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STRONG UNIQUENESS OF A CLASS OF BEST SIMULTANEOUS APPROXIMATION FROM RS-SETS IN NORMED SPACES

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Abstract. This paper is concerned with the problem of best simultaneous approximations from RS-sets in Banach spaces X. It is shown that the best simultaneous approximations from an RS-set is strongly unique in the case when X is a real Banach space, and strongly unique of order $\alpha \geq 2$ in the case when X is a complex Banach space.

1. Introduction

Let X be a real or complex normed linear space with norm $\|\cdot\|$ and let B be a normed linear space consisting of some real sequences with norm $\|\cdot\|_B$. Let G be a nonempty subset of X and let $\hat{x} = (x_i)$ be a sequence of X satisfying $\sup_{\mathbf{a}\in U} \|\sum_{i=1}^{\infty} a_i x_i\| < +\infty$, where and in the sequel $\mathbf{a} = (a_i)$ and U stands for the closed unit ball of B. Then the problem considered here is of finding $g_0 \in G$ such that

$$\sup_{\mathbf{a}\in U} \left\| \sum_{i=1}^{\infty} a_i(x_i - g_0) \right\| \leq \sup_{\mathbf{a}\in U} \left\| \sum_{i=1}^{\infty} a_i(x_i - g) \right\| \quad \text{for each } g \in G. \tag{1.1}$$

Such an element g_0 is called a best simultaneous approximation to $\hat{x}$ from G. The set of all best simultaneous approximation to $\hat{x}$ from G is denoted by $P_G(\hat{x})$, that is

$$P_G(\hat{x}) = \{g_0 : \ g_0 \text{ satisfies } (1.1)\}.$$

The study of the best simultaneous approximation problem has a long history and continues generating much interest. For example, in the case when B is a space consisting of all n-dimensional vectors and G is a subspace of X, the problem was studied

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by Watson in [17] and characterization results of the best simultaneous approximation were established. In the case when G is nonlinear sets, the characterization results were obtained by Li and Watson in paper [9], where some uniqueness and strong uniqueness results for the best simultaneous approximation from so-called interpolating subspaces and/or from convex sets were also provided. These results were then extended in [11] to the case when B is a (infinite-dimensional) normed linear space consisting of some real sequences with norm $\|\cdot\|_B$ satisfying (2.5) below; while for the special case when X is the continuous function space endowed with the uniform norm, the alternative characterization results was considered in [10]. Moreover, best simultaneous approximation problems in the other sense can be founded in [12, 13, 14, 15].

The family of RS-sets in normed linear spaces, which was introduced in [2] for the real case and in [7] for the complex case, plays a key role in characterizing the uniqueness of the best approximation and/or the Chebyshev centers; see for example [7, 8, 12, 13]. In [8], Li established the strong and/or the generalized strong uniqueness of the Chebyshev center relative to RS-sets. This idea was used in [13] to the study of the best simultaneous approximation problem in the other sense. In the present paper, we further utilize this interesting idea to investigate the uniqueness and the (generalized) strong uniqueness of the best simultaneous approximation to countable many elements. Our main theorems are presented in sections 3 and 4, where the strong uniqueness result and the generalized strong uniqueness results for RS-sets/strict RS-sets are respectively established in real and complex normed linear spaces. It should be noted that results obtained in the present paper are new even for the special case of best simultaneous approximations to finite many elements.

2. Preliminaries and Auxiliary Results

Let X be a normed space (with the norm $\|\cdot\|$) over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Through the whole paper, we always assume that $y_1, y_2, \ldots, y_n$ are n linearly independent elements in X and let G denote the convex set defined by

$$G := \left\{ g = \sum_{i=1}^{n} c_i y_i : \ c_i \in J_i \right\}, \tag{2.1}$$

where each J_i is a subset of the field $\mathbb{F}$ of one of the following types:

(I) the whole of $\mathbb{F}$,

(II) a nontrivial proper closed convex (bounded or unbounded) subset with nonempty interior in $\mathbb{F}$,

(III) a singleton of $\mathbb{F}$.

Let Y_n denote the n-dimensional subspace of X spanned by $y_1, \ldots, y_n$ and set $I := \{1, 2, \ldots, n\}$. We use I_0 and I_1 to denote the index sets of all i such that J_i is of the type (III) and the type (II), respectively. For each $i \in I$, define the linear functional $\tilde{c}_i$ on Y_n by

$$\tilde{c}_i(y) := c_i \quad \text{for each } y = \sum_{i=1}^{n} c_i y_i \in Y_n.$$

Given $g_0 \in G$, set

$$I(g_0) := \{i \in I_1 : \tilde{c}_i(g_0) \in \mathrm{bd}J_i\},$$

where and in the sequel, $\mathrm{bd}J$ and $\mathrm{int}J$ stand for the boundary and interior of a subset J of $\mathbb{F}$, respectively.

To prepare the auxiliary results that are used in the next sections, we first give two lemmas, which were actually proved in [8, Theorem 2.1] and [8, Theorem 3.1] (cf. the proofs given there), respectively. Let Ω be a compact Hausdorff space and let $C(\Omega)$ be the space of all real or complex-valued continuous functions on Ω with the uniform norm. Given a nonempty closed convex subset J of $\mathbb{F}$ and a point $z_0 \in \mathbb{F}$, the normal cone of J at z_0 is denoted by $N_J(z_0)$ and defined by

$$N_J(z_0) := \{u \in \mathbb{F} : \mathrm{Re}(z - z_0)\bar{u} \le 0 \text{ for each } z \in J\}.$$

Lemma 2.1. *Let $\varphi_1, \ldots, \varphi_n$ be n linearly independent elements in $C(\Omega)$ and let*

$$\widetilde{G} := \left\{ \sum_{i=1}^{n} c_i \varphi_i : c_i \in J_i \right\}, \tag{2.2}$$

where each J_i is of types (I), (II) *or* (III). *Let $\phi_0 \in \widetilde{G}$ and Φ be a upper semicontinuous real function on Ω. Then the following statements are equivalent.*

(i) *The following inequalities hold:*

$$\max_{t\in\Omega}[\Phi(t) - \mathrm{Re}\,\phi_0(t)] \le \max_{t\in\Omega}[\Phi(t) - \mathrm{Re}\,\phi(t)] \quad \text{for each } \phi \in \widetilde{G}. \tag{2.3}$$

