TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 1, pp. 361-378, February 2013
DOI: 10.11650/tjm.17.2013.2074
This paper is available online at http://journal.taiwanmathsoc.org.tw

# GENERALIZATIONS OF STURM-PICONE THEOREM FOR SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

J. Tyagi

**Abstract.** The goal of this paper is to show a generalization to Sturm–Picone theorem for a pair of second-order nonlinear differential equations

$$(p_1(t)x'(t))' + q_1(t)f_1(x(t)) = 0.$$

$$(p_2(t)y'(t))' + q_2(t)f_2(y(t)) = 0,\ t_1 < t < t_2.$$

This work generalizes well-known comparison theorems [C. Sturm, J. Math. Pu res. Appl. **1** (1836), 106–186; M. Picone, Ann. Scoula Norm. Sup. Pisa **11** (1909) 39; W. Leighton, Proc. Amer. Math. Soc.**13** (1962), 603–610], which play a key role in the qualitative behavior of solutions. We establish the generalization to a pair of nonlinear singular differential equations and elliptic partial differential equations also. We show generalization via the quadratic functionals associated to the above pair of equations. The celebrated Sturm–Picone theorem for a pair of linear differential equations turns out to be a particular case of our result.

## 1. INTRODUCTION

In the qualitative theory of ordinary differential equations (ODEs), celebrated Sturm–Picone theorem plays a crucial role. In 1836, the first important comparison theorem was established by C.Sturm [19], which deals with a pair of linear ODEs

$$(p_1(t)x'(t))' + q_1(t)x(t) = 0. \tag{1.1}$$

$$(p_2(t)y'(t))' + q_2(t)y(t) = 0, \tag{1.2}$$

on a bounded interval $(t_1,\, t_2)$, where $p_1,\, p_2,\, q_1,\, q_2$ are real-valued continuous functions and $p_1(t) > 0$, $p_2(t) > 0$ on $[t_1,\, t_2]$. The original Sturm's comparison theorem [19] reads as

Received May 27, 2012, accepted August 7, 2012.
Communicated by Eiji Yanagida.
2010 *Mathematics Subject Classification*: Primary 34C10, 35J15; Secondary 35B105, 34C15.
*Key words and phrases*: Singular equation, Elliptic partial differential equations, Zeros, Comparison theorem, *Oscillatory* as well as *nonoscillatory* behavior.

**Theorem 1.1.** (Sturm's comparison theorem). *Suppose* $p_1(t) = p_2(t)$ *and* $q_1(t) > q_2(t), \forall\, t \in (t_1, t_2)$. *If there exists a nontrivial real solution* $y$ *of* (1.2) *such that* $y(t_1) = 0 = y(t_2)$, *then every real solution of* (1.1) *has at least one zero in* $(t_1, t_2)$.

In 1909, M.Picone [17] modified Sturm's theorem. The modification reads as

**Theorem 1.2.** (Sturm-Picone theorem). *Suppose that* $p_2(t) \geq p_1(t)$ *and* $q_1(t) \geq q_2(t), \forall\, t \in (t_1, t_2)$. *If there exists a nontrivial real solution* $y$ *of* (1.2) *such that* $y(t_1) = 0 = y(t_2)$, *then every real solution of* (1.1) *unless a constant multiple of* $y$ *has at least one zero in* $(t_1, t_2)$.

Theorem 1.2 is in fact a special case of Leighton's theorem (see [15]). For a detailed study and earlier developments of this subject, we refer the reader to Swanson's book [20]. Though Sturm–Picone theorem is extended in several directions, see, S. Ahmad and A. C.Lazer [2] and S. Ahmad [3] for linear systems, E.Müller–Pfeiffer [16] for non-selfadjoint differential equations, the present author and V.Raghavendra [22] for implicit differential equations, W.Allegretto [6] for degenerate elliptic equations, C.Zhang and S.Sun [25] for linear equations on time scales, there is no natural generalization of Sturm–Picone theorem for a pair of nonlinear differential equations. To obtain nonoscillation results for perturbed nonlinear differential equations, J.Graef and P.Spikes [11] established Sturm–Picone type comparison theorem for the same class of equations. This comparison theorem works nicely in getting nonoscillation results but it cannot be viewed as a natural generalization of Sturm–Picone theorem as the zeros of the solutions of a pair of equations may coincide. We emphasize that the classical proof of Sturm–Picone theorem heavily depends on a variational lemma due to W.Leighton [15] (see [20] also). Since when it was proved, it has been extended in different contexts, see, for instance, Jaros et. al. [12], V.Komkov [14], O.Doslý and J.Jaros [8]. As far as our understanding goes, there is no natural generalization of Leighton's variational lemma for nonlinear differential equations.

Since 1962, when W.Leighton proved a theorem ([15]), so called "Leighton's theorem", there has been a good interest to generalize Leighton's theorem for a class of nonlinear differential equations. In this article, we prove a nonlinear analogue of Leighton's theorem. In fact, via this analogue, we give a generalization to Sturm–Picone theorem. In order to give a nonlinear analogue of Leighton's theorem, our strategy is to first establish a nonlinear version of Leighton's variational lemma.

When $p_1, p_2, q_1, q_2$ (some of them or all) are not continuous at $t_1$ or $t_2$ or at $t_1$ and $t_2$ both, then (1.1), (1.2) are called singular differential equations. Analogs of Theorems 1.1, 1.2 and other related theorems for singular differential equations have been obtained earlier (see [20]). Recently, D.Aharonov and U.Elias [1] proved Sturm's theorem for a pair of singular linear differential equations assuming that the solution of minorant equation is principal at both end points of the interval. By the older approach, we give the generalization of these theorems to a pair of nonlinear singular differential equations also.

There is also a good amount of interest in the qualitative theory of PDEs to determine whether the given equation is oscillatory or not. In this direction, Sturm–Picone theorem plays an important role. We also give a generalization to Sturm–Picone comparison theorem to nonlinear elliptic equations. There is an enormous excellent work about Sturm's comparison theorem/oscillation theory but for convenience, we just name a few articles. For the earlier developments about Sturm–Picone comparison theorem and oscillation theory, we refer to [17, 19, 20] and for recent developments, we refer to Yoshida's book [23]. For sturm comparison theorem to quasilinear elliptic equation, we refer to [4, 5, 6] and for Picone type identities, we refer to [7, 10, 13, 21, 24].

Let us consider a pair of second-order nonlinear ODEs

$$lx \equiv (p_1(t)x'(t))' + q_1(t)f_1(x(t)) = 0. \tag{1.3}$$

$$Ly \equiv (p_2(t)y'(t))' + q_2(t)f_2(y(t)) = 0,\ t_1 < t < t_2, \tag{1.4}$$

where $p_1, p_2 \in C^1([t_1, t_2], (0, \infty))$, $q_1, q_2 \in C([t_1, t_2], \mathbb{R})$, $f_1, f_2 \in C(\mathbb{R}, \mathbb{R})$, $l$ and $L$ are differential operators or mappings whose domains consist of all real-valued functions $x \in C^1[t_1, t_2]$ such that $p_1x'$ and $p_2x' \in C^1[t_1, t_2]$, respectively.

We make the following hypotheses on nonlinearity $f_1,\ f_2$ and $q_2$:

(H1) Let $f_1 \in C^1(\mathbb{R}, \mathbb{R})$ and there exist $\alpha_1 > 0,\ M > 0$ such that $0 < \alpha_1 \le f_1'(y) \le M,\ \forall\, 0 \ne y \in \mathbb{R}$.

