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WEIGHTED ESTIMATES FOR VECTOR-VALUED COMMUTATORS OF MULTILINEAR OPERATORS

Zengyan Si* and Qingying Xue

Abstract. Let T be the multilinear Calderón-Zygmund operator and $T_q(\vec{f})$ be the vector-valued version of T given by $T_q(\vec{f})(x) = \left(\sum_{k=1}^{\infty} |T(f_{1k}, \cdots, f_{mk})(x)|^q\right)^{1/q}$. In this paper, the weighted strong type and weighted end-point weak type estimates for the commutators of $T_q(\vec{f})$ were established respectively.

1. INTRODUCTION

Multilinear Calderón-Zygmund operators were introduced and first studied by Coifman and Meyer [1-3], and later on by Grafakos and Torres [6, 7]. In analogy with the linear theory, the class of multilinear singular integrals with standard Calderón-Zygmund kernels provides a fundamental topic of investigation within the framework of the general theory. The study of this subject was recently enjoyed a resurgence of renewed interest and activity.

Let $K(x, y_1, \cdots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$. For constants $A > 0$ and $\varepsilon \in (0, 1]$, we say that K is a kernel in m-$CZK(A, \varepsilon)$ if it satisfies

(1) the size condition

$$\left|K(x, y_1, \cdots, y_m)\right| \leq \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}}$$

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*Corresponding author.

for all $(x, y_1, \cdots, y_m) \in (\mathbb{R}^n)^{m+1}$ satisfying $x \neq y_j$ for some $1 \leq j \leq m$;

(2) the regularity conditions

$$|K(x, y_1, ..., y_m) - K(x', y_1, ..., y_m)| \leq \frac{A|x - x'|^{\varepsilon}}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\varepsilon}}$$

whenever $2|x - x'| \leq \max_{1 \leq k \leq m} |x - y_k|$ and, for each fixed $k \in \{1, \cdots, m\}$,

$$|K(x, y_1, ..., y_k, ..., y_m) - K(x, y_1, ..., y_k', ..., y_m)| \leq \frac{A|y_k - y_k'|^{\varepsilon}}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\varepsilon}}$$

whenever $2|y_k - y_k'| \leq \max_{1 \leq j \leq m} |x - y_j|$. An operator T, defined on m-fold product of Schwartz spaces and taking values into the space of tempered distributions, is said to be an m-linear Calderón-Zygmund operator with kernel K if

(a) T is m-linear;

(b) for $q_1, \cdots, q_m \in [1, \infty]$ and $q \in (0, \infty)$ with $1/q = \sum_{k=1}^m 1/q_k$, T can be extended to be a bounded operator from $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$;

(c) for $f_1, \cdots, f_m \in L^2(\mathbb{R}^n)$ with compact support and $x \notin \bigcap_{k=1}^m \mathrm{supp} f_k$,

$$T\vec{f}(x) = T(f_1, \cdots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m,$$

where K is in m-$CZK(A, \gamma)$ for some constant A and ε.

Given a collection of locally integrable functions $\vec{b} = (b_1, \cdots, b_l)$, where $1 \leq l \leq m$. The commutators associated with T is defined by

$$T_{\Pi\vec{b}}(\vec{f})(x) = [b_l, [b_{l-1}, \cdots, [b_1, T]^1 \cdots]^{l-1}]^l (f_1, \cdots, f_m)(x),$$

where b is a suitable function and

$$[b, T]^k(\vec{f})(x) = b(x)T(f_1, \cdots, f_m)(x) - T(f_1, \cdots, f_{k-1}, bf_k, f_{k+1}, \cdots, f_m)(x).$$

If T is associated with a distribution kernel, which coincides with the function K defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, then, at formal level,

$$T_{\Pi\vec{b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{l} [b_j(x) - b_j(y_j)] K(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m.$$

whenever $x \notin \bigcap_{j=1}^m \mathrm{supp} f_j$ and $f_1, \cdots, f_m$ are C^∞ functions with compact support.

Recently, Lerner, Ombrosi, Pérez, Torres, and Trujillo-González [9] developed a multiple weight theory. Precisely, for $\vec{p} = (p_1, \cdots, p_m)$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{P_m}$ with $1 \le p_1, \cdots, p_m < \infty$. Given $\vec{\omega} = (\omega_1, \cdots, \omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^{m} \omega_i^{p/p_i}$. We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \prod_{i=1}^{m} \omega_i^{\frac{p}{p_i}} \right)^{\frac{1}{p}} \prod_{i=1}^{m} \left(\frac{1}{|Q|} \int_Q \omega_i^{1-p_i'} \right)^{\frac{1}{p_i'}} < \infty,$$

when $p_i = 1$, $\left(\frac{1}{|Q|} \int_Q \omega_i^{1-p_i'} \right)^{1/p_i'}$ is understood as $(\inf_Q \omega_i)^{-1}$.

The weighted strong type and end-point estimates for $T_{\Pi\vec{b}}$ with multiple weights were established.