(ii) *There exist $\{t_j\}_{j=1}^{k} \subseteq \Omega_{\Phi-\phi_0}$, $\{i_j\}_{j=1}^{l} \subseteq I(\phi_0)$, $\tau_{i_j} \in -N_{J_{i_j}}(\tilde{c}_{i_j}(\phi_0)) \setminus \{0\}$ for each $j = 1, 2, \ldots, l$ with $1 + l \le k + l \le 2n + 1$ if $\mathbb{F} = \mathbb{C}$ and $1 + l \le k + l \le n + 1$ if $\mathbb{F} = \mathbb{R}$, $\{\lambda_i\}_{i=1}^{k} \subseteq (0, +\infty)$ such that*

$$\sum_{j=1}^{k} \lambda_j \phi(t_j) + \sum_{j=1}^{l} \tilde{c}_{i_j}(\phi)\bar{\tau}_{i_j} = 0 \quad \text{for each } \phi \in \widetilde{Y}_n, \tag{2.4}$$

where

$$\Omega_{\Phi-\phi_0} := \left\{ t \in \Omega : \Phi(t) - \mathrm{Re}\phi_0(t) = \max_{t\in\Omega}[\Phi(t) - \mathrm{Re}\phi_0(t)] \right\}$$

and

$$\widetilde{Y}_n := \left\{ \sum_{i=1}^{n} c_i \varphi_i : \ c_i = 0 \text{ for each } i \in I_0 \right\}.$$

Lemma 2.2. *Let $\widetilde{G}$ be as in Lemma 2.1 and $\phi_0 \in \widetilde{G}$. Let $\Omega_0 \subseteq \Omega$ be a nonempty closed subset and suppose that*

$$\max_{t\in\Omega_0} \mathrm{Re}\,(\phi_0 - \phi)(t) > 0 \quad \textit{for each } \phi \in \widetilde{G} \setminus \{\phi_0\}.$$

Then we have that

$$\inf_{\phi\in\widetilde{G}\setminus\{\phi_0\}} \max_{t\in\Omega_0} \frac{\mathrm{Re}\,(\phi_0 - \phi)(t)}{\|\phi_0 - \phi\|} > 0.$$

Let B be a space consisting of some sequences in $\mathbb{R}$ with the norm $\|\cdot\|_B$ and U be its closed unit ball. Following [11], we assume throughout this paper that

$$\sup\left\{\sum_{i=1}^{\infty} |a_i| : \ \mathbf{a} = (a_i) \in U\right\} < +\infty. \tag{2.5}$$

Then U is a bounded subset of l_1 and the weak* closure of U in l_1 is denoted by $\overline{U}$ (noting that l_1 is the dual of c, the space of all real infinite convergent sequences with the sup-norm). Without loss of generality, we may assume that $\mathbf{e}^n \in B$ for each $n \in \mathbb{N}$, where the nth coordinate of $\mathbf{e}^n$ is 1 while the other coordinates are 0.

Let

$$\mathcal{F} := \left\{\hat{x} = (x_i) : \ \sup_{\mathbf{a}\in U} \left\|\sum_{i=1}^{\infty} a_i x_i\right\| < +\infty \text{ with each } x_i \in X\right\}. \tag{2.6}$$

We endow $\mathcal{F}$ with the norm defined by

$$\|\hat{x}\|_{\mathcal{F}} := \sup_{\mathbf{a}\in U} \left\|\sum_{i=1}^{\infty} a_i x_i\right\| \quad \text{for each } \hat{x} \in \mathcal{F}.$$

Then $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ is a normed space. Let $\mathcal{F}_0$ denote the subspace of $\mathcal{F}$ consisting of all convergent sequences, that is

$$\mathcal{F}_0 := \left\{\hat{x} = (x_i) \in \mathcal{F} : \ \lim_{n\to\infty} x_i \text{ exists}\right\}.$$

Then $X \subseteq \mathcal{F}_0 \subseteq \mathcal{F}$, where we adopt the convention that $x = (x, x, \dots) \in \mathcal{F}_0$ for each $x \in X$. Clearly, we have that $g_0 \in P_G(\hat{x})$ for $\hat{x} \in \mathcal{F}$ if and only if

$$\|\hat{x} - g_0\|_{\mathcal{F}} \le \|\hat{x} - g\|_{\mathcal{F}} \quad \text{for each } g \in G.$$

Let W be the closed unit ball of X^*, the dual space of X, with the restricted weak* topology, and let $\overline{U} \times W$ be endowed the product topology. Then $\overline{U} \times W$ is a compact Hausdorff space. Let $\hat{x} = (x_i) \in \mathcal{F}$ and let $\phi_{\hat{x}}$ be the function defined by

$$\phi_{\hat{x}}(\mathbf{a}, w) := \sum_{i=1}^{\infty} a_i w(x_i) \quad \text{for each } (\mathbf{a}, w) \in \overline{U} \times W,$$

which plays important roles in this paper. Clearly,

$$\text{Re}\phi_{\hat{x}}(\mathbf{a},w) \le |\phi_{\hat{x}}(\mathbf{a},w)| \le \|\hat{x}\|_{\mathcal{F}} \quad \text{for each } (\mathbf{a},w) \in \overline{U} \times W \tag{2.7}$$

and

$$\sup_{(\mathbf{a},w)\in\overline{U}\times W} \text{Re}\phi_{\hat{x}}(\mathbf{a},w) = \|\hat{x}\|_{\mathcal{F}}. \tag{2.8}$$

