.6) if and only if L is an ellipsoid centered at the origin. This together with the condition under which equality holds in inequality (3.4), shows that equality holds in inequality (3.3) if and only if K and L are dilates of the same ellipsoid centered at the origin. ■

Proof of Theorem 1.1. For $K \in \mathcal{F}_o^n$, $L \in \mathcal{K}_o^n$, inequality (3.3) states that

$$\Omega_p(K)V(L)^{\frac{n-p}{n+p}} \leq n\omega_n^{\frac{n}{n+p}}V_p(K,\Pi_p^\tau L)^{\frac{n}{n+p}}. \tag{3.5}$$

Using (3.1), we have

$$\Omega_p(K)V(L)^{\frac{n-p}{n+p}} \leq n\omega_n^{\frac{n}{n+p}}V_p(L,\Pi_p^\tau K)^{\frac{n}{n+p}}, \tag{3.6}$$

where equality holds in (3.5) and (3.6) if and only if K and L are dilates of the same ellipsoid centered at the origin by the condition under which equality holds in (3.3).

Taking $L = \Pi_p^\tau K$ in inequality (3.6), and using (2.3), we get inequality (1.8). According to the conditions under which equality holds in (3.6), we easily see that equality holds in (1.8) if and only if K is an ellipsoid centered at the origin. ■

Proof of Theorem 1.2. For $L \in \mathcal{K}_o^n$, taking $Q = \Pi_p^\tau L$ in (2.7) and using (3.1), we have

$$\begin{aligned}\omega_n^{\frac{p}{n}}G_p(K) &\leq nV_p(K,\Pi_p^\tau L)V(\Pi_p^{\tau,*}L)^{\frac{p}{n}}\\ &= nV_p(L,\Pi_p^\tau K)V(\Pi_p^{\tau,*}L)^{\frac{p}{n}}.\end{aligned}$$

Taking $L = \Pi_p^\tau K$ and using (2.3), we get

$$\omega_n^{\frac{p}{n}}G_p(K) \leq nV(\Pi_p^\tau K)V(\Pi_p^{\tau,*}\Pi_p^\tau K)^{\frac{p}{n}},$$

which together with the general L_p-Petty projection inequality (1.6), then yields

$$\begin{aligned}\omega_n G_p(K)^{\frac{n}{p}} &\leq n^{\frac{n}{p}}V(\Pi_p^\tau K)[V(\Pi_p^\tau K)^{\frac{n-p}{p}}V(\Pi_p^{\tau,*}\Pi_p^\tau K)]\\ &\leq n^{\frac{n}{p}}\omega_n^{\frac{n}{p}}V(\Pi_p^\tau K).\end{aligned} \tag{3.7}$$

From (3.7), we now obtain inequality (1.9).

According to the conditions of equality in inequality (1.6), we know that equality holds in the second inequality of (3.7) if and only if $\Pi_p^\tau K$ is an ellipsoid centered at the origin. ■

Proof of Corollary 1.1. Using inequalities (1.9) and (2.8), we immediately get inequality (1.10). ■

Proof of Theorem 1.3. By Theorem 1.2, if $K \in \mathcal{K}_c^n$, then inequality (1.11) is true, and equality holds in (1.11) if and only if $\Pi_p^\tau K$ is an ellipsoid centered at the origin. In this case the Blaschke-Santaló inequality (2.1) yields

$$V(\Pi_p^\tau K)V(\Pi_p^{\tau,*}K) = \omega_n^2. \tag{3.8}$$

But equality in inequality (1.11) implies that

$$G_p(K) = n\omega_n^{\frac{n-p}{n}} V(\Pi_p^\tau K)^{\frac{p}{n}},$$

from this and definition (2.7), we obtain for any $Q \in \mathcal{K}_o^n$,

$$\omega_n^{\frac{p}{n}} n\omega_n^{\frac{n-p}{n}} V(\Pi_p^\tau K)^{\frac{p}{n}} \le nV_p(K,Q)V(Q^*)^{\frac{p}{n}}. \tag{3.9}$$