Theorem A. ([10]). *Let $\vec{\omega} \in A_{\vec{p}}, 1/p = 1/p_1 + \cdots + 1/p_m$ with $1 < p_j < \infty, j = 1, \cdots, m$; and $\vec{b} \in (BMO)^m$. Then there is a constant $C > 0$ independent of $\vec{b}$ and $\vec{f}$ such that*

$$||T_{\Pi\vec{b}}(\vec{f})||_{L^p(\nu_{\vec{\omega}})} \le C \prod_{j=1}^{m} ||b_j||_{BMO} \prod_{j=1}^{m} ||f_j||_{L^p(\omega_j)},$$

where $\vec{b} = (b_1, \cdots, b_m)$.

Theorem B. ([10]). *Let $\vec{\omega} \in A_{(1,\cdots,1)}$ and $\vec{b} \in (BMO)^m$. Then there exists a constant C depending on $\vec{b}$ such that*

$$\nu_{\vec{\omega}}\left(\left\{ x \in \mathbb{R}^n : T_{\Pi\vec{b}}(\vec{f})(x) > t^m \right\} \right) \le C \left(\prod_{j=1}^{m} \int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j(y_j)|}{t}\right) \omega_j(y_j) dy_j \right)^{1/m},$$

where $\Phi(t) = t(1 + \log^+ t)$ and $\Phi^{(m)} = \overbrace{\Phi \circ \cdots \circ \Phi}^{m}$.

We will sometime use the notation $\vec{f} = (f_1, \cdots, f_m)$, with $f_j = \{f_{jk}\}_{k=1}^{\infty}$, and $\vec{y} = (y_1, \cdots, y_m)$, $d\vec{y} = dy_1 \cdots dy_m$. The vector-valued multilinear Calderón-Zygmund operator T_q associated with the operator T was defined and studied by Grafakos and Martell in [8].

$$T_q(\vec{f})(x) = |T(f_1, \cdots, f_m)(x)|_q = ||T(f_{1\cdot}, \cdots, f_{m\cdot})(x)||_{l^q} = \left(\sum_{k=1}^{\infty} |T(f_{1k}, \cdots, f_{mk})(x)|^q \right)^{1/q},$$

The commutators associated with T_q can be defined by

$$T_{\Pi\vec{b},q}(\vec{f})(x) = |T_{\Pi\vec{b}}(\vec{f})(x)|_q = ||T_{\Pi\vec{b}}(f_{1\cdot}, \cdots, f_{m\cdot})(x)||_{l^q} = \left(\sum_{k=1}^{\infty} |T_{\Pi\vec{b}}(\vec{f}_k)(x)|^q \right)^{1/q}.$$

where $f_i = \{f_{ik}\}_{k=1}^{\infty}$ for $i = 1, \cdots, m$. Grafakos and Martell [8] obtained the following results.

Theorem C. ([8]). Let T be a multilinear Calderón-Zygmund operators, and let $1/m < p < \infty, 1/p = 1/p_1 + \cdots + 1/p_m$ with $1 < p_1, \cdots, p_m < \infty, 1/m < q < \infty$ and $1/q = 1/q_1 + \cdots + 1/q_m$ with $1 < q_1, \cdots, q_m < \infty$. Then there exists a constant $C > 0$ such that

$$||T_q(\vec{f})||_{L^p(\mathbb{R}^n)} \leq C \prod_{j=1}^{m} |||f_j|_{q_j}||_{L^{p_j}(\mathbb{R}^n)}.$$

Cruz-Uribe, Martell and Pérez [4] obtained a weak version of Theorem C as follows:

Theorem D. ([4]). Let T be a multilinear Calderón-Zygmund operators, and let $1/m \leq p < \infty, 1/p = 1/p_1 + \cdots + 1/p_m$ with $1 \leq p_1, \cdots, p_m < \infty, 1/m < q < \infty$ and $1/q = 1/q_1 + \cdots + 1/q_m$ with $1 < q_1, \cdots, q_m < \infty$. Then there exists a constant $C > 0$ such that

$$||T_q(\vec{f})||_{L^{p,\infty}(\mathbb{R}^n)} \leq C \prod_{j=1}^{m} |||f_j|_{q_j}||_{L^{p_j}(\mathbb{R}^n)}.$$

In this paper, we consider the vector-valued version of Theorem A and B. The following are the main results:

Theorem 1.1. *Let* $1/m < p < \infty, \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, *with* $1 < p_1, \cdots, p_m < \infty, 1/m < q < \infty$ *and* $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$ *with* $1 < q_1, \cdots, q_m < \infty$. *If* $\vec{\omega} \in A_{\vec{p}}, \nu_{\vec{\omega}} = \prod_{i=1}^{m} \omega_i^{\frac{p}{p_i}}$, *and* $\vec{b} \in (BMO)^l$. *Then there exists a constant* $C > 0$ *such that*

$$||T_{\Pi\vec{b},q}(\vec{f})||_{L^p(\nu_{\vec{\omega}})} \leq \prod_{j=1}^{l} \|b_j\|_{BMO} \prod_{j=1}^{m} \||f_j|_{q_j}\|_{L^{p_j}(w_j)}.$$

Theorem 1.2. *Let* $1/m < q < \infty$ *and* $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$ *with* $1 < q_1, \cdots, q_m < \infty$. *If* $\vec{\omega} \in A_{(1,\cdots,1)}$ *and* $\vec{b} \in (BMO)^l$. *Then there exists a constant* $C > 0$ *depending on* $\vec{b}$ *such that*

$$\nu_{\vec{\omega}}\left(\left\{x \in \mathbb{R}^n : T_{\Pi\vec{b},q}(\vec{f})(x) > t^m\right\}\right) \leq C\left(\prod_{j=1}^{m} \int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j|_{q_j}(y_j)}{t}\right)\omega_j(y_j)dy_j\right)^{1/m},$$

where $\Phi(t) = t(1 + \log^+ t)$ *and* $\Phi^{(m)} = \overbrace{\Phi \circ \cdots \circ \Phi}^{m}$.