Moreover, in the case when $\hat{x} \in \mathcal{F}_0$, $\phi_{\hat{x}}$ is continuous thanks to [11, Lemma 1] (it was stated for a real normed space but the proof works for the complex case). Now, let $\phi^+_{\hat{x}}$ be the upper envelop of $\text{Re}\phi_{\hat{x}}$, that is

$$\phi^+_{\hat{x}}(\mathbf{a},w) := \inf_{O\in N_{(\mathbf{a},w)}} \sup_{(\mathbf{a}',w')\in O} \text{Re}\phi_{\hat{x}}(\mathbf{a}',w'), \tag{2.9}$$

where $N_{(\mathbf{a},w)}$ denotes the set of all open neighborhoods of $(\mathbf{a},w)$ in $\overline{U}\times W$. Then $\phi^+_{\hat{x}}$ is upper semicontinuous on $\overline{U}\times W$ and

$$\max_{(\mathbf{a},w)\in\overline{U}\times W} \phi^+_{\hat{x}}(\mathbf{a},w) = \sup_{(\mathbf{a},w)\in\overline{U}\times W} \text{Re}\phi_{\hat{x}}(\mathbf{a},w) = \|\hat{x}\|_{\mathcal{F}}, \tag{2.10}$$

where the first equality holds by [5, Remark 4] and the last one by (2.8). Furthermore,

$$\phi^+_{\hat{x}-x} = \phi^+_{\hat{x}} - \text{Re}\phi_x \quad \text{for each } x \in X. \tag{2.11}$$

For the main theorem of this section, we set $W_0 := \text{ext}W$ and let $\overline{W}_0$ denote the weak* closure of W_0 in X^*, where and in the sequel, $\text{ext}A$ stands for the set of all extreme points of the set A. For $\hat{x} \in \mathcal{F}$ and $g \in G$, write

$$M^+_{\hat{x}-g} := \{(\mathbf{a},w) \in \overline{U}\times\overline{W}_0 :\ (\phi^+_{\hat{x}} - \text{Re}\phi_g)(\mathbf{a},w) = \|\hat{x}-g\|_{\mathcal{F}}\}.$$

Then $M^+_{\hat{x}-g}$ is a nonempty compact subset of $\overline{U}\times W$. In fact, by (2.10) and (2.11), one has that

$$\|\hat{x}-g\|_{\mathcal{F}} \ge \max_{(\mathbf{a},w)\in\overline{U}\times\overline{W}_0} (\phi^+_{\hat{x}} - \text{Re}\phi_g)(\mathbf{a},w). \tag{2.12}$$

On the other hand, by (2.11) and the definition of $\|\hat{x}-g\|_{\mathcal{F}}$, we obtain that

$$\begin{aligned} \|\hat{x}-g\|_{\mathcal{F}} &= \sup_{(\mathbf{a},w)\in\overline{U}\times W} \text{Re}\phi_{\hat{x}-g}(\mathbf{a},w) \\ &= \sup_{(\mathbf{a},w)\in\overline{U}\times W_0} \text{Re}\phi_{\hat{x}-g}(\mathbf{a},w) \\ &\le \max_{(\mathbf{a},w)\in\overline{U}\times\overline{W}_0} \phi^+_{\hat{x}-g}(\mathbf{a},w). \end{aligned}$$

This together with (2.12) implies that

$$\|\hat{x}-g\|_{\mathcal{F}} = \max_{(\mathbf{a},w)\in\overline{U}\times\overline{W}_0} (\phi^+_{\hat{x}} - \mathrm{Re}\phi_g)(\mathbf{a},w). \tag{2.13}$$

Hence, $M^+_{\hat{x}-g}$ is a nonempty compact subset of $\overline{U}\times W$ because $\phi^+_{\hat{x}} - \mathrm{Re}\phi_g$ is upper semicontinuous on $\overline{U}\times W$.

Let

$$\Omega := \overline{U}\times\overline{W}_0 \quad \text{and} \quad \widetilde{G} := \left\{ \sum_{i=1}^{n} c_i\phi_{y_i} \,:\, c_i \in J_i \right\}. \tag{2.14}$$

Then $\phi^+_{\hat{x}}$ is upper semicontinuous on Ω and $\phi_x \in C(\Omega)$ for each $x \in X$. Moreover, by (2.13) we have the following equivalence:

$$g_0 \in P_G(\hat{x}) \Longleftrightarrow [(2.3) \text{ holds with } \Phi := \phi^+_{\hat{x}} \text{ and } \phi_0 := \phi_{g_0}]. \tag{2.15}$$

In the rest of the present paper, without loss of generality, we always assume that $I_0 = \emptyset$. Indeed, otherwise, G can be expressed as $G = G_0 + \bar{g}$, where G_0 is such that the corresponding $I_0 = \emptyset$ and $\bar{g}$ is a fixed element, and approximating $\hat{x}$ from G is equivalent to approximating $\hat{x} - \bar{g}$ from G_0. Under the assumption that $I_0 = \emptyset$, we have that $Y_n = \widetilde{Y}_n$. Now we are ready to give the main result of this section.

Theorem 2.1. *Let G be defined by* (2.1) *and $g_0 \in G$. Let $\hat{x} \in \mathcal{F}$. Then $g_0 \in P_G(\hat{x})$ if and only if there exist $\{(\mathbf{a}^j, w^j)\}_{j=1}^k \subseteq M^+_{\hat{x}-g_0}$ with $k \geq 1$, $\{i_j\}_{j=1}^l \subseteq I(g_0)$, $\{\tau_{i_j}\}_{j=1}^l$ with each $\tau_{i_j} \in -N_{J_{i_j}}(\tilde{c}_{i_j}(g_0)) \setminus \{0\}$ and $\{\lambda_i\}_{i=1}^k \subseteq (0,+\infty)$ such that*

$$\sum_{j=1}^{k} \lambda_j \phi_g(\mathbf{a}^j, w^j) + \sum_{j=1}^{l} \tilde{c}_{i_j}(g)\overline{\tau}_{i_j} = 0 \quad \textit{for each } g \in Y_n. \tag{2.16}$$

Moreover, if $\hat{x} \in \mathcal{F}_0$, then $M^+_{\hat{x}-g_0}$ can be replaced by $E_{\hat{x}-g_0}$, which is defined by

$$E_{\hat{x}-g_0} := \{(\mathbf{a}, w) \in \mathrm{ext}U \times \mathrm{ext}W \,:\, (\phi_{\hat{x}} - \phi_{g_0})(\mathbf{a}, w) = \|\hat{x} - g_0\|_{\mathcal{F}}\}.$$

Proof. Consider the sets Ω and $\widetilde{G}$ defined by (2.14). Let $\Phi := \phi^+_{\hat{x}}$ and $\phi_0 := \phi_{g_0}$. Then it is routine to check that $M^+_{\hat{x}-g_0} = \Omega_{\Phi-\phi_0}$. Thus the first conclusion of this theorem follows directly from (2.15) and Lemma 2.1.