(H2) $f_1(y) \ne 0,\ \forall\, 0 \ne y \in \mathbb{R},\ f_1(0) = 0$.

(H3) Let $f_2 \in C(\mathbb{R}, \mathbb{R})$ and there exist $\alpha_2, \alpha_3 \in (0, \infty)$ such that $\alpha_3 y^2 \le f_2(y)y \le \alpha_2 y^2,\ \forall\, 0 \ne y \in \mathbb{R}$.

**Remark 1.3.** (H1) motivates us to take the nonlinearity of the form

$$f_1(y) = \text{“linear part in } y\text{”} \pm \text{“nonlinear part in } y\text{”},$$

where "nonlinear" part is decaying at $\infty$. One can take the following examples of $f_1$ like, $f_1(y) = 2y - \frac{y}{y^2+1}$; $y + ye^{-y^2}$ etc.

**Remark 1.4.** (H3) simply says that $\frac{f_2(y)}{y}$ is bounded, $\forall\ 0 \ne y \in \mathbb{R}$.

We organize this paper as follows. Section 2 deals with the generalizations of comparison theorems to nonlinear ODEs.

Section 3 contains the generalizations to singular ODEs. In Section 4, we show the generalizations to nonlinear elliptic equations.

## 2. Generalizations

We begin with the following quadratic functionals corresponding to (1.3) and (1.4), respectively

$$j[u] = \int_{t_1}^{t_2} [p_1(t)(u'(t))^2 - \alpha_1 q_1(t)(u(t))^2]dt.$$
$$J[u] = \int_{t_1}^{t_2} [p_2(t)(u'(t))^2 - (\alpha_2 q_2^+(t) - \alpha_3 q_2^-(t))(u(t))^2]dt,$$

where the domain $D$ of $j$ and $J$ is defined to be the set of all real-valued functions $u \in C^1[t_1, t_2]$ such that $u(t_1) = u(t_2) = 0$ ($t_1, t_2$ are consecutive zeros of $u$) and $q_2^+ = \max\{q_2, 0\}$ and $q_2^- = \max\{-q_2, 0\}$. The variation $V(u)$ is defined as $V[u] = J[u] - j[u]$, i.e.,

$$V[u] = \int_{t_1}^{t_2} [(p_2(t)-p_1(t))(u'(t))^2+(\alpha_1 q_1(t)-(\alpha_2 q_2^+(t)-\alpha_3 q_2^-(t)))(u(t))^2]dt. \tag{2.1}$$

The next lemma deals with a generalization of Leighton's variational lemma.

**Lemma 2.1.** (Generalization of Leighton's variational lemma). *Let $u \in D$ and $j[u] \leq 0$. Let $x$ be a nontrivial solution of* (1.3)*, then under hypotheses* (H1)-(H2)*, $x$ vanishes at least once in $(t_1, t_2)$ unless $f_1(x)$ is a constant multiple of $u$.*

*Proof.* We establish this result by contradiction. Suppose $x(t) \neq 0$, $\forall\, t \in (t_1, t_2)$. By (H2), $f_1(x(t)) \neq 0$, $\forall t \in (t_1, t_2)$. We observe that the following is valid on $(t_1, t_2)$ :

$$\begin{aligned}
&\left[\frac{(u(t))^2}{f_1(x(t))} p_1(t)x'(t)\right]' \\
&= \frac{(u(t))^2}{f_1(x(t))}(p_1(t)x'(t))' + p_1(t)x'(t)\left[\frac{2f_1(x(t))u(t)u'(t) - (u(t))^2 f_1'(x(t))x'(t)}{(f_1(x(t)))^2}\right] \\
&= -q_1(t)(u(t))^2 + \frac{2p_1(t)u(t)u'(t)x'(t)}{f_1(x(t))} - \frac{p_1(t)(u(t))^2(x'(t))^2 f_1'(x(t))}{(f_1(x(t)))^2} \\
&= -q_1(t)(u(t))^2 - p_1(t)\left(\frac{u(t)x'(t)\sqrt{f_1'(x(t))}}{f_1(x(t))} - \frac{u'(t)}{\sqrt{f_1'(x(t))}}\right)^2 + \frac{p_1(t)(u'(t))^2}{f_1'(x(t))} \\
&\leq -q_1(t)(u(t))^2 - p_1(t)\left(\frac{u(t)x'(t)\sqrt{f_1'(x(t))}}{f_1(x(t))} - \frac{u'(t)}{\sqrt{f_1'(x(t))}}\right)^2 + \frac{p_1(t)(u'(t))^2}{\alpha_1}.
\end{aligned} \tag{2.2}$$

This implies that

$$\begin{aligned}
&p_1(t)(u'(t))^2 - \alpha_1 q_1(t)(u(t))^2 \\
&\geq \alpha_1\left[\frac{(u(t))^2}{f_1(x(t))} p_1(t)x'(t)\right]' + \alpha_1\, p_1(t)\left(\frac{u(t)x'(t)\sqrt{f_1'(x(t))}}{f_1(x(t))} - \frac{u'(t)}{\sqrt{f_1'(x(t))}}\right)^2.
\end{aligned} \tag{2.3}$$

So an integration of (2.3) over $(t_1,\, t_2)$ yields

$$\begin{aligned}(2.4)\quad &\int_{t_1}^{t_2}(p_1(t)(u'(t))^2-\alpha_1\, q_1(t)(u(t))^2)dt\\ &\geq \alpha_1\left[\frac{(u(t))^2p_1(t)x'(t)}{f_1(x(t))}\right]_{t_1}^{t_2}+\alpha_1\int_{t_1}^{t_2}p_1(t)\left(\frac{u(t)x'(t)\sqrt{f_1'(x(t))}}{f_1(x(t))}-\frac{u'(t)}{\sqrt{f_1'(x(t))}}\right)^2dt.\end{aligned}$$

Now there are three cases.

**Case 1.** If $x(t_1)\neq 0$ and $x(t_2)\neq 0$, it follows from (2.4) and $u(t_1)=0=u(t_2)$ that $j[u]\geq 0$ and

$$\int_{t_1}^{t_2}p_1(t)\left(\frac{u(t)x'(t)\sqrt{f_1'(x(t))}}{f_1(x(t))}-\frac{u'(t)}{\sqrt{f_1'(x(t))}}\right)^2dt=0 \text{ if and only if}$$

$$\frac{u(t)x'(t)\sqrt{f_1'(x(t))}}{f_1(x(t))}-\frac{u'(t)}{\sqrt{f_1'(x(t))}}\equiv 0.$$

This implies that

$$\left[\frac{u(t)}{f_1(x(t))}\right]'=0,\ \text{ i.e.,}$$

$$u(t)=Cf_1(x(t)),\ \forall\, t\in(t_1,\, t_2) \text{ for some constant } C.$$

$u\in C^1[t_1,\, t_2]$ is such that $u(t_1)=u(t_2)=0$ ($t_1,\, t_2$ are consecutive zeros of $u$). This implies that $u(t)\neq 0,\ \forall\ t\in(t_1,\, t_2)$. So $C$ is a non-zero constant. Using this fact, one can obtain that

$$(2.5)\quad f_1(x(t))=C_1u(t),\ \forall\, t\in(t_1,\, t_2) \text{ for some another non-zero constant } C_1=\frac{1}{C}.$$

Now as $t\to t_1$ or $t\to t_2$, L. H. S of (2.5) is non-zero while R. H. S. is zero. Therefore, $j[u]>0$, which leads a contradiction. This contradiction shows that $x$ vanishes at least once in $(t_1,\, t_2)$.