2. Notations and Main Lemmas

We first introduce some notations. For $1 \leq l \leq m$, we define some multilinear maximal operators as follows:

$$\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x) = \sup_{Q \ni x} \prod_{j=1}^{l} \| |f_j|_{q_j} \|_{L(\log L),Q} \prod_{j=l+1}^{m} \frac{1}{|Q|} \int_Q |f_j|_{q_j},$$

$$\mathcal{M}_{L(\log L)}(|\vec{f}|_q)(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \| |f_j|_{q_j} \|_{L(\log L),Q},$$

and

$$\mathcal{M}(|\vec{f}|_q)(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j|_{q_j},$$

where the supremum is taken over all the cubes containing x.

It is easy to see that

$$\mathcal{M}(|\vec{f}|_q)(x) \le \mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x) \le \mathcal{M}_{L(\log L)}(|\vec{f}|_q)(x).$$

Throughout the paper, M denotes the Hardy-Littlewood maximal operator. For $\delta > 0$, M_δ is the maximal function defined by

$$M_\delta f(x) = M(|f|^\delta)^{\frac{1}{\delta}}(x) = \left(\sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|^\delta dy \right)^{\frac{1}{\delta}}.$$

In addition, $M^\sharp$ is the sharp maximal function of Feffeman and Stein,

$$M^\sharp f(x) = \sup_{Q \ni x} \inf_c \frac{1}{|Q|} \int_Q |f(y) - c| dy \approx \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy$$

and a variant of $M^\sharp$ is given by

$$M^\sharp_\delta f(x) = M^\sharp(|f|^\delta)^{\frac{1}{\delta}}(x).$$

To prove the main theorems, we need the following lemmas.

Lemma 2.1. *Let $0 < \delta < 1/m$, $1/m < q < \infty$ and $1/q = 1/q_1 + \cdots + 1/q_m$ with $1 < q_1, \cdots, q_m < \infty$. Then there exists a constant $C > 0$ such that*

$$M^\sharp_\delta(T_q(\vec{f}))(x) \le C\mathcal{M}(|\vec{f}|_q)(x)$$

for any smooth vector function $\{\vec{f}_k\}_{k=1}^\infty$ and any $x \in \mathbb{R}^n$.

Proof. For simplicity, we only prove for the case $m = 2$, since there is no essential difference for the general case. Fix $x \in \mathbb{R}^n$ and let Q be a cube of side length r centered at x. For any smooth vector function sequence $\{\vec{f}_k\}_{k=1}^\infty$, set $\vec{f}_k^\infty = \vec{f}_k - \vec{f}_k^0$, where $\vec{f}_k^0 = \vec{f}_k \chi_{2Q} = (f_{1k}\chi_{2Q}, \cdots, f_{mk}\chi_{2Q})$. Since $0 < \delta < 1/2 < 1$, we have

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left||T_q(\vec{f})(y)|^\delta - |C|^\delta\right|dy\right)^{\frac{1}{\delta}}\\
&\leq C\left(\frac{1}{|Q|}\int_Q |T_q(\vec{f})(y) - C|^\delta dy\right)^{\frac{1}{\delta}}\\
&\leq C\left(\frac{1}{|Q|}\int_Q \left|T(\vec{f}^0)(y) + T(\vec{f}\chi_{(2Q)^c})(y) - c\right|_q^\delta dy\right)^{\frac{1}{\delta}}\\
&\leq C\left(\frac{1}{|Q|}\int_Q \left|T_q(\vec{f}^0)(y)\right|^\delta dy\right)^{\frac{1}{\delta}} + \left(\frac{1}{|Q|}\int_Q \left|T(\vec{f}\chi_{(2Q)^c})(y) - c\right|_q^\delta dy\right)^{\frac{1}{\delta}}\\
&= U_1 + U_2,
\end{aligned}
$$

where $C = \left|c\right|_q = \left(\sum_{k\geq 1}\left|c_k\right|^q\right)^{1/q}$.

For U_1, we applying Kolmogorov's inequality and Theorem D to get

$$
\begin{aligned}
\left(\frac{1}{|Q|}\int_Q \left|T_q(\vec{f}^0)(y)\right|^\delta dy\right)^{\frac{1}{\delta}} &\leq C||T_q(\vec{f}^0)||_{L^{1/2,\infty}(Q,\frac{dx}{|Q|})} \leq C\prod_{j=1}^{2}\frac{1}{|Q|}\int_Q |f_j|_{q_j}(z)dz\\
&\leq C\mathcal{M}(|\vec{f}|_q)(x).
\end{aligned}
$$

To estimate U_2, we choose $c = \sum_{i=1}^{3} c_i$, where

$$
c_1 = T(f_{1k}^\infty, f_{2k}^\infty)(x), \quad c_2 = T(f_{1k}^0, f_{2k}^\infty)(x), \quad c_3 = T(f_{1k}^\infty, f_{2k}^0)(x).
$$