Now we assume that $\hat{x} \in \mathcal{F}_0$. We below prove that $M^+_{\hat{x}-g_0}$ can be replaced by $E_{\hat{x}-g_0}$. To do this, suppose that $\hat{x} \in \mathcal{F}_0$ and $g_0 \in P_G(\hat{x})$. Then (2.16) holds with appropriate $\{(\mathbf{a}^j, w^j)\}_{j=1}^k \subseteq M^+_{\hat{x}-g_0}$, $\{i_j\}_{j=1}^l$, $\{\tau_{i_j}\}_{j=1}^l$, $\{\lambda_i\}_{i=1}^k$ as stated in Theorem 2.1. Hence

$$\sum_{j=1}^{k} \lambda_j \mathrm{Re}\phi_g(\mathbf{a}^j, w^j) + \sum_{j=1}^{l} \mathrm{Re}\tilde{c}_{i_j}(g)\overline{\tau}_{i_j} = 0 \quad \text{for each } g \in Y_n. \tag{2.17}$$

Set

$$\mathcal{P}^+ := \{\mathbf{p}(\mathbf{a}, w) : (\mathbf{a}, w) \in M^+_{\hat{x}-g_0}\} \quad \text{and} \quad \mathcal{P} := \{\mathbf{p}(\mathbf{a}, w) : (\mathbf{a}, w) \in E_{\hat{x}-g_0}\},$$

where

$$\mathbf{p}(\mathbf{a}, w) := (\mathrm{Re}\phi_{y_1}(\mathbf{a}, w), \ldots, \mathrm{Re}\phi_{y_n}(\mathbf{a}, w)) \quad \text{for each } (\mathbf{a}, w) \in \overline{U} \times W.$$

Then $\mathcal{P}^+$ is a nonempty compact subset of $\mathbb{R}^n$ because $M^+_{\hat{x}-g_0}$ is nonempty compact. Furthermore, let $\lambda := \sum_{j=1}^k \lambda_j$ and let

$$\mathbf{q} := -\frac{1}{\lambda} \sum_{j=1}^{l} (\mathrm{Re}\tilde{c}_{i_j}(y_1)\bar{\tau}_{i_j}, \ldots, \mathrm{Re}\tilde{c}_{i_j}(y_n)\bar{\tau}_{i_j}). \tag{2.18}$$

It then follows from (2.17) that $\mathbf{q} \in \mathrm{co}\mathcal{P}^+$, where and in the sequel, $\mathrm{co}A$ stands for the convex hull of the set A. Below we prove that $\mathcal{P}^+ \subseteq \mathrm{co}\mathcal{P}$. Granting this, we have that $\mathbf{q} \in \mathrm{co}\mathcal{P}$. Note by (2.18) that $\mathbf{q} \in \mathrm{co}\mathcal{P}$ if and only if there exist $\{(\mathbf{a}^j, w^j)\}_{j=1}^k \subseteq E_{\hat{x}-g_0}$, together with $\{i_j\}_{j=1}^l$, $\{\tau_{i_j}\}_{j=1}^l$, $\{\lambda_i\}_{i=1}^k$, such that (2.17) holds, which is clearly equivalent to that (2.16) holds. Therefore, the conclusion follows.

To show that $\mathcal{P}^+ \subseteq \mathrm{co}\mathcal{P}$, let $\mathbf{p}(\mathbf{a}_0, w_0) \in \mathcal{P}^+$ for some fixed $(\mathbf{a}_0, w_0) \in M^+_{\hat{x}-g_0}$. Define

$$M^1_{\hat{x}-g_0}(\mathbf{a}_0) := \{w \in W : (\mathrm{Re}\phi_{\hat{x}} - \mathrm{Re}\phi_{g_0})(\mathbf{a}_0, w) = \|\hat{x} - g_0\|_{\mathcal{F}}\}$$

and

$$\mathcal{P}^+(\mathbf{a}_0) := \{\mathbf{p}(\mathbf{a}_0, w) : w \in M^1_{\hat{x}-g_0}(\mathbf{a}_0)\}.$$

Then $M^1_{\hat{x}-g_0}(\mathbf{a}_0) \subseteq W$ is nonempty compact convex subset as $\phi_{\hat{x}}$ is continuous and, so is $\mathcal{P}^+(\mathbf{a}_0) \subseteq \mathbb{R}^n$. Moreover, $w_0 \in M^1_{\hat{x}-g_0}(\mathbf{a}_0)$. This means that

$$\mathbf{p}(\mathbf{a}_0, w_0) \in \mathcal{P}^+(\mathbf{a}_0) = \mathrm{co}\left(\mathrm{ext}(\mathcal{P}^+(\mathbf{a}_0))\right), \tag{2.19}$$

where the equality holds by the Carathéodory theorem (cf. [3, p. 22]). Since

$$\mathrm{ext}(\mathcal{P}^+(\mathbf{a}_0)) \subseteq \left\{\mathbf{p}(\mathbf{a}_0, w) : w \in \mathrm{ext}M^1_{\hat{x}-g_0}(\mathbf{a}_0)\right\}$$

(the proof of which is straightforward) and since $\mathrm{ext}M^1_{\hat{x}-g_0}(\mathbf{a}) = M^1_{\hat{x}-g_0}(\mathbf{a}) \cap \mathrm{ext}W$ by [6, Lemma (d), p. 32] (as $M^1_{\hat{x}-g_0}(\mathbf{a})$ is an extremal subset of W), it follows from (2.19) that

$$\mathbf{p}(\mathbf{a}_0, w_0) \in \mathrm{co}\left\{\mathbf{p}(\mathbf{a}_0, w) : w \in M^1_{\hat{x}-g_0}(\mathbf{a}) \cap \mathrm{ext}W\right\}.$$

Therefore, without loss of generality, we may assume that $w_0 \in \mathrm{ext}W$. Write

$$M^2_{\hat{x}-g_0}(w_0) := \left\{\mathbf{a} \in \overline{U} : (\mathrm{Re}\phi_{\hat{x}} - \mathrm{Re}\phi_{g_0})(\mathbf{a}, w_0) = \|\hat{x} - g_0\|_{\mathcal{F}}\right\}.$$