**Case 2.** If $x(t_1)=0$ and $x(t_2)=0$ then $x'(t_1)\neq 0$ and $x'(t_2)\neq 0$. Suppose if $x'(t_1)=0$ or $x'(t_2)=0$, then by (H2), $x(t)=0$ is a solution of (1.3) and by uniqueness theorem (in view of (H1)), $x(t)\equiv 0$, which is not possible as $x$ is a nontrivial solution of (1.3). An application of L'Hospital rule implies that

$$\lim_{t\to t_1^+}\frac{(u(t))^2p_1(t)x'(t)}{f_1(x(t))}=\lim_{t\to t_1^+}\frac{(u(t))^2(p_1(t)x'(t))'+2p_1(t)x'(t)u(t)u'(t)}{f_1'(x(t))x'(t)}=0$$

and

$$\lim_{t\to t_2^-}\frac{(u(t))^2p_1(t)x'(t)}{f_1(x(t))}=\lim_{t\to t_2^-}\frac{(u(t))^2(p_1(t)x'(t))'+2p_1(t)x'(t)u(t)u'(t)}{f_1'(x(t))x'(t)}=0.$$

Therefore, we obtain from (2.4) that $j[u] \geq 0$ and hence we get a contradiction $j[u] > 0$ unless $f_1(x)$ is a constant multiple of $u$.

**Case 3.** If $x(t_1) = 0$, $x(t_2) \neq 0$ or $x(t_1) \neq 0$, $x(t_2) = 0$, then from the proof of Case 1, it is clear that $j[u] > 0$, which leads a contradiction and hence $x$ vanishes at least once in $(t_1, t_2)$. This completes the proof. ∎

**Corollary 2.2.** *Let* $f_1(x) = x$ *in* (1.3) *and*

$$\int_{t_1}^{t_2} [p_1(t)(u'(t))^2 - q_1(t)(u(t))^2]dt \leq 0,$$

*where* $u \in C^1[t_1, t_2]$ *such that* $u(t_1) = u(t_2) = 0$ $(t_1, t_2$ *are consecutive zeros of* $u)$. *Let* $x$ *be any nontrivial solution of* (1.3)*, then* $x$ *vanishes at least once in* $(t_1, t_2)$ *unless* $x$ *is a constant multiple of* $u$.

*Proof.* It is trivial to see that $f_1$ satisfies (H1)–(H2). In this case, $\alpha_1 = 1 = M$ and $j[u] \leq 0$. An application of Lemma 2.1 implies that $x$ vanishes at least once in $(t_1, t_2)$ unless $x$ is a constant multiple of $u$. For a proof of this corollary, we refer the reader to [15, 20]. ∎

Lemma 2.1 plays a very crucial role to establish the following

**Theorem 2.3.** (Generalization of Leighton's theorem). *Suppose there exists a nontrivial solution* $u$ *of* $Lu = 0$ *in* $(t_1, t_2)$ *such that* $u(t_1) = 0 = u(t_2)$. *Let* (H1)–(H3) hold and $V[u] \geq 0$, then every nontrivial solution $v$ of $lv = 0$ has at least one zero in $(t_1, t_2)$ unless $f_1(v)$ is a constant multiple of $u$.

*Proof.* Since $u(t_1) = 0 = u(t_2)$ and $Lu = 0$, so by an application of Green's identity, we have

$$\int_{t_1}^{t_2} (q_2(t)f_2(u(t))u(t) - p_2(t)(u'(t))^2)dt = 0. \tag{2.6}$$

In view of (H3), one can see that

$$\int_{t_1}^{t_2} (q_2(t)f_2(u(t))u(t) - (\alpha_2 q_2^+(t) - \alpha_3 q_2^-(t))(u(t))^2)dt \leq 0. \tag{2.7}$$

By (2.6), (2.7), we get $J[u] \leq 0$. Since $V[u] \geq 0$, so this implies that

$$j[u] \leq J[u] \leq 0$$

and hence by an application of Lemma 2.1, every nontrivial solution $v$ of $lv = 0$ has at least one zero in $(t_1, t_2)$ unless $f_1(v)$ is a constant multiple of $u$. This completes the proof. ∎

**Corollary 2.4.** (Leighton's theorem). *Let us consider* (1.3) *and* (1.4) *with* $f_1(u) = u = f_2(u)$. *Let*

$$V_1[u] = \int_{t_1}^{t_2} [(p_2(t) - p_1(t))(u'(t))^2 + (q_1(t) - q_2(t))(u(t))^2]dt \geq 0. \tag{2.8}$$

*If there exists a nontrivial solution* $u$ *of* (1.4) *in* $(t_1,\, t_2)$ *such that* $u(t_1) = 0 = u(t_2)$, *then every nontrivial solution* $v$ *of* (1.3) *has at least one zero in* $(t_1,\, t_2)$ *unless* $v$ *is a constant multiple of* $u$.

*Proof.* Since $f_1(u) = u = f_2(u)$, it is easy to see that $\alpha_1 = \alpha_2 = \alpha_3 = 1 = M$. In view of (2.8), $V[u] \geq 0$ and hence by an application of Theorem 2.3, the required conclusion follows. For a proof of this corollary, we refer the reader to [15]. ∎

The following generalization is a special case of Theorem 2.3.

**Theorem 2.5.** (Generalization of Sturm-Picone theorem). *Suppose there exists a nontrivial solution* $u$ *of* $Lu = 0$ *in* $(t_1,\, t_2)$ *such that* $u(t_1) = 0 = u(t_2)$. *Let* (H1)–(H3) hold. Suppose $p_2(t) \geq p_1(t)$ and

$$\alpha_1 q_1(t) - (\alpha_2 q_2(t) - (\alpha_3 - \alpha_2) q_2^-(t)) \geq 0, \ \forall\, t \in (t_1, t_2), \tag{2.9}$$

then every nontrivial solution $v$ of $lv = 0$ has at least one zero in $(t_1,\, t_2)$ unless $f_1(v)$ is a constant multiple of $u$.

*Proof.* In view of (2.9), it is easy to see that $V[u] \geq 0$ and the proof of this theorem follows from Theorem 2.3. ∎

The celebrated Sturm-Picone theorem can be seen as a particular case of Theorem 2.5 in

**Corollary 2.6.** (Celebrated Sturm–Picone theorem). *Consider* (1.3) *and* (1.4) *with* $f_1(u) = u = f_2(u)$. *Let* $p_2(t) \geq p_1(t)$ *and* $q_1(t) \geq q_2(t)$, $\forall\, t \in (t_1, t_2)$. *If there exists a nontrivial solution* $u$ *of* (1.4) *in* $(t_1,\, t_2)$ *such that* $u(t_1) = 0 = u(t_2)$, *then every nontrivial solution* $v$ *of* (1.3) *has at least one zero in* $(t_1,\, t_2)$ *unless* $v$ *is a constant multiple of* $u$.