We may split U_2 as $U_2 \leq U_{21} + U_{22} + U_{23}$, where

$$
\begin{aligned}
U_{21} &= \left(\frac{1}{|Q|}\int_Q \left|T(f_{1k}^\infty, f_{2k}^\infty)(y) - T(f_{1k}^\infty, f_{2k}^\infty)(x)\right|_q^\delta dy\right)^{\frac{1}{\delta}};\\
T_{22} &= \left(\frac{1}{|Q|}\int_Q \left|T(f_{1k}^0, f_{2k}^\infty)(y) - T(f_{1k}^0, f_{2k}^\infty)(x)\right|_q^\delta dy\right)^{\frac{1}{\delta}};\\
T_{23} &= \left(\frac{1}{|Q|}\int_Q \left|T(f_{1k}^\infty, f_{2k}^0)(y) - T(f_{1k}^\infty, f_{2k}^0)(x)\right|_q^\delta dy\right)^{\frac{1}{\delta}}.
\end{aligned}
$$

For the first term U_{21}, we have

$$
\begin{aligned}
&|T(f_{1k}^\infty, f_{2k}^\infty)(y) - T(f_{1k}^\infty, f_{2k}^\infty)(x)|\\
&\leq C\int_{(\mathbb{R}^n\setminus 2Q)^2}\frac{|Q|^{\varepsilon/n}}{|(x-y_1, x-y_2)|^{2n+\varepsilon}}\prod_{j=1}^{2}|f_{jk}^\infty(y_j)|d\vec{y}\\
&\leq C\sum_{s=1}^{\infty}\int_{(2^{s+1}Q)^2\setminus(2^sQ)^2}\frac{|Q|^{\varepsilon/n}}{(2^{s+1}|Q|^{1/n})^{2n+\varepsilon}}\prod_{j=1}^{2}|f_{jk}(y_j)|d\vec{y}\\
&\leq C\sum_{s=1}^{\infty}2^{-\varepsilon s}\prod_{j=1}^{2}\frac{1}{2^{(s+1)n}|Q|}\int_{2^{s+1}Q}|f_{jk}(y_j)|dy_j.
\end{aligned}
$$

Now let $g_{sjk} = \frac{1}{2^{(s+1)n}|Q|}\int_{2^{s+1}Q}|f_{jk}(y_j)|dy_j$ and $G_{sk} = \prod_{j=1}^{2} g_{sjk}$. Then we obtain

$$|T(f_{1k}^{\infty}, f_{2k}^{\infty})(y) - T(f_{1k}^{\infty}, f_{2k}^{\infty})(x)| \le C\sum_{s=1}^{\infty} 2^{-\varepsilon s}G_{sk}.$$

For any two positive real numbers ε and q, let $\rho = \min\{1, q\}$. It is easy to show that

$$\left|\sum_{s=1}^{\infty} 2^{-\varepsilon s}\mu_s\right|^q \le C_{(\varepsilon,q)}\sum_{s=1}^{\infty} 2^{-\varepsilon\rho s}|\mu_s|^q$$

for any sequence μ_s. Applying this inequality to $\mu_s = G_{sk}$, we get

$$|T(f_{1k}^{\infty}, f_{2k}^{\infty})(y) - T(f_{1k}^{\infty}, f_{2k}^{\infty})(x)|_q \le C\bigg(\sum_{s=1}^{\infty} 2^{-\varepsilon\rho s}\sum_k G_{sk}^q\bigg)^{1/q}.$$

Next, Minkowski's inequality gives

$$\bigg(\sum_k g_{sjk}^{q_j}\bigg)^{1/q_j} \le \frac{1}{2^{(s+1)n}|Q|}\int_{2^{s+1}Q}|f_j|_{q_j}(y_j)dy_j$$

and then Hölder's inequality gives

$$\bigg(\sum_k G_{sk}^q\bigg)^{1/q} \le \prod_{j=1}^{2}\bigg(\sum_k g_{sjk}^{q_j}\bigg)^{1/q_j} \le \mathcal{M}(|\vec{f}|_q)(x).$$

From which we deduce $|T(f_{1k}^{\infty}, f_{2k}^{\infty})(z) - T(f_{1k}^{\infty}, f_{2k}^{\infty})(x)|_q \le C\mathcal{M}(|\vec{f}|_q)(x)$. Since $0 < \delta < 1/2$, we have

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q |T(f_{1k}^{\infty}, f_{2k}^{\infty})(y) - T(f_{1k}^{\infty}, f_{2k}^{\infty})(x)|_q^{\delta}dy\right)^{\frac{1}{\delta}}\\ &\le C\frac{1}{|Q|}\int_Q |T(f_{1k}^{\infty}, f_{2k}^{\infty})(y) - T(f_{1k}^{\infty}, f_{2k}^{\infty})(x)|_q dy\\ &\le C\mathcal{M}(|\vec{f}|_q)(x).\end{aligned}$$