Then, with the similar argument, one can check that

$$\begin{aligned}
&\mathbf{p}(\mathbf{a}_0, w_0) \in \left\{\mathbf{p}(\mathbf{a}, w_0) : \ \mathbf{a} \in M^2_{\hat{x}-g_0}(w_0)\right\} \\
&= \mathrm{co}\left\{\mathbf{p}(\mathbf{a}, w_0) : \ \mathbf{a} \in M^2_{\hat{x}-g_0}(w_0) \cap \mathrm{ext}U\right\} \\
&\subseteq \mathrm{co}\mathcal{P}.
\end{aligned}$$

This shows that $\mathcal{P}^+ \subseteq \mathrm{co}\mathcal{P}$ and completes the proof. ■

Given a nonempty subset Z of X, we write

$$d_Z(\hat{x}) := \inf_{g \in Z} \|\hat{x} - g\|_{\mathcal{F}}.$$

Then the following lemma is useful in the next sections.

Lemma 2.3. *Let $Z \subseteq X$ and $\hat{x} \in \mathcal{F}$ be such that $d_Z(\hat{x}) > d_X(\hat{x})$. If $g_0 \in P_Z(\hat{x})$, then the following assertion holds:*

$$\inf_{(\mathbf{a},w) \in M^+_{\hat{x}-g_0}} \left| \sum_{i=1}^{\infty} a_i \right| > 0. \tag{2.20}$$

Proof. Suppose on the contrary that there exists $\{(\mathbf{a}^j, w^j)\}_{j=1}^{\infty} \subseteq M^+_{\hat{x}-g_0}$ such that $\sum_{i=1}^{\infty} a_i^j \to 0$ as $j \to \infty$. Then by the assumption and (2.11), one has that $\phi^+_{\hat{x}-g_0}(\mathbf{a}^j, w^j) = \|\hat{x} - g_0\|_{\mathcal{F}} = d_G(\hat{x})$. Thus, for each $x \in X$, we obtain from (2.10) and (2.11) that

$$\begin{aligned}
\|\hat{x} - x\|_{\mathcal{F}} \geq \phi^+_{\hat{x}-x}(\mathbf{a}^j, w^j) &= \phi^+_{\hat{x}-g_0}(\mathbf{a}^j, w^j) + \mathrm{Re}\phi_{g_0-x}(\mathbf{a}^j, w^j) \\
&= d_G(\hat{x}) + \left(\sum_{i=1}^{\infty} a_i^j\right) \mathrm{Re}w^j(g_0 - x).
\end{aligned}$$

Letting $j \to \infty$ gives that $\|\hat{x} - x\|_{\mathcal{F}} \geq d_G(\hat{x})$ for each $x \in X$. Hence, $d_X(\hat{x}) \geq d_G(\hat{x})$, which contradicts that $d_G(\hat{x}) > d_X(\hat{x})$ and the proof is complete. ■

The following definition is taken from [1].

Definition 2.1. An n-dimensional subspace Y_n of a normed space X is called an interpolating subspace (*resp.* a strictly interpolating subspace) if, for any $f_1, \cdots, f_n \in W_0$ (*resp.* $\overline{W}_0$), any n numbers $c_1, \cdots, c_n$ in $\mathbb{F}$, there exists a unique $y \in Y_n$ such that $f_i(y) = c_i, \ i = 1, \cdots, n$.

The notion of an RS-set in the following definition was introduced by Amir in [2] for the case when $\mathbb{F} = \mathbb{R}$ and by Li in [7, 8] for the case when $\mathbb{F} = \mathbb{C}$.

Definition 2.2. Let X be a normed linear space on the field $\mathbb{F}$ and let $y_1, y_2, \ldots, y_n$ be a n linear independent elements of X. The set G defined by (2.1) is called an RS-set (resp. a strict RS-set) if every subset of $\{y_1, y_2, \ldots, y_n\}$ consisting of all y_i with J_i of type (I) and some y_i with J_i of type (II) spans an interpolating subspace (resp. a strictly interpolating subspace).

We end this section with the following useful lemma.

Lemma 2.4. *Let G be a strict RS-set (resp. an RS-set) in X over the field $\mathbb{F}$. Let $\hat{x} \in \mathcal{F}$ (resp. $\hat{x} \in \mathcal{F}_0$) and $g_0 \in P_G(\hat{x})$. Suppose that $\{(\mathbf{a}^j, w^j)\}_{j=1}^k \subseteq M^+_{\hat{x}-g_0}$ (resp. $\{(\mathbf{a}^j, w^j)\}_{j=1}^k \subseteq E_{\hat{x}-g_0}$) with $k \geq 1$, $\{i_j\}_{j=1}^l \subseteq I(g_0)$, $\{\tau_{i_j}\}_{j=1}^l$ with each $\tau_{i_j} \in -N_{J_{i_j}}(\tilde{c}_{i_j}(g_0))$ and $\{\lambda_i\}_{i=1}^k \subseteq (0, +\infty)$ satisfy* (2.16). *Then $\{w^j\}_{j=1}^k$ contains at least $n-l$ linear independent elements provided that $d_G(\hat{x}) < d_X(\hat{x})$.*

Proof. We only prove this lemma for the case when G is a strict RS-set and $\mathbb{F} = \mathbb{C}$ as it is similar for other cases. Suppose that the stated condition is satisfied and let

$$Q := \{g \in Y_n : \tilde{c}_{i_j}(g) = 0, j = 1, 2, \ldots, l\}. \tag{2.21}$$

Then it follows from the assumption made on G that Q is a strict interpolating subspace of dimension $n-l$. Without loss of generality, we assume that $\{w^j\}_{j=1}^k$ is linearly independent, and suppose on the contrary that $k < n-l$. Then there exists $\bar{g} \in Q$ such that $w^j(\bar{g}) = \overline{\beta_j}$, where $\beta_j := \lambda_j \sum_{i=1}^{\infty} a_i^j$ for each $j = 1, 2, \ldots, k$. It follows from (2.17) and (2.21) that

$$\sum_{j=1}^k \lambda_j \mathrm{Re}\phi_{\bar{g}}(\mathbf{a}^j, w^j) = \sum_{j=1}^k |\beta_j|^2 = 0 \quad \text{for each } 1 \leq j \leq k;$$

hence $\beta_1 = \cdots = \beta_k = 0$. Consequently,

$$\sum_{j=1}^k \lambda_j \phi_x(\mathbf{a}^j, w^j) = \sum_{j=1}^k \beta_j w^j(x) = 0 \quad \text{for each } x \in X.$$

Thus by Theorem 2.1, $g_0 \in P_X(\hat{x})$ and so $d_X(\hat{x}) = d_G(\hat{x})$. This contradicts that $d_G(\hat{x}) > d_X(\hat{x})$ and the proof is complete. ■

3. Strong Uniqueness From Real RS-Sets

We assume for the whole section that X is a real normed space. Then the main result of this section is as follows.