Take $Q = K$ in (3.9) and use the Blaschke-Santaló inequality (2.1) to get

$$\omega_n V(\Pi_p^\tau K)^{\frac{p}{n}} \le V(K)V(K^*)^{\frac{p}{n}} \le \omega_n^{\frac{2p}{n}} V(K)^{\frac{n-p}{n}}.$$

Thus

$$V(\Pi_p^\tau K)^{\frac{p}{n}} \le \omega_n^{\frac{2p-n}{n}} V(K)^{\frac{n-p}{n}}. \tag{3.10}$$

From (3.8) and (3.10), we see that

$$\left(\frac{\omega_n^2}{V(\Pi_p^{\tau,*}K)}\right)^{\frac{p}{n}} \le \omega_n^{\frac{2p-n}{n}} V(K)^{\frac{n-p}{n}},$$

i.e.,

$$V(K)^{\frac{n-p}{p}} V(\Pi_p^{\tau,*}K) \ge \omega_n^{\frac{n}{p}}.$$

This together with the general L_p-Petty projection inequality (1.6) yields

$$V(K)^{\frac{n-p}{p}} V(\Pi_p^{\tau,*}K) = \omega_n^{\frac{n}{p}},$$

which is possible only if K is an ellipsoid centered at the origin.

Therefore, we know that equality holds in inequality (1.11) if and only if K is an ellipsoid centered at the origin. ■

From inequalities (1.11) and (2.8), we easily obtain Corollary 1.2.

Acknowledgment

The authors wish to thank the referees for their very helpful comments and suggestions on this paper.

References

1. E. D. Bolker, A class of convex bodies, *Trans. Amer. Math. Soc.*, **145** (1969), 323-345.
2. J. Bourgain and J. Lindenstrauss, Projection bodies, Geometric Aspects of Functional Analysis, *Springer Lecture Notes in Math.*, **1317** (1988), 250-270.
3. N. S. Brannen, Volumes of projection bodies, *Mathematika*, **43** (1996), 255-264.
4. G. D. Chakerian and E. Lutwak, Bodies with similar projections, *Trans. Amer. Math. Soc.*, **349** (1997), 1811-1820.
5. A. Cianchi, E. Lutwak, D. Yang and G. Zhang, Affine Moser-Trudinger and Morrey-Sobolev inequalities, *Calc. Var. Partial Differ. Equ.*, **36** (2009), 419-436.
6. W. J. Firey, p-means of convex bodies, *Math. Scand.*, **10** (1962), 17-24.
7. R. J. Gardner, *Geometric Tomography*, Cambridge Univ. Press, Cambridge, UK, 2nd ed., 2006.
8. C. Haberl and F. Schuster, General L_p affine isoperimetric inequalities, *J. Differential Geom.*, **83** (2009), 1-26.
9. C. Haberl and F. E. Schuster, Asymmetric affine L_p Sobolev inequalities, *J. Funct. Anal.*, **257** (2009), 641-658.
10. C. Haberl, F. E. Schuster and J. Xiao, An Asymmetric Affine Pólya-Szegö Principle, *Math. Ann.*, **352** (2012), 517-542.
11. C. Q. Hu, X. N. Ma and Ch. L. Shen, On the Christoffel-Minkowski problem of Firey's p-sum, *Calc. Var. Partial Differential Equations*, **21** (2004), 137-155.
12. G. S. Leng, C. J. Zhao, B. W. He and X. Y. Li, Inequalities for polars of mixed projection bodies, *Sci. China Ser. A*, **47(2)** (2004), 175-186.
13. M. Ludwig, Projection bodies and valuations, *Adv. Math.*, **172** (2002), 158-168.
14. M. Ludwig, Minkowski valuations, *Trans. Amer. Math. Soc.*, **357** (2005), 4191-4213.
15. M. Ludwig, General affine surface areas, *Adv. Math.*, **224** (2010), 2346-2360.
16. M. Ludwig and M. Reitzner, A characterization of affine surface area, *Adv. Math.*, **147** (1999), 138-172.
17. M. Ludwig and M. Reitzner, A classification of $SL(n)$ invariant valuations, *Ann. of Math.*, **172** (2010), 1223-1271.
18. E. Lutwak, Mixed projection inequalities, *Trans. Amer. Math. Soc.*, **287** (1985), 91-106.
19. E. Lutwak, Extended affine surface area, *Adv. Math.*, **85** (1991), 39-68.