For U_{22}, we observe the following fact

$$\begin{aligned}&|T(f_{1k}^{0}, f_{2k}^{\infty})(z) - T(f_{1k}^{0}, f_{2k}^{\infty})(x)|\\ &\le C\int_{2Q}|f_{1k}(y_1)|dy_1\int_{(2Q)^c}\frac{|x-z|^{\varepsilon}|f_{2k}(y_2)|dy_2}{(|z-y_1|+|z-y_2|)^{2n+\varepsilon}}\\ &\le C\sum_{s=1}^{\infty}\frac{|Q|^{\varepsilon/n}}{(2^s|Q|^{1/n})^{2n+\varepsilon}}\int_{2^{s+1}Q}|f_{1k}(y_1)|dy_1\int_{2^{s+1}Q}|f_{2k}(y_2)|dy_2\\ &\le C\sum_{s=1}^{\infty}2^{-\varepsilon s}\prod_{j=1}^{2}\frac{1}{2^{(s+1)n}|Q|}\int_{2^{s+1}Q}|f_{jk}(y_j)|dy_j.\end{aligned}$$

The dealing of the other part is almost the same as U_{21}, we omit the detail. U_{23} can be estimated in the same way. Hence we proved Lemma 2.1. ■

For positive integers m and j with $1 \leq j \leq m$, we denote by $\mathcal{C}_j^m$ the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements and the associated complementary sequence $\sigma^{'}$ is given by $\sigma^{'} = \{1, \cdots, m\} \setminus \sigma$.

Lemma 2.2. *Let* $0 < \delta < \varepsilon < 1/m$. *Then there exists a constant* $C > 0$ *depending only on* δ *and* ε *such that*

$$M_\delta^\sharp(T_{\Pi\vec{b},q}\vec{f})(x) \leq C\prod_{j=1}^{l}||b_j||_{BMO}\Big(\mathcal{M}_{L(\log L)}^l(|\vec{f}|_q)(x) + M_\varepsilon(T_q\vec{f})(x)\Big) \tag{2.1}$$
$$+ C\sum_{j=1}^{l-1}\sum_{\sigma\in\mathcal{C}_j^l}\prod_{i\in\sigma}||b_i||_{BMO}M_\varepsilon(T_{\Pi b_{\sigma'},q}\vec{f})(x)$$

for any smooth vector function $\{\vec{f}_k\}_{k=1}^\infty$ *and for any* $x \in \mathbb{R}^n$, *where* $\sigma^{'} = \{1, \cdots, l\}\setminus\sigma$.

Proof. Let $F(\vec{y}) = f_1(y_1)\cdots f_m(y_m)$, for any $\lambda = (\lambda_1, \cdots, \lambda_m)$ we have

$$T_{\Pi\vec{b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m}(b_1(x) - b_1(y))\cdots(b_l(x) - b_l(y))K(x,\vec{y})F(\vec{y})d\vec{y}$$
$$= \int_{(\mathbb{R}^n)^m}((b_1(x) - \lambda_1) - (b_1(y) - \lambda_1))\cdots((b_l(x) - \lambda_l) - (b_l(y) - \lambda_l))$$
$$\times K(x,\vec{y})F(\vec{y})d\vec{y}$$
$$= \sum_{i=0}^{l}\sum_{\sigma\in\mathcal{C}_i^l}(-1)^{l-j}\prod_{j\in\sigma}(b_j(x) - \lambda_j)\int_{(\mathbb{R}^n)^m}\prod_{j\in\sigma'}(b_j(y_j) - \lambda_j)K(x,\vec{y})F(\vec{y})d\vec{y}$$
$$= (b_1(x) - \lambda_1)\cdots(b_l(x) - \lambda_l)T(\vec{f})(x) + T((b_1(\cdot_1) - \lambda_1)\cdots(b_l(\cdot_l) - \lambda_l)\vec{f})(x)$$
$$+ \sum_{i=1}^{l-1}\sum_{\sigma\in\mathcal{C}_i^l}(-1)^{l-j}\prod_{j\in\sigma}(b_j(x) - \lambda_j)\int_{(\mathbb{R}^n)^m}\prod_{j\in\sigma'}(b_j(y_j) - \lambda_j)K(x,\vec{y})F(\vec{y})d\vec{y}.$$

Noting the fact $\prod_{j\in\sigma'}(b_j(y_j) - \lambda_j) = \prod_{j\in\sigma'}[(b_j(y_j) - b_j(x)) + (b_j(x) - \lambda_j)]$. Then we obtain

$$T_{\Pi\vec{b},q}\vec{f}(x) \leq |(b_1(x) - \lambda_1)\cdots(b_l(x) - \lambda_l)|T_q(\vec{f})(x)$$
$$+ |T((b_1(\cdot_1) - \lambda_1)\cdots(b_l(\cdot_l) - \lambda_l)\vec{f})(x)|_q$$
$$+ C\sum_{i=1}^{l-1}\sum_{\sigma\in\mathcal{C}_i^l}\prod_{j\in\sigma}|b_j(x) - \lambda_j|T_{\Pi b_{\sigma'},q}\vec{f}(x),$$

Now fix $x \in \mathbb{R}^n$, for any cube Q of side length r and centered at x. Let $\lambda_j = \frac{1}{|2Q|}\int_{2Q} b_j(z)dz$, for $j = 1, \cdots, l$. Since $0 < \delta < 1/m < 1$, it follows that