*Proof.* Since $f_1(y) = y = f_2(y)$ in (1.3) and (1.4) so in this case $\alpha_1 = \alpha_2 = \alpha_3 = 1 = M$. It is easy to see that $f_1$ and $f_2$ satisfy (H1)–(H2).
An application of Theorem 2.5 leads the required conclusion. For a proof of Corollary 2.6, we refer the reader to [17] or Theorem 1.6 [20]. ∎

**Remark 2.7.** Let $p_1(t) = p_2(t)$, $q_1(t) > q_2(t)$, $\forall\, t \in (t_1,\, t_2)$ in Corollary 2.6, then Corollary 2.6 is indeed original Sturm's theorem (see [19]).

## 3. Singular Sturm-picone Theorem for Nonlinear Equations

In this section, we consider a pair of singular equations (1.3) and (1.4). More precisely, we consider a pair of singular nonlinear ODEs

$$l_s x \equiv (p_1(t)x'(t))' + q_1(t)f_1(x(t)) = 0. \tag{3.1}$$

$$L_s y \equiv (p_2(t)y'(t))' + q_2(t)f_2(y(t)) = 0,\ t_1 < t < t_2, \tag{3.2}$$

where $p_1,\, p_2 \in C^1((t_1,\, t_2),\, (0,\, \infty))$, $q_1,\, q_2 \in C((t_1,\, t_2),\, \mathbb{R})$, $p_1,\, p_2,\, q_1,\, q_2$ (some of them or all) may not be continuous at $t_1$ or $t_2$ or at $t_1$ and $t_2$ both. Let $f_1,\, f_2 \in C(\mathbb{R},\, \mathbb{R})$, $l_s$ and $L_s$ are differential operators or mappings whose domains consist of all real-valued functions $x \in C^1(t_1,\, t_2)$ such that $p_1 x'$ and $p_2 x' \in C^1(t_1,\, t_2)$, respectively. We make the following hypotheses on nonlinearity $f_1$:

(H1)' Let $f_1 \in C^1(\mathbb{R},\, \mathbb{R})$ and there exists $\alpha_1 > 0$ such that
$0 < \alpha_1 \le f_1'(y),\ \forall\, 0 \ne y \in \mathbb{R}$.

(H2)' $f_1(y) \ne 0,\ \forall\, 0 \ne y \in \mathbb{R}$.

We begin with the following quadratic functionals corresponding to (3.1) and (3.2), respectively. Let $t_1 < \xi < \eta < t_2$ and let

$$j_{\xi\eta}[u] = \int_\xi^\eta [p_1(t)(u'(t))^2 - \alpha_1 q_1(t)(u(t))^2]dt \quad \text{and}$$

$$J_{\xi\eta}[u] = \int_\xi^\eta [p_2(t)(u'(t))^2 - (\alpha_2 q_2^+(t) - \alpha_3 q_2^-(t))(u(t))^2]dt.$$

Let us define $j_s[u] = \lim_{\xi \to t_1^+,\, \eta \to t_2^-} j_{\xi\eta}[u]$, $J_s[u] = \lim_{\xi \to t_1^+,\, \eta \to t_2^-} J_{\xi\eta}[u]$, whenever the limits exist. The domain $D_{j_s}$ of $j_s$ and $D_{J_s}$ of $J_s$ are defined to be the set of all real-valued continuous functions $u \in C^1(t_1,\, t_2)$ with $u(t_1) = 0 = u(t_2)$ such that $j_s[u]$ and $J_s[u]$ exist. Let us define

$$A_{t_1 t_2}[u,\, x] = \lim_{t \to t_2^-} \frac{(u(t))^2 p_1(t) x'(t)}{f_1(x(t))} - \lim_{t \to t_1^+} \frac{(u(t))^2 p_1(t) x'(t)}{f_1(x(t))},$$

whenever the limits on the right side exist. The variation $V_s(u)$ is defined as $V_s[u] = J_s[u] - j_s[u]$, i.e.,

$$V_s[u] = \int_{t_1}^{t_2} [(p_2(t) - p_1(t))(u'(t))^2 + (\alpha_1 q_1(t) - (\alpha_2 q_2^+(t) - \alpha_3 q_2^-(t)))(u(t))^2]dt. \tag{3.3}$$

The next lemma deals with a generalization of Leighton's variational lemma.

**Lemma 3.1.** (Generalization of singular Leighton's variational lemma). *Suppose there exists a function $u \in D_{j_s}$ not identically zero in any open subinterval of $(t_1,\, t_2)$ such that $j_s[u] \leq 0$. Let $x$ be any nontrivial solution of* (3.1) $(l_s x = 0)$ *and* $A_{t_1 t_2}[u,\, x] \geq 0$, *then under hypotheses* (H1)', (H2)', *$x$ has at least one zero in $(t_1,\, t_2)$ unless $f_1(x)$ is a constant multiple of $u$.*

*Proof.* We establish this result by contradiction. Suppose $x(t) \neq 0,\ \forall\ t \in (t_1,\, t_2)$. By (H2)', $f_1(x(t)) \neq 0,\ \forall t \in (t_1,\, t_2)$. Along the same lines of proof of Lemma 2.1, we see that the following is valid on $(t_1,\, t_2)$ :

$$\begin{aligned} & p_1(t)(u'(t))^2 - \alpha_1 q_1(t)(u(t))^2 \\ & \geq \alpha_1 \left[\frac{(u(t))^2}{f_1(x(t))}\, p_1(t) x'(t)\right]' + \alpha_1\, p_1(t) \left(\frac{u(t)x'(t)\sqrt{f_1'(x(t))}}{f_1(x(t))} - \frac{u'(t)}{\sqrt{f_1'(x(t))}}\right)^2 . \end{aligned} \tag{3.4}$$

An integration of (3.4) over $(\xi,\, \eta)$ yields

$$\begin{aligned} & \int_\xi^\eta (p_1(t)(u'(t))^2 - \alpha_1\, q_1(t)(u(t))^2)dt \\ & \geq \alpha_1 \left[\frac{(u(t))^2 p_1(t) x'(t)}{f_1(x(t))}\right]_\xi^\eta + \alpha_1 \int_\xi^\eta p_1(t) \left(\frac{u(t)x'(t)\sqrt{f_1'(x(t))}}{f_1(x(t))} - \frac{u'(t)}{\sqrt{f_1'(x(t))}}\right)^2 dt \end{aligned}$$

or we have

$$j_{\xi\eta}[u] \geq \alpha_1 \left[\frac{(u(t))^2 p_1(t) x'(t)}{f_1(x(t))}\right]_\xi^\eta + \alpha_1 \int_\xi^\eta p_1(t) \left(\frac{u(t)x'(t)\sqrt{f_1'(x(t))}}{f_1(x(t))} - \frac{u'(t)}{\sqrt{f_1'(x(t))}}\right)^2 dt.$$

Letting $\xi \to t_1^+$, $\eta \to t_2^-$ and using $A_{t_1 t_2}[u,\, x] \geq 0$, we get

$$j_s[u] \geq \alpha_1 \int_{t_1}^{t_2} p_1(t) \left(\frac{u(t)x'(t)\sqrt{f_1'(x(t))}}{f_1(x(t))} - \frac{u'(t)}{\sqrt{f_1'(x(t))}}\right)^2 dt \tag{3.5}$$

and

$$\int_{t_1}^{t_2} p_1(t) \left(\frac{u(t)x'(t)\sqrt{f_1'(x(t))}}{f_1(x(t))} - \frac{u'(t)}{\sqrt{f_1'(x(t))}}\right)^2 dt = 0 \text{ if and only if}$$

$$\frac{u(t)x'(t)\sqrt{f_1'(x(t))}}{f_1(x(t))} - \frac{u'(t)}{\sqrt{f_1'(x(t))}} \equiv 0.$$