Theorem 3.1. *Let G be a strict RS-set (resp. an RS-set) on $\mathbb{R}$, $\hat{x} \in \mathcal{F}$ (resp. $\mathcal{F}_0$) and $g_0 \in P_G(\hat{x})$. If $d_G(\hat{x}) > d_X(\hat{x})$, then g_0 is strongly unique, that is, there exists $r > 0$ such that*

$$\|\hat{x} - g\|_{\mathcal{F}} \geq \|\hat{x} - g_0\|_{\mathcal{F}} + r\|g - g_0\| \quad \textit{for each } g \in G.$$

Proof. As before, we only prove the theorem for the case when $\hat{x} \in \mathcal{F}$ and G is a strict RS-set. To do this, we will prove that

$$r := \inf_{g \in G \setminus \{g_0\}} \max_{(\mathbf{a},w) \in M^+_{\hat{x}-g_0}} \frac{\phi_{g_0-g}(\mathbf{a}, w)}{\|g_0 - g\|} > 0. \tag{3.1}$$

Granting this, for any $g \in G \setminus \{g_0\}$, we take $(\mathbf{a}^0, w^0) \in M^+_{\hat{x}-g_0}$ such that

$$\phi_{g_0-g}(\mathbf{a}^0, w^0) = \max_{(\mathbf{a},w) \in M^+_{\hat{x}-g_0}} \phi_{g_0-g}(\mathbf{a}, w) \geq r\|g_0 - g\| \tag{3.2}$$

and then, we conclude that

$$\|\hat{x}-g\|_{\mathcal{F}} \geq \phi^+_{\hat{x}-g}(\mathbf{a}^0, w^0) = \phi^+_{\hat{x}-g_0}(\mathbf{a}^0, w^0) + \phi_{g_0-g}(\mathbf{a}^0, w^0) \geq \|\hat{x}-g_0\|_{\mathcal{F}} + r\|g_0-g\|,$$

which completes the proof.

To show (3.1), we consider, as in the proof of Theorem 2.1, the set Ω and $\widetilde{G}$ defined by (2.14). We assert that

$$\max\Big\{(\phi_{g_0} - \phi_g)(\mathbf{a}, w) : \ (\mathbf{a}, w) \in M^+_{\hat{x}-g_0}\Big\} > 0 \quad \text{for each } \phi_g \in \widetilde{G} \setminus \{\phi_{g_0}\}. \tag{3.3}$$

In fact, otherwise, we can choose $g_1 \in G \setminus \{g_0\}$ such that

$$\max\Big\{(\phi_{g_0} - \phi_{g_1})(\mathbf{a}, w) : \ (\mathbf{a}, w) \in M^+_{\hat{x}-g_0}\Big\} \leq 0. \tag{3.4}$$

Since $g_0 \in P_G(\hat{x})$, it follows from Theorem 2.1 that there exist $\{(\mathbf{a}^j, w^j)\}_{j=1}^k \subseteq M^+_{\hat{x}-g_0}$ with $k \geq 1$, $\{i_j\}_{j=1}^l \subseteq I(g_0)$, $\{\tau_{i_j}\}_{j=1}^l$ with each $\tau_{i_j} \in -N_{J_{i_j}}(\tilde{c}_{i_j}(g_0)) \setminus \{0\}$ and $\{\lambda_i\}_{i=1}^k \subseteq (0, +\infty)$ such that

$$\sum_{j=1}^k \lambda_j \phi_{g_1-g_0}(\mathbf{a}^j, w^j) + \sum_{j=1}^l \tilde{c}_{i_j}(g_1 - g_0)\tau_{i_j} = 0. \tag{3.5}$$

Noting that $\phi_{g_0-g_1} = \phi_{g_0} - \phi_{g_1}$ and that $\tilde{c}_{i_j}(g_1 - g_0)\tau_{i_j} \geq 0$ for each $j = 1, 2, \ldots, l$, we have by (3.4) and (3.5) that

$$0 \geq \sum_{j=1}^k \lambda_j \phi_{g_0-g_1}(\mathbf{a}^j, w^j) = \sum_{j=1}^l \tilde{c}_{i_j}(g_1 - g_0)\tau_{i_j} \geq 0.$$

Consequently,

$$\phi_{g_0-g_1}(\mathbf{a}^j, w^j) = 0 \quad \text{for each } j = 1, 2, \ldots, k \tag{3.6}$$

and

$$\tilde{c}_{i_j}(g_0 - g_1) = 0 \quad \text{for each } j = 1, 2, \ldots, l. \tag{3.7}$$

Then one has that

$$w^j(g_0 - g_1) = 0 \quad \text{for each } j \text{ with } 1 \le j \le k \tag{3.8}$$

because $\left(\sum_{j=1}^k a_i^j\right) w^j(g_0 - g_1) = 0$ by (3.6) and $\left(\sum_{j=1}^k a_i^j\right) \neq 0$ by Lemma 2.3. On the other hand, one sees that $g_0 - g_1 \in Q$ by (3.7), where Q is defined by (2.21). Since Q is a strict interpolating subspace of dimension $n - l$ by assumption and since $\{w^j\}_{j=1}^k$ contains at least $n - l$ linear independent elements by Lemma 2.4, it follows that $g_0 - g_1 = 0$. This is a contradiction and so assertion (3.3) is verified.