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left||T_{\Pi\vec{b},q}(\vec{f}^{\infty})(z)|^{\delta} - |C|^{\delta}\right| dz\right)^{1/\delta} \\
&\le C\left(\frac{1}{|Q|}\int_Q \left|T_{\Pi\vec{b}}(\vec{f})(z) - c\right|_q^{\delta} dz\right)^{1/\delta} \\
&\le C\left(\frac{1}{|Q|}\int_Q \left|(b_1(z)-\lambda_1)\cdots(b_l(z)-\lambda_l)T(\vec{f})(z)\right|_q^{\delta} dz\right)^{1/\delta} \\
&\quad + C\sum_{i=1}^{l-1}\sum_{\sigma\in\mathcal{C}_i^l}\left(\frac{1}{|Q|}\int_Q \left(\prod_{j\in\sigma}|b_j(z)-\lambda_j| T_{\Pi b_{\sigma'},q}\vec{f}(z)\right)^{\delta} dz\right)^{1/\delta} \\
&\quad + C\left(\frac{1}{|Q|}\int_Q \left|T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f})(z) - c\right|_q^{\delta} dz\right)^{1/\delta} \\
&= I + II + III,
\end{aligned}$$

where $C = \left|c\right|_q$.

We can choose $1 < p_1, \cdots, p_l < \infty$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_l} + \frac{1}{\varepsilon} = \frac{1}{\delta}$. Since $0 < \delta < \varepsilon < 1/m$. Hölder's inequality gives

$$I \le C\prod_{j=1}^{l}||b_j||_{BMO} M_{\varepsilon}(T_q\vec{f})(x).$$

Similarly, we have

$$II \le C\sum_{i=1}^{l-1}\sum_{\sigma\in\mathcal{C}_i^l}\prod_{j\in\sigma}||b_j||_{BMO} M_{\varepsilon}(T_{\Pi b_{\sigma'},q}\vec{f})(x).$$

For III, let $\vec{f_j} = \vec{f}_j^0 + \vec{f}_j^{\infty}$, where $\vec{f}_j^0 = \vec{f}_j\chi_{2Q}$, and we choose $c = \sum_{\alpha_1,\cdots,\alpha_m} |(T((b_1(\cdot_1) - \lambda_1)\cdots(b_l(\cdot_l) - \lambda_l)f_1^{\alpha_1}\cdots f_m^{\alpha_m}))(x)|$.

We use the notation $\vec{f}^{\alpha}$ to denote $f_1^{\alpha_1}\cdots f_m^{\alpha_m}$. Obviously, we have

$$\begin{aligned}
&\left|T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f})(z) - c\right|_q \\
&\le T_q((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^0)(z) \\
&\quad + C\sum_{\alpha_1,\cdots,\alpha_m}|(T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^{\alpha}))(z) \\
&\quad - (T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^{\alpha}))(x)|_q,
\end{aligned}$$

where in the last sum each $\alpha_j = 0$ or ∞ and in each term there is at least one $\alpha_j = \infty$.

For the first term, we applying Kolmogorov's estimate and Theorem D to get

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|T_q((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^0)(z)\right|^\delta dz\right)^{1/\delta}\\
&\le C||T_q((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^0)||_{L^{1/m,\infty}(Q,\frac{dz}{|Q|})}\\
&\le C\prod_{j=1}^{l}\frac{1}{|Q|}\int_Q |b_j(y_j)-\lambda_j||f_j(z)|_{q_j}dz\prod_{j=l+1}^{m}\frac{1}{|Q|}\int_Q|f_j(z)|_{q_j}dz\\
&\le C\prod_{j=1}^{l}||b_j||_{BMO}|||f_j|_{q_j}||_{L(\log L),Q}\prod_{j=l+1}^{m}\frac{1}{|Q|}\int_Q|f_j(z)|_{q_j}dz\\
&\le C\prod_{j=1}^{l}||b_j||_{BMO}\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x).
\end{aligned}
$$

If all the $\alpha_j=\infty$, we have

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \Big|(T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^\alpha))(z)\right.\\
&\left.\quad-(T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^\alpha))(x)\Big|_q^\delta dz\right)^{1/\delta}\\
&\le \frac{1}{|Q|}\int_Q \Big|(T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^\alpha))(z)\\
&\quad-(T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^\alpha))(x)\Big|_q dz\\
&\le C\int_{(\mathbb{R}^n\setminus 2Q)^m}\frac{|x-z|^\varepsilon|(b_1(y_1)-\lambda_1)\cdots(b_l(y_l)-\lambda_l)||f_1(y_j)|_{q_j}\cdots|f_m(y_m)|_{q_m}d\vec{y}}{(|z-y_1|+\cdots+|z-y_m|)^{mn+\varepsilon}}\\
&\le C\sum_{k=1}^{\infty}\frac{1}{2^{k\varepsilon}}\prod_{j=1}^{l}\frac{1}{2^{(k+1)n}|Q|}\int_{2^{k+1}Q}|b_j(y_j)\\
&\quad-\lambda_j||f_j|_{q_j}dy_j\prod_{j=l+1}^{m}\frac{1}{2^{(k+1)n}|Q|}\int_{2^{k+1}Q}|f_j|_{q_j}dy_j\\
&\le C\sum_{k=1}^{\infty}\frac{1}{2^{k\varepsilon}}\prod_{j=1}^{l}||b_j-\lambda_j||_{BMO}|||f_j|_{q_j}||_{L(\log L),2^{k+1}Q}\prod_{j=l+1}^{m}\frac{1}{2^{(k+1)n}|Q|}\int_{2^{k+1}Q}|f_j|_{q_j}dy_j\\
&\le C\sum_{k=1}^{\infty}\frac{1}{2^{k\varepsilon}}k^l\prod_{j=1}^{l}||b_j||_{BMO}|||f_j|_{q_j}||_{L(\log L),2^{k+1}Q}\prod_{j=l+1}^{m}\frac{1}{2^{(k+1)n}|Q|}\int_{2^{k+1}Q}|f_j|_{q_j}dy_j\\
&\le C\prod_{j=1}^{l}||b_j||_{BMO}\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x).
\end{aligned}
$$