This implies that

$$\left[\frac{u(t)}{f_1(x(t))}\right]' = 0, \text{ i.e.,}$$

$$u(t) = Cf_1(x(t)),\ \forall\, t \in (t_1,\, t_2) \text{ for some constant } C.$$

Since $u \in C^1(t_1,\, t_2)$ such that $u(t_1) = u(t_2) = 0$ ($t_1,\, t_2$ are consecutive zeros of $u$). This implies that $u(t) \neq 0,\ \forall\ t \in (t_1,\, t_2)$. So $C$ is a non-zero constant. Using this fact, one can obtain that

$$f_1(x(t)) = C_1 u(t),\ \forall\, t \in (t_1,\, t_2) \text{ for some another non-zero constant } C_1 = \frac{1}{C}$$

and unless $f_1(x)$ is a constant multiple of $u$, by (3.5), we have $j_s[u] > 0$, which leads a contradiction. This contradiction shows that $x$ vanishes at least once in $(t_1,\, t_2)$. This completes the proof. ■

**Corollary 3.2.** *Let* $f_1(x) = x$ *in* (3.1) *and*

$$\lim_{\xi \to t_1^+,\, \eta \to t_2^-} \int_\xi^\eta [p_1(t)(u'(t))^2 - q_1(t)(u(t))^2]dt$$

*exists and is nonpositive, where* $u \in C^1(t_1,\, t_2)$ *not identically zero in any open subinterval of* $(t_1,\, t_2)$ *with* $u(t_1) = 0 = u(t_2)$. *Let* $x$ *be any nontrivial solution of* (3.1) *and* $A_{t_1 t_2}[u,\, x] \geq 0$, *then* $x$ *vanishes at least once in* $(t_1,\, t_2)$ *unless* $x$ *is a constant multiple of* $u$.

*Proof.* It is trivial to see that $f_1$ satisfies (H1)’, (H2)’. In this case, $\alpha_1 = 1$ and $j_s[u] \leq 0$. An application of Lemma 3.1 implies that $x$ vanishes at least once in $(t_1,\, t_2)$ unless $x$ is a constant multiple of $u$. For a proof of this corollary, we refer the reader to [15, 20]. ■

Lemma 3.1 plays an important role to establish the following

**Theorem 3.3.** (Generalization of singular Leighton’s theorem). *Suppose there exists a nontrivial solution* $u$ *of* (3.2) *(*$L_s u = 0$*) in* $(t_1,\, t_2)$ *such that* $u(t_1) = 0 = u(t_2)$. *Let* $x$ *be any nontrivial solution of* (3.1) *(*$l_s x = 0$*). Let* $A_{t_1 t_2}[u,\, x] \geq 0$, *and*

$$\lim_{t \to t_1^+} p_2(t)u(t)u'(t) \geq 0,\ \lim_{t \to t_2^-} p_2(t)u(t)u'(t) \leq 0. \tag{3.6}$$

*Let* (H1)’, (H2)’, (H3), *hold and* $V_s[u] \geq 0$, *then* $x$ *has at least one zero in* $(t_1,\, t_2)$ *unless* $f_1(x)$ *is a constant multiple of* $u$.

*Proof.* Since $u$ is a solution of $L_s u = 0$, so by an application of Green’s identity, we have

$$\int_\xi^\eta u L_s u\, dt = [u(t)p_2(t)u'(t)]_\xi^\eta - \int_\xi^\eta p_2(t)(u'(t))^2 dt + \int_\xi^\eta q_2(t) f_2(u(t))u(t)dt. \tag{3.7}$$

In view of (H3), one can see that

$$\int_{\xi}^{\eta} q_2(t) f_2(u(t))u(t)dt \leq \int_{\xi}^{\eta} (\alpha_2 q_2^+(t) - \alpha_3 q_2^-(t))(u(t))^2 dt. \tag{3.8}$$

From (3.7) and (3.8), we get

$$J_{\xi\eta}[u] \leq [u(t)p_2(t)u'(t)]_{\xi}^{\eta}. \tag{3.9}$$

Letting $\xi \to t_1^+$, $\eta \to t_2^-$ in (3.9) and by (3.6), we get $J_s[u] \leq 0$. Since $V_s[u] \geq 0$, we get $j_s[u] \leq 0$ and hence by an application of Lemma 3.1, every solution $x$ of $l_s x = 0$ has at least one zero in $(t_1,\, t_2)$ unless $f_1(x)$ is a constant multiple of $u$. This completes the proof. ∎

**Corollary 3.4.** (Singular Leighton's theorem). *Let us consider* (3.1) *and* (3.2) *with* $f_1(u) = u = f_2(u)$. *Let* $x$ *be any nontrivial solution of* (3.1). *Let*

$$\begin{aligned} &A_{t_1t_2}[u,\, x] \geq 0,\ \lim_{t\to t_1^+} p_2(t)u(t)u'(t) \geq 0,\ \lim_{t\to t_2^-} p_2(t)u(t)u'(t) \leq 0 \text{ and} \\ &\bar{V}_s[u] = \int_{t_1}^{t_2} [(p_2(t) - p_1(t))(u'(t))^2 + (q_1(t) - q_2(t))(u(t))^2]dt \geq 0. \end{aligned} \tag{3.10}$$

*Suppose there exists a nontrivial solution* $u$ *of* (3.2) *in* $(t_1,\, t_2)$ *such that* $u(t_1) = 0 = u(t_2)$, *then* $x$ *has at least one zero in* $(t_1,\, t_2)$ *unless* $x$ *is a constant multiple of* $u$.

*Proof.* Since $f_1(u) = u = f_2(u)$, it is easy to see that $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and (H1)', (H2)', (H3) are satisfied. In view of (3.10), $V_s[u] \geq 0$ and hence by an application of Theorem 3.3, the required conclusion follows. For a proof of this corollary, we refer the reader to [15], Theorem 1.19[20]. ∎

The following generalization is a special case of Theorem 3.3.

**Theorem 3.5.** (Generalization of singular Sturm-Picone theorem). *Suppose there exists a nontrivial solution* $u$ *of* (3.2) *in* $(t_1,\, t_2)$ *such that* $u(t_1) = 0 = u(t_2)$. *Let* (H1)', (H2)', (H3), hold. Suppose $p_2(t) \geq p_1(t)$. Let $x$ be any nontrivial solution of (3.1).

$$\begin{aligned} &\text{Let } A_{t_1t_2}[u,\, x] \geq 0,\ \lim_{t\to t_1^+} p_2(t)u(t)u'(t) \geq 0,\ \lim_{t\to t_2^-} p_2(t)u(t)u'(t) \leq 0 \text{ and} \\ &\alpha_1 q_1(t) - (\alpha_2 q_2(t) - (\alpha_3 - \alpha_2)q_2^-(t)) \geq 0,\ \forall\, t \in (t_1, t_2), \end{aligned} \tag{3.11}$$

then every solution $x$ of (3.1) has at least one zero in $(t_1,\, t_2)$ unless $f_1(x)$ is a constant multiple of $u$.