Now we apply Lemma 2.2 (to ϕ_{g_0} and $M^+_{\hat{x}-g_0}$ in place of g_0 and Ω_0) to conclude that

$$\inf_{\phi_g \in \widetilde{G}\setminus\{\phi_{g_0}\}} \max_{(\mathbf{a},w)\in M^+_{\hat{x}-g_0}} \frac{(\phi_{g_0} - \phi_g)(\mathbf{a}, w)}{\|\phi_{g_0} - \phi_g\|} > 0. \tag{3.9}$$

Note that

$$\|\phi_{g_0} - \phi_g\| = \sup_{\mathbf{a}\in\overline{U}} \left|\sum_{i=1}^{\infty} a_i\right| \sup_{w\in\overline{W}_0} |w(g_0 - g)| \ge \inf_{(\mathbf{a},w)\in M^+_{\hat{x}-g_0}} \left|\sum_{i=1}^{\infty} a_i\right| \|g_0 - g\|.$$

This together with (3.9) implies that (3.1) holds and then the proof is complete. ■

4. Strong Uniqueness From Complex RS-Sets

We assume for the whole section that X is a complex normed linear space. This section is devoted to the study of the uniqueness and strong uniqueness of best simultaneous approximation from an RS-set in the complex space X. We begin with the following lemma.

Lemma 4.1. *Let $\hat{x} \in \mathcal{F}$, $g_0 \in P_G(\hat{x})$ and $(\mathbf{a}, w) \in M^+_{\hat{x}-g_0}$. Then, for any $g \in G$, we have the following inequality:*

$$\|\hat{x} - g\|^2_{\mathcal{F}} \ge d_G(\hat{x})^2 + |\phi_{g_0-g}(\mathbf{a}, w)|^2 + 2d_G(\hat{x})\mathrm{Re}\phi_{g_0-g}(\mathbf{a}, w). \tag{4.1}$$

Proof. By the choice of g_0 and $(\mathbf{a}, w)$, we obtain that

$$\phi^+_{\hat{x}-g_0}(\mathbf{a}, w) = \|\hat{x} - g_0\|_{\mathcal{F}} = d_G(\hat{x}). \tag{4.2}$$

Fix $k = 1, 2, \ldots.$ In view of (2.9) and the continuity of the function ϕ_{g_0-g}, we can choose $O_k \in N_{(\mathbf{a},w)}$ such that

$$\phi^+_{\hat{x}-g_0}(\mathbf{a}, w) \leq \sup_{(\mathbf{a}',w')\in O_k} \mathrm{Re}\phi_{\hat{x}-g_0}(\mathbf{a}', w') < \phi^+_{\hat{x}-g_0}(\mathbf{a}, w) + \frac{1}{k} \tag{4.3}$$

and

$$|\phi_{g_0-g}(\mathbf{a}', w') - \phi_{g_0-g}(\mathbf{a}, w)| < \frac{1}{k} \quad \text{for each } (\mathbf{a}', w') \in O_k. \tag{4.4}$$

It follows from (4.2) and (4.3) that

$$d_G(\hat{x}) \leq \sup_{(\mathbf{a}',w')\in O_k} \mathrm{Re}\phi_{\hat{x}-g_0}(\mathbf{a}', w') < d_G(\hat{x}) + \frac{1}{k}.$$

Take a sequence $\{(\mathbf{a}^k, w^k)\}$ with each $(\mathbf{a}^k, w^k) \in O_k$ such that

$$d_G(\hat{x}) - \frac{1}{k} < \mathrm{Re}\phi_{\hat{x}-g_0}(\mathbf{a}^k, w^k) < d_G(\hat{x}) + \frac{1}{k}.$$

By (2.7), we get that

$$|\phi_{\hat{x}-g_0}(\mathbf{a}^k, w^k)| \leq \|\hat{x} - g_0\|_{\mathcal{F}} = d_G(\hat{x}).$$

It follows that

$$\begin{aligned}(d_G(\hat{x}) - \tfrac{1}{k})^2 &\leq |\mathrm{Re}\phi_{\hat{x}-g_0}(\mathbf{a}^k, w^k)|^2 + |\mathrm{Im}\phi_{\hat{x}-g_0}(\mathbf{a}^k, w^k)|^2 \\ &= |\phi_{\hat{x}-g_0}(\mathbf{a}^k, w^k)|^2 \leq d_G(\hat{x})^2.\end{aligned}$$

Passing to the limits, one has that $\lim_{k\to\infty} \mathrm{Im}\phi_{\hat{x}-g_0}(\mathbf{a}^k, w^k) = 0$; hence

$$\lim_{k\to\infty} \phi_{\hat{x}-g_0}(\mathbf{a}^k, w^k) = d_G(\hat{x}). \tag{4.5}$$

Moreover, by (4.4), we conclude that

$$\lim_{k\to\infty} \phi_{g_0-g}(\mathbf{a}^k, w^k) = \phi_{g_0-g}(\mathbf{a}, w). \tag{4.6}$$

Note that

$$\begin{aligned}\|\hat{x} - g\|^2_{\mathcal{F}} &\geq |\phi_{\hat{x}-g}(\mathbf{a}^k, w^k)|^2 \\ &= |\phi_{\hat{x}-g_0}(\mathbf{a}^k, w^k) + \phi_{g_0-g}(\mathbf{a}^k, w^k)|^2 \\ &= |\phi_{\hat{x}-g_0}(\mathbf{a}^k, w^k)|^2 + |\phi_{g_0-g}(\mathbf{a}^k, w^k)|^2 \\ &\quad +2\mathrm{Re}\left[\phi_{\hat{x}-g_0}(\mathbf{a}^k, w^k)\overline{\phi_{g_0-g}(\mathbf{a}^k, w^k)}\right].\end{aligned}$$

Thus, taking the limits and making use of (4.6) and (4.5), one checks (4.1) and completes the proof. ■

Recall that a convex subset J of $\mathbb{C}$ is called strictly convex if, for any two distinct points $z_1, z_2 \in J$, $\frac{1}{2}(z_1 + z_2) \in \text{int}J$.