Now we estimate the typical representative of III. By Minkowski's inequality, we have

$$\left(\frac{1}{|Q|}\int_Q \left|T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)f_1^\infty,\cdots,f_i^\infty,f_{i+1}^0,\cdots,f_m^0)(z)-c\right|_q^\delta dz\right)^{1/\delta}$$

$$\leq \frac{C}{|Q|}\int_Q \Big|T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)f_1^\infty,\cdots,f_i^\infty,f_{i+1}^0,\cdots,f_m^0)(z)$$

$$-T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)f_1^\infty,\cdots,f_i^\infty,f_{i+1}^0,\cdots,f_m^0)(x)\Big|_q dz$$

$$\leq C\int_{(\mathbb{R}^n\setminus 2Q)^l}\frac{|x-z|^\varepsilon|(b_1(y_1)-\lambda_1)\cdots(b_l(y_l)-\lambda_l)||f_1(y_1)|_{q_1}\cdots|f_l(y_l)|_{q_l}dy_1\cdots dy_l}{(|z-y_1|+\cdots+|z-y_m|)^{mn+\varepsilon}}$$

$$\times\prod_{j=l+1}^m\int_{2Q}|f_j(y_j)|_{q_j}dy_j$$

$$\leq C\sum_{k=1}^m\frac{|Q|^{\frac{\varepsilon}{n}}}{(2^k|Q|^{\frac{1}{n}})^{mn+\varepsilon}}\prod_{j=1}^l\int_{2^{k+1}Q}|b_j(y_j)-\lambda_j||f_j(y_j)|_{q_j}dy_j\prod_{j=l+1}^m\int_{2^{k+1}Q}|f_j(y_j)|_{q_j}dy_j$$

$$\leq C\sum_{k=1}^m\frac{k^l}{2^{k\varepsilon}}\prod_{j=1}^l||b_j||_{BMO}|||f_j|_{q_j}||_{L(\log L),2^{k+1}Q}\prod_{j=l+1}^m\frac{1}{2^{(k+1)n}|Q|}\int_{2^{k+1}Q}|f_j|_{q_j}dy_j$$

$$\leq C\prod_{j=1}^l||b_j||_{BMO}\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x).$$

In other cases, we can also deduce the same estimates with little modifications on the above argument. Then we proved Lemma 2.2. ∎

3. Proof of the Main Results

By similar arguments used in the proof of Theorem 3.1 in [11] we get the following estimates for $T_{\Pi\vec{b},q}$. Since the the main ideas are almost the same, we omit the proof.

Lemma 3.1. *Let* $0<p<\infty, 1/m<q<\infty$, *and* $\frac{1}{q_1}+\cdots+\frac{1}{q_m}=\frac{1}{q}$ *with* $1<q_1,\cdots,q_m<\infty$. *Suppose that* $\vec{b}\in(BMO)^l$ *and* $w\in A_\infty$, *then there exists a constant* $C>0$, *such that*

$$\int_{\mathbb{R}^n}|T_{\Pi\vec{b},q}\vec{f}|^p w(x)dx\leq C\prod_{j=1}^l||b_j||^p_{BMO}\int_{\mathbb{R}^n}\left(\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x)\right)^p w(x)dx, \tag{3.1}$$

and

$$\begin{aligned}
&\sup_{t>0}\frac{1}{\Phi^{(m)}(1/t)}\omega\left(\left\{y\in\mathbb{R}^n : |T_{\Pi\vec{b},q}\vec{f}(y)|>t^m\right\}\right)\\
&\leq C\sup_{t>0}\frac{1}{\Phi^{(m)}(1/t)}\omega\left(\left\{y\in\mathbb{R}^n : \mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(y)>t^m\right\}\right),
\end{aligned}\tag{3.2}$$

for any smooth function $\vec{f}$ with compact support.