20. E. Lutwak, Inequalities for mixed projection bodies, *Amer. Math. Soc.*, **339** (1993), 901-916.

21. E. Lutwak, The Brunn-Minkowski-Firey theory I: mixed volumes and the Minkowski problem, *J. Differential Geom.*, **38** (1993), 131-150.

22. E. Lutwak, The Brunn-Minkowski-Firey theory II: affine and geominimal surface areas, *Adv. Math.*, **118** (1996), 244-294.

23. E. Lutwak, D. Yang and G. Y. Zhang, L_p affine isoperimetric inequalities, *J. Differential Geom.*, **56** (2000), 111-132.

24. E. Lutwak, D. Yang and G. Y. Zhang, Sharp affine L_p Sobolev inequalities, *J. Differential Geom.*, **62** (2002), 17-38.

25. E. Lutwak, D. Yang and G. Y. Zhang, Orlicz projection bodies, *Adv. Math.*, **223** (2010), 220-242.

26. S. J. Lv and G. S. Leng, The L_p-curvature images of convex bodies and L_p-projection bodies, *Proc. Indian Acad. Sci. Math. Sci.*, **118** (2008), 413-424.

27. C. M. Petty, Projection bodies, *Proc. Coll. Convexity*, Copenhagen, 1965, Kϕbenhavns Univ. Math. Inst., 1967, pp. 234-241.

28. C. M. Petty, Isoperimetric problems, *Proc. Conf. Convexty and Combinatorial Geometry* (*Univ. Oklahoma, 1971*), University of Oklahoma, 1972, pp. 26-41.

29. C. M. Petty, Affine isoperimetric problems, *Ann. N. Y. Acad. Sci.*, **440** (1985), 113-127.

30. D. Ryabogin and A. Zvavitch, The Fourier transform and Firey projections of convex bodies, *Indiana. Univ. Math. Journal*, **53** (2004), 667-682.

31. F. E. Schuster, Volume inequalities and additive maps of convex bodies, *Mathematika*, **53** (2006), 211-234.

32. F. E. Schuster and T. Wannerer, $GL(n)$ contravariant Minkowski valuations, *Trans. Amer. Math. Soc.*, **364** (2012), 815-826.

33. A. Stancu and E. Werner, New higher-order equiaffine invariants, *Israel J. Math.*, **171** (2009), 221-235.

34. W. D. Wang, F. H. Lu and G. S. Leng, A type of monotonicity on the L_p centroid body and L_p projection body, *Math. Inequal. Appl.*, **8(4)** (2005), 735-742.

35. W. D. Wang and G. S. Leng, The Petty projection inequality for L_p-mixed projection bodies, *Acta Math. Sinica (English Series)*, **23(8)** (2007), 1485-1494.

36. W. D. Wang and G. S. Leng, On the L_p-versions of the Petty's conjectured projection inequality and applications, *Taiwan J. Math.*, **12(5)** (2008), 1067-1086.

37. W. D. Wang and G. S. Leng, Some affine isoperimetric inequalities associated with L_p-affine surface area, *Houston J. Math.*, **34(2)** (2008), 443-453.

38. W. D. Wang and C. Qi, L_p-Dual geominimal surface area, *J. Inequal. Appl.*, **6** (2011), 1-10.

39. W. D. Wang and X. Y. Wan, Shephard type problems for general L_p-projection bodies, *Taiwan J. Math.*, **16(5)** (2012), 1749-1762.

40. E. Werner, On L_p-affine surface areas, *Indiana Univ. Math. J.*, **56** (2007), 2305-2323.

41. E. Werner and D. Ye, New L_p affine isoperimetric inequalities, *Adv. Math.*, **218** (2008), 762-780.

42. G. Y. Zhang, Restricted chord projection and affine inequalities, *Geom. Dedicata*, **39** (1991), 213-222.

43. B. Zhu, L. Ni and J. Zhou, Isoperimetric inequalities for L_p geominimal surface area, *Glasg. Math. J.*, **53** (2011), 717-726.

Wang Weidong and Feng Yibin
Department of Mathematics
China Three Gorges University
Yichang 443002
P. R. China
E-mail: wdwxh722@163.com