*Proof.* In view of (3.11), it is easy to see that $V_s[u] \geq 0$ and the proof of this theorem follows from Theorem 3.3. ∎

The singular Sturm-Picone theorem can be seen as a particular case of Theorem 3.5 in next corollary.

**Corollary 3.6.** (Singular Sturm-Picone theorem). *Consider* (3.1) *and* (3.2) *with* $f_1(u) = u = f_2(u)$. *Let* $p_2(t) \geq p_1(t)$ *and* $q_1(t) \geq q_2(t),\ \forall\, t \in (t_1, t_2)$. *Let* $x$ *be any nontrivial solution of* (3.1). *Let* $A_{t_1 t_2}[u,\, x] \geq 0$ *and*

$$\lim_{t \to t_1^+} p_2(t)u(t)u'(t) \geq 0, \quad \lim_{t \to t_2^-} p_2(t)u(t)u'(t) \leq 0.$$

*Suppose there exists a nontrivial solution* $u$ *of* (3.2) *in* $(t_1,\, t_2)$ *such that* $u(t_1) = 0 = u(t_2)$, *then every solution* $x$ *of* (3.1) *has at least one zero in* $(t_1,\, t_2)$ *unless* $x$ *is a constant multiple of* $u$.

*Proof.* Since $f_1(u) = u = f_2(u)$ in (3.1) and (3.2) so in this case $\alpha_1 = \alpha_2 = \alpha_3 = 1$. It is easy to see that $f_1$ and $f_2$ satisfy (H1)', (H2)', (H3) and Inequality (3.11) is also satisfied. An application of Theorem 3.5 leads the required conclusion. For a proof of Corollary 3.6, we refer the reader to [17] or a singular version of Theorem 1.6 [20]. ∎

## 4. Nonlinear Elliptic Version of Sturm-picone Theorem

In this section, we give a nonlinear analogue of Leighton's theorem to $n$-dimensions. In fact, via this analogue, we give a generalization to Sturm–Picone theorem for semilinear elliptic PDEs in $n$-dimensions. In order to prove a nonlinear analogue of Leighton's theorem, we first establish a nonlinear version of Leighton's variational lemma.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ having a piecewise continuous unit normal. Let $a_i,\, b_i \in C^{\mu}(\bar{\Omega},\, \mathbb{R}),\ f_i,\, g_i \in C^1(\mathbb{R}, \mathbb{R}),\ \forall\, i = 1,\, 2$ where $0 < \mu \leq 1$, $a_i's$ are of indefinite sign $\forall\ i = 1,\, 2$ and $b_1(x) \geq 0,\ \forall\, x \in \bar{\Omega}$.

Let us consider a pair of second-order nonlinear elliptic PDEs

$$-\Delta u = a_1(x) f_1(u) + b_1(x) g_1(u). \tag{4.1}$$

$$-\Delta v = a_2(x) f_2(v) + b_2(x) g_2(v), \tag{4.2}$$

where $f_i$ and $g_i$ satisfy the following assumptions:

(A1) $\exists\ \beta \geq 0$ such that $\frac{g_1(u)}{f_1(u)} \geq \beta,\ \forall\, 0 \neq u \in \mathbb{R}$.

(A2) There exist $\alpha_2,\, \alpha_3,\, \alpha_4 \in (0,\, \infty)$ such that
$g_2(u)u \leq \alpha_4 u^2,\ \ \alpha_3 u^2 \leq f_2(u)u \leq \alpha_2 u^2,\ \forall\, 0 \neq u \in \mathbb{R}$.

In this study, we are interested in non-trivial classical solutions of (4.1) and (4.2). (4.1) and (4.2) can be rewritten in the operator form $l_e u = 0 = L_e u$, where

$$l_e u \equiv \Delta u + a_1(x) f_1(u) + b_1(x) g_1(u), \ \ L_e u \equiv \Delta u + a_2(x) f_2(u) + b_2(x) g_2(u).$$

Let us consider the following quadratic functionals corresponding to (4.1) and (4.2), respectively

$$(4.3)\quad j_e[u] = \int_\Omega [|\nabla u(x)|^2 - \alpha_1 (u(x))^2 (a_1(x) + \beta b_1(x))]dx.$$

$$(4.4)\quad J_e[u] = \int_\Omega [|\nabla u(x)|^2 - (\alpha_2 a_2^+(x) - \alpha_3 a_2^-(x) + \alpha_4 b_2(x))(u(x))^2]dx,$$

where the domain $D_e$ of $j_e$ and $J_e$ is defined to be the set of all real-valued continuous functions defined on $\bar{\Omega}$ which vanish on $\partial\Omega$ and have uniformly continuous first partial derivatives on $\Omega$.

The variation $V_e[u]$ is defined as $V_e[u] = J_e[u] - j_e[u]$, i.e.,

$$(4.5)\quad V_e[u] = \int_\Omega (u(x))^2 [\alpha_1(a_1(x) + \beta b_1(x)) - (\alpha_2 a_2^+(x) - \alpha_3 a_2^-(x) + \alpha_4 b_2(x))]dx.$$

The next lemma deals with a generalization of Leighton's variational lemma.

**Lemma 4.1.** (Generalization of n-dimensional Leighton's variational lemma). *Assume that there exists a nontrivial function $u \in D_e$ such that $j_e[u] \le 0$. Then under the hypotheses/assumption* (H1)$'$, (H2)$'$, (A1)*, every solution $v$ of $l_e v = 0$ vanishes at some point of $\bar{\Omega}$.*

*Proof.* Suppose to the contrary that there exists a solution $v$ of (4.1) such that $v(x) \neq 0$, $\forall x \in \bar{\Omega}$. By (H2)', we have $f_1(v(x)) \neq 0$, $\forall x \in \bar{\Omega}$. Then for $u \in D_e$, we have

$$\begin{aligned}
&\nabla.\left[\frac{(u(x))^2}{f_1(v(x))}\nabla v(x)\right] \\
&= \frac{(u(x))^2}{f_1(v(x))}\Delta v(x) + \frac{\nabla v(x)}{(f_1(v(x)))^2}.[2f_1(v(x))u(x)\nabla u(x) - (u(x))^2 f_1'(v(x))\nabla v(x)] \\
&= \frac{(u(x))^2}{f_1(v(x))}\Delta v(x) + \frac{2u(x)\nabla u(x).\nabla v(x)}{f_1(v(x))} - \frac{(u(x))^2|\nabla v(x)|^2 f_1'(v(x))}{(f_1(v(x)))^2} \\
&= -a_1(x)(u(x))^2 - b_1(x)(u(x))^2\frac{g_1(v(x))}{f_1(v(x))} \\
&\quad - \left[\frac{(u(x))^2|\nabla v(x)|^2 f_1'(v(x))}{(f_1(v(x)))^2} + \frac{|\nabla u(x)|^2}{f_1'(v(x))} - \frac{2u(x)\nabla u(x).\nabla v(x)}{f_1(v(x))}\right] + \frac{|\nabla u(x)|^2}{f_1'(v(x))} \qquad (4.6)\\
&= -a_1(x)(u(x))^2 - b_1(x)(u(x))^2\frac{g_1(v(x))}{f_1(v(x))} \\
&\quad - \left(\frac{u(x)\nabla v(x)\sqrt{f_1'(v(x))}}{f_1(v(x))} - \frac{\nabla u(x)}{\sqrt{f_1'(v(x))}}\right)^2 + \frac{|\nabla u(x)|^2}{f_1'(v(x))} \\
&\le -a_1(x)(u(x))^2 - \beta b_1(x)(u(x))^2 \\
&\quad - \left(\frac{u(x)\nabla v(x)\sqrt{f_1'(v(x))}}{f_1(v(x))} - \frac{\nabla u(x)}{\sqrt{f_1'(v(x))}}\right)^2 + \frac{|\nabla u(x)|^2}{\alpha_1} \quad \text{(by (H1)}', \text{(A1))}.
\end{aligned}$$