Lemma 3.2. ([11]). *Let $1/m<q<\infty$ and $1<q_1,\cdots,q_m<\infty$ with $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$.*

(i) Let $1/m<p<\infty$ and $1<p_1,\cdots,p_m<\infty$ with $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, and $\vec{\omega}$ satisfy the $A_{\vec{p}}$ condition. Then there exists a constant $C>0$ such that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n}\left|\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x)\right|^p\nu_{\vec{\omega}}(x)dx\right)^{1/p}\\
&\leq C\left(\int_{\mathbb{R}^n}\left||f_j|_{q_j}(x)\right|^{p_j}\omega_j(x)dx\right)^{1/p_j}.
\end{aligned}$$

(ii) Let $\vec{\omega}\in A_{(1,\cdots,1)}$. Then there exists a constant $C>0$ such that

$$\begin{aligned}
&\nu_{\vec{\omega}}\left(\left\{x\in\mathbb{R}^n : \mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x)>t^m\right\}\right)\\
&\leq C\left(\prod_{j=1}^m\int_{\mathbb{R}^n}\Phi^{(m)}\Big(\frac{|f_j|_{q_j}(y_j)}{t}\Big)\omega_j(y_j)dy_j\right)^{1/m},
\end{aligned}$$

where $\Phi(t)=t(1+\log^+t)$ and $\Phi^{(m)}=\overbrace{\Phi\circ\cdots\circ\Phi}^{m}$.

Lemma 3.2 was proved by Si and Xue in [11].

Proof of Theorem 1.1 - 1.2. Theorem 1.1 is a consequence of (3.1) and Lemma 3.2. We now prove Theorem 1.2. By homogeneity we may assume $t=1$. Since Φ is submultiplicative, Lemma 3.1 and Lemma 3.2 yields

$$\begin{aligned}
&\nu_{\vec{\omega}}\left(\left\{x\in\mathbb{R}^n : T_{\Pi\vec{b},q}\vec{f}(x)>1\right\}\right)^m\\
&\leq C\sup_{t>0}\frac{1}{\Phi^{(m)}(1/t)^m}\nu_{\vec{\omega}}\left(\left\{x\in\mathbb{R}^n : T_{\Pi\vec{b},q}\vec{f}(x)>t^m\right\}\right)^m\\
&\leq C\sup_{t>0}\frac{1}{\Phi^{(m)}(1/t)^m}\nu_{\vec{\omega}}\left(\left\{x\in\mathbb{R}^n : \mathcal{M}^l_{L(\log L)}\vec{f}(x)>t^m\right\}\right)^m\\
&\leq C\sup_{t>0}\frac{1}{\Phi^{(m)}(1/t)^m}\prod_{j=1}^m\int_{\mathbb{R}^n}\Phi^{(m)}\Big(\frac{|f_j|_{q_j}(y_j)}{t}\Big)\omega_j(y_j)dy_j\\
&\leq C\sup_{t>0}\frac{1}{\Phi^{(m)}(1/t)^m}\prod_{j=1}^m\int_{\mathbb{R}^n}\Phi^{(m)}(|f_j|_{q_j}(y_j))\Phi^{(m)}\Big(\frac{1}{t}\Big)\omega_j(y_j)dy_j
\end{aligned}\tag{3.3}$$

$$\leq C \prod_{j=1}^{m} \int_{\mathbb{R}^n} \Phi^{(m)}(|f_j|_{q_j}(y_j))\omega_j(y_j)dy_j.$$

This complete the proof of Theorem 1.2.

References

1. R. R. Coifman and Y. Meyer, On commutators of singular integrals and bilinear singular integrals, *Trans. Amer. Math. Soc.*, **212** (1975), 315-331.
2. R. R. Coifman and Y. Meyer, Commutateurs d'intégrales singuliéres et opèrateurs multilinéaires, *Ann. Inst. Fourier, Grenoble*, **28** (1978), 177-202.
3. R. R. Coifman and Y. Meyer, *Au-delà des opérateurs pseudo-différentiels*, Asterisque, 57, 1978.
4. D. Cruz-Uribe, J. M. Martell and C. Pérez, Extrapolation from A_∞ weights and applications, *J. Func. Anal.*, **213** (2004), 412-439.
5. C. Fefferman and E. Stein, Some maximal inequalities, *Am. J. Math.*, **93** (1971), 107-115.
6. L. Grafakos and R. Torres, Multilinear Calderón-Zygmund theory, *Adv. Math.*, **165** (2002), 124-164.
7. L. Grafakos and R. H. Torres, Maximal operator and weighted norm inequalities for multilinear singular integrals, *Indiana Univ. Math. J.*, **51(5)** (2002), 1261-1276.
8. L. Grafakos and J. M. Martell, Extrapolation of weighted norm inequalities for multivariable operators, *J. Geom Anal.*, **14(1)** (2004), 19-46.
9. A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres and R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, *Adv. Math.*, **220(4)** (2009), 1222-1264.
10. C. Pérez, G. Pradolini, R. H. Torres and R. Trujillo-González, *End-point estimates for iterated commutators of multilinear singular integrals*, preprint, arXiv:1004.4976.
11. Z. Si and Q. Xue, *Weighted estimates for commutators of vector-valued maximal multilinear operators*, preprint.
12. L. Tang, Weighted estimates for vector-valued commutators of multilinear operators, *Proceedings of the Rayal Society of Edingburgh*, **138A** (2008), 897-922.

Zengyan Si
School of Mathematics and Information Science
Henan Polytechnic University
Jiaozuo 454000
People's Republic of China
E-mail: sizengyan@yahoo.cn

Qingying Xue
School of Mathematical Sciences
Beijing Normal University
Laboratory of Mathematics and Complex Systems
Ministry of Education
Beijing 100875
People's Republic of China
E-mail: qyxue@bnu.edu.cn