This implies that

$$
\begin{aligned}
&|\nabla u(x)|^2 - \alpha_1(u(x))^2(a_1(x)+\beta b_1(x))\\
&\geq \alpha_1 \nabla.\left[\frac{(u(x))^2}{f_1(v(x))}\nabla v(x)\right] + \alpha_1\left(\frac{u(x)\nabla v(x)\sqrt{f_1'(v(x))}}{f_1(v(x))} - \frac{\nabla u(x)}{\sqrt{f_1'(v(x))}}\right)^2. \qquad (4.7)
\end{aligned}
$$

An integration of (4.7) yields

$$
\begin{aligned}
&\int_\Omega (|\nabla u(x)|^2 - \alpha_1(u(x))^2(a_1(x)+\beta b_1(x)))dx\\
&\geq \alpha_1 \int_\Omega \nabla.\left[\frac{(u(x))^2}{f_1(v(x))}\nabla v(x)\right]dx \qquad (4.8)\\
&+\alpha_1 \int_\Omega \left(\frac{u(x)\nabla v(x)\sqrt{f_1'(v(x))}}{f_1(v(x))} - \frac{\nabla u(x)}{\sqrt{f_1'(v(x))}}\right)^2 dx.
\end{aligned}
$$

Since $u$ vanishes on $\partial\Omega$, so an application of Gauss–Green's theorem (see, [9]) implies that

$$\int_\Omega \nabla.\left[\frac{(u(x))^2}{f_1(v(x))}\nabla v(x)\right]dx = 0$$

and

$$\int_\Omega \left(\frac{u(x)\nabla v(x)\sqrt{f_1'(v(x))}}{f_1(v(x))} - \frac{\nabla u(x)}{\sqrt{f_1'(v(x))}}\right)^2 dx = 0 \text{ if and only if}$$

$$\frac{u(x)\nabla v(x)\sqrt{f_1'(v(x))}}{f_1(v(x))} \equiv \frac{\nabla u(x)}{\sqrt{f_1'(v(x))}}, \quad \text{i.e.,}$$

$\nabla.\left(\frac{u(x)}{f_1(v(x))}\right) \equiv 0$ or $u(x) \equiv Cf_1(v(x)),\ \forall\, x \in \bar{\Omega}$ for some non-zero constant $C$. This is not possible because $u = 0$ on $\partial\Omega$ but $f_1(v) \neq 0$ on $\partial\Omega$ ($v \neq 0$ on $\partial\Omega$). This implies that

$$j_e[u] > 0, \text{ which is a contradiction}$$

and hence every solution $v$ of $l_e v = 0$ vanishes at some point of $\bar{\Omega}$. This completes the proof. ■

**Corollary 4.2.** (n-dimensional Leighton's variational lemma). *Let* $f_1(u) = u$ *and either* $b_1(x) = 0$ *or* $g_1(u) = 0$ *in* (4.1). *Let*

$$\int_\Omega [|\nabla u(x)|^2 - a_1(x)(u(x))^2]dx \leq 0,$$

*where* $u$ *is a real-valued continuous functions defined on* $\bar{\Omega}$ *which vanish on* $\partial\Omega$ *and have uniformly continuous first partial derivatives on* $\Omega$, *then every nontrivial solution* $v$ *of* $l_e v = 0$ *vanishes at some point of* $\bar{\Omega}$.

*Proof.* In this case $\alpha_1 = 1$ and it is easy to see that $f_1$ satisfies (H1)', (H2)' and $j_e[u] \le 0$. By an application of Lemma 4.1, the conclusion follows easily. For a proof of this corollary, we refer the reader to Lemma 5.3 [20]. ■

**Remark 4.3.** In fact, one can consider the following nonlinear PDE with nonlinear damping

$$-\Delta u = a_1(x)f_1(u) + b_1(x)g_1(u) + c_1(x)H(\nabla u), \tag{4.9}$$

where $a_1$, $b_1$, $f_1$, $g_1$ are defined earlier. Let $c_1 \in C^{\mu}(\bar{\Omega}, \mathbb{R}^+ = [0, \infty))$, where $0 < \mu \le 1$ and $H \in C^1(M, \mathbb{R}^+)$, $M \subseteq \mathbb{R}^N$. For the existence of classical solutions to Eq. (4.9), we refer the reader to a survey paper [18] and references cited therein. In this case, let us assume that $f_1(s) > 0$, $\forall\, 0 \ne s \in \mathbb{R}$. Assume that there exists a nontrivial function $u \in D_e$ such that $j_e[u] \le 0$. Then every solution $v$ of (4.9) vanishes at some point of $\bar{\Omega}$. The proof goes exactly same as the proof of Lemma 2.1 in view of the positivity of $c_1$, $H$ and $f_1$. For the sake of brevity, we omit the details.

Lemma 2.1 plays a crucial role to establish the following

**Theorem 4.4.** (Generalization of n-dimensional Leighton's theorem). *Suppose there exists a nontrivial solution $u$ of $L_e u = 0$ in $\bar{\Omega}$ such that $u = 0$ on $\partial\Omega$. Let* (H1)', (H2)', (A1) -(A2) hold and $V_e[u] \ge 0$, then every nontrivial solution $v$ of $l_e v = 0$ vanishes at some point of $\bar{\Omega}$.

*Proof.* Since $u$ is a solution of $L_e u = 0$ and $u = 0$ on $\partial\Omega$, so by an application of Green's theorem, we have

$$\int_{\Omega} [a_2(x)f_2(u)u + b_2(x)g_2(u)u - |\nabla u(x)|^2]dx = 0. \tag{4.10}$$

In view of (A2), one can see that

$$\begin{aligned}\int_{\Omega} &(a_2(x)f_2(u(x))u(x) + b_2(x)g_2(u(x))u(x) \\ &-(\alpha_2 a_2^+(x) - \alpha_3 a_2^-(x) + \alpha_4 b_2(x))(u(x))^2)dx \le 0.\end{aligned} \tag{4.11}$$

By (4.10), (4.11), we get $J_e[u] \le 0$. Since $V_e[u] \ge 0$, so this implies that

$$j_e[u] \le J_e[u] \le 0$$

and hence by an application of Lemma 4.1, every nontrivial solution $v$ of $l_e v = 0$ vanishes at some point of $\bar{\Omega}$. This completes the proof. ■

**Corollary 4.5.** (n-dimensional Leighton's theorem). *Let us consider* (4.1) *and* (4.2) *with* $f_1(u) = u = f_2(u)$, $g_1(u) = g_2(u) = 0$. *Let*

$$\bar{V_e}[u] = \int_\Omega (u(x))^2[a_1(x) - (\alpha_2 a_2^+(x) - \alpha_3 a_2^-(x))]dx \geq 0. \tag{4.12}$$

*If there exists a nontrivial solution* $u$ *of* (4.2) *in* $\Omega$ *such that* $u = 0$ *on* $\partial\,\Omega$, *then every nontrivial solution* $v$ *of* (4.1) *vanishes at some point of* $\bar{\Omega}$.

*Proof.* Since $f_1(u) = u = f_2(u)$, it is easy to see that $\alpha_1 = 1,\ \alpha_2 = \alpha_3 = 1,\ \alpha_4 = 0,\ \beta = 0$ and therefore (H1)', (H2)', (A1), (A2) of Theorem 4.4 are satisfied. In view of (4.12), $V_e[u] \geq 0$ and hence by an application of Theorem 4.4, the required conclusion follows. ■

The following generalization is a special case of Theorem 4.4.

**Theorem 4.6.** (Generalization of n-dimensional Sturm-Picone theorem). *Suppose there exists a nontrivial solution* $u$ *of* $L_e u = 0$ *in* $\bar{\Omega}$ *such that* $u = 0$ *on* $\partial\Omega$. *Let* (H1)', (H2)', (A1)-(A2) hold and

$$\alpha(a_1(x) + \beta b_1(x)) - (\alpha_2 a_2(x) - (\alpha_2 - \alpha_3)a_2^-(x) + \alpha_1 b_2(x)) \geq 0,\ \forall\, x \in \bar{\Omega}, \tag{4.13}$$

then every nontrivial solution $v$ of $l_e v = 0$ vanishes at some point of $\bar{\Omega}$.

*Proof.* In view of (4.13), it is easy to observe that $V_e[u] \geq 0$ and therefore the conclusion follows from Theorem 4.4. ■

n-dimensional Sturm–Picone theorem can be seen as a particular case of Theorem 4.6 in

**Corollary 4.7.** (n-dimensional Sturm-Picone theorem). *Consider* (4.1) *and* (4.2) *with* $f_1(u) = u = f_2(u)$, $g_1(u) = 0 = g_2(u)$. *Let* $a_1(x) \geq a_2(x),\ \forall\, x \in \bar{\Omega}$. *If there exists a nontrivial solution* $u$ *of* (4.2) *in* $\bar{\Omega}$ *such that* $u = 0$ *on* $\partial\Omega$, *then every nontrivial solution* $v$ *of* (4.1) *vanishes at some point of* $\bar{\Omega}$.

*Proof.* Since $f_1(u) = u = f_2(u)$ in (4.1) and (4.2) so in this case $\alpha_1 = 1,\ \alpha_2 = \alpha_3 = 1,\ \alpha_4 = 0 = \beta$. It is easy to see that $f_1$ and $f_2$ satisfy (H1)', (H2)', (A1), (A2) and Inequality (4.13) is also satisfied. An application of Theorem 4.6 leads the required conclusion. For a proof of Corollary 4.7, we refer the reader to Theorem 5.5 [20]. ■

## Acknowledgment

The author wants to thank Prof. V. Raghavendra for introducing him this problem and useful discussions, also wants to thank the referee for his/her constructive comments. Financial support under Summer Research Fellowship-2012 from IIT Gandhinagar is gratefully acknowledged.

## REFERENCES


1. D. Aharonov and U. Elias, Singular Sturm comparison theorems, *J. Math. Anal. Appl.*, **371(2)** (2010), 759-763.
2. S. Ahmad and A. C. Lazer, A new generalization of the Sturm comparison theorem to selfadjoint systems, *Proc. Amer. Math. Soc.*, **68** (1978), 185-188.
3. S. Ahmad, On Sturmian theory for second order systems, *Proc. Amer. Math. Soc.*, **87** (1983), 661-665.
4. W. Allegretto, Sturm type theorems for solutions of elliptic nonlinear problems, *Nonlinear Differ. Equ. Appl.*, **7** (2000), 309-321.
5. W. Allegretto and Y.-X. Huang, A Picones identity for the $p$-Laplacean and applications, *Nonlinear Analysis T.M.A.*, **32** (1998), 819-830.
6. W. Allegretto, Sturm theorems for degenerate elliptic equations, *Proc. Amer. Math. Soc.*, **129** (2001), 3031-3035.
7. O. Došlý, The Picone identity for a class of partial differential equations, *Mathematica Bohemica*, **127** (2002), 581-589.
8. O. Došlý and J. Jaroš, A singular version of Leighton's comparison theorem for forced quasilinear second-order differential equations, *Arch. Math. (BRNO)*, **39** (2003), 335-345.
9. L. C. Evans, Partial Differential Equations, Graduate studies in Mathematics, *AMS*, **19** (1999).
10. S. Fišnarová and R. Mašík, Generalized Picone and Riccati inequalities for half-linear differential operators with arbitrary elliptic matrices, *Electronic J. Diff. Equations*, **111** (2010), 1-13.
11. J. Graef and P. Spikes, Comparison and nonoscillation results for perturbed nonlinear differential equations, *Ann. Mat. Pura. Appl.*, **116(4)** (1978), 135-142.
12. J. Jaroš, T. Kusano and N. Yoshida, Forced superlinear oscillations via Picone's identity, *Acta Math. Univ. Comenianae*, **LXIX** (2000), 107-113.
13. J. Jaroš, T. Kusano and N. Yoshida, Picone type inequalities for half-linear elliptic equations and their applications, *Adv. Math. Sci. Appl.*, **12(2)** (2002), 709-724.
14. V. Komkov, A generalization of Leighton's variational theorem, *Applicable Analysis*, **2** (1972), 377-383.
15. W. Leighton, Comparison theorems for linear differential equations of second order, *Proc. Amer. Math. Soc.*, **13** (1962), 603-610.
16. E. Müller-Pfeiffer, Sturm comparison theorems for non-selfadjoint differential equations on non-compact intervals, *Math. Nachr.*, **159** (1992), 291-298.
17. M. Picone, Sui valori eccezionalle di un parametro da cui dipende unâ™ equatione differenzialle lineare del secondo ordine, *Ann. Scuola Norm. Sup. Pisa*, **11** (1910), 1-141.

18. K. Schmitt, Boundary value problems for quasilinear second-order elliptic equations, *Nonlinear Anal.*, **2** (1978), 263-309.

19. C. Sturm, Sur les équations différentielles linéaries du second ordere, *J. Math. Pures. Appl.*, **1** (1836), 106-186.

20. C. A. Swanson, *Comparison and oscillation theory of linear differential equations*, Vol. 48, Academic Prees, New York and London, 1968.

21. Tadie, Oscillation criteria for semilinear elliptic equations with a damping term in $\mathbb{R}^n$, *Electronic J. Diff. Equations*, **51** (2010), 1-5.

22. J. Tyagi and V. Raghavendra, A note on a generalization of Sturm's comparison theorem, *Nonlinear Dyn. Syst. Theory*, **8(2)** (2008), 213-216.

23. N. Yoshida, Oscillation theory of partial differential equations, World Scientific, 2008.

24. N. Yoshida, A Picone identity for half-linear elliptic equations and its applications to oscillation theory, *Nonlinear Anal.*, **71** (2009), 4935-4951.

25. C. Zhang and S. Sun, *Sturm-Picone comparison theorem of second-order linear equations on time scales*, Adv. Diff. Eqs., 2009, 12 pp.

J. Tyagi
Indian Institute of Technology Gandhinagar
Vishwakarma Government Engineering College Complex Chandkheda
Visat-Gandhinagar Highway
Ahmedabad Gujarat 382424
India
E-mail: jtyagi@iitgn.ac.in
jtyagi1@gmail.com