Theorem 4.1. *Let G be a strict RS-set (resp. an RS-set) and let $\hat{x} \in \mathcal{F}$ (resp. $\hat{x} \in \mathcal{F}_0$) with $d_G(\hat{x}) > d_X(\hat{x})$. If each J_i ($i \in I_1$) is strictly convex, then the best simultaneous approximation to $\hat{x}$ from G is unique.*

Proof. We only prove the conclusion for the case when G is a strict RS-set. Suppose on the contrary that $P_G(\hat{x})$ contains two distinct points g_1 and g_2. Then $g_0 := \frac{1}{2}(g_1 + g_2) \in P_G(\hat{x})$. Let $(\mathbf{a}, w) \in M^+_{\hat{x}-g_0}$ and fix $i = 1, 2$. Then, we apply Lemma 4.1 (applied to g_i in place of g) to conclude that

$$\frac{1}{4}|\phi_{g_2-g_1}(\mathbf{a}, w)|^2 + d_G(\hat{x})\text{Re}\phi_{g_2-g_1}(\mathbf{a}, w) \leq 0$$

and

$$\frac{1}{4}|\phi_{g_1-g_2}(\mathbf{a}, w)|^2 + d_G(\hat{x})\text{Re}\phi_{g_1-g_2}(\mathbf{a}, w) \leq 0$$

(noting that $\|\hat{x} - g_1\|_{\mathcal{F}} = \|\hat{x} - g_2\|_{\mathcal{F}} = d_G(\hat{x})$). Summing the above two inequalities implies that $\phi_{g_1-g_2}(\mathbf{a}, w) = 0$, that is $(\sum_{i=1}^{\infty} a_i)w(g_1 - g_2) = 0$. Since $\sum_{i=1}^{\infty} a_i \neq 0$ Lemma 2.3, it follows that $w(g_1 - g_2) = 0$. Thus, we complete the proof of the following assertion:

$$w(g_1 - g_2) = 0 \quad \text{for each } w \in A, \tag{4.7}$$

where

$$A := \left\{ w \in \overline{W}_0 : \ (\mathbf{a}, w) \in M^+_{\hat{x}-g_0} \text{ for some } \mathbf{a} \in \overline{U} \right\}.$$

Below we prove that $g_1 = g_2$, which is a contradiction and then completes the proof. To do this, we first note that $I(g_0) \subseteq I(g_1) \cap I(g_2)$ and that each J_i ($i \in I_1$) is strict convex. Then one has that $\tilde{c}_i(g_1) = \tilde{c}_i(g_2)$ for each $i \in I(g_0)$, that is $g_1 - g_2 \in Q$, where

$$Q := \{g \in Y_n : \ c_i(g) = 0, \ \forall\, i \in I(g_0)\}.$$

Since Q is a strictly interpolating subspace of dimension $n - \text{card}I(g_0)$ by assumption (as G is a strict RS-set) and since A contains at least $n - \text{card}I(g_0)$ linearly independent elements by Lemma 2.4, it follows from (4.7) that $g_1 = g_2$, which completes the proof. ■

In the rest of this paper, we will establish the generalized strong uniqueness results for the best simultaneous approximation from an RS-sets. Let $\alpha > 0$. Recall that

$g_0 \in P_G(\hat{x})$ is a strongly unique best simultaneous approximation of order α if there exists a constant number $\lambda_\alpha > 0$ such that

$$\|\hat{x} - g\|_{\mathcal{F}}^{\alpha} \geq \|\hat{x} - g_0\|_{\mathcal{F}}^{\alpha} + \lambda_\alpha \|g - g_0\|^{\alpha} \quad \text{for each } g \in G.$$

(cf. [16, 18]).

Our first theorem on the generalized strong uniqueness is as follows.

Theorem 4.2. *Let G be a strict RS-set (resp. an RS-set) and let $\hat{x} \in \mathcal{F}$ (resp. $\hat{x} \in \mathcal{F}_0$) with $d_G(\hat{x}) > d_X(\hat{x})$. Suppose that, for each $i \in I_1$ and each $z \in \mathrm{bd}J_i$, $\mathrm{bd}J_i$ has a positive curvature at z. Then the best simultaneous approximation to $\hat{x}$ from G is strongly unique of order 2.*

The proofs for Theorem 4.2 and Theorem 4.3 below are standard; see for example [8, Theorems 4.2 and 4.3]. Here we will keep the proof for Theorem 4.3 while the one for Theorem 4.2 is omitted.

For Theorem 4.3 below, we recall from [19] the notion of the uniform convexity and a relative fact.

Definition 4.1. Let $f : \mathbb{C} \to \mathbb{R}$ be a function. Let $z_0 \in \mathbb{C}$ and $p > 0$. The function f is said to be

(a) uniformly convex at z_0 if there exists $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\delta(t) > 0$ for all $t > 0$ such that, for each $z \in \mathbb{C}$ and each $\lambda \in (0, 1)$,

$$f(\lambda z_0 + (1-\lambda)z) \leq \lambda f(z_0) + (1-\lambda)f(z) - \lambda(1-\lambda)\delta(|z_0 - z|). \tag{4.8}$$

(b) p-uniformly convex at z_0 if there is $\mu_p > 0$ such that for each $t > 0$, $z \in \mathbb{C}$ with $|z_0 - z| = t$ and each $\lambda \in (0, 1)$, the following inequality holds:

$$\frac{\lambda f(z_0) + (1-\lambda)f(z) - f(\lambda z_0 + (1-\lambda)z)}{\lambda(1-\lambda)} > \mu_p t^p. \tag{4.9}$$

As usual, we let $\partial f(z_0)$ denote the subdifferential of f at z_0 defined by

$$\partial f(z_0) := \{\tau \in \mathbb{C} : \operatorname{Re}(z - z_0)\bar{\tau} \leq f(z) - f(z_0), \ \forall z \in \mathbb{C}\}.$$

Then we have the following characterization for the p-uniform convexity:

Proposition 4.1. *Let $z_0 \in \mathbb{C}$. The function $f : \mathbb{C} \to \mathbb{R}$ is p-uniformly convex at z_0 if and only if there exists $\mu_p > 0$ such that*

$$f(z) \geq f(z_0) + \operatorname{Re}(z - z_0)\bar{\tau} + \mu_p |z - z_0|^p \quad \textit{for each } z \in \mathbb{C} \textit{ and each } \tau \in \partial f(z_0). \tag{4.10}$$

Note that, if each J_i has nonempty interior in $\mathbb{C}$, then there exists a convex function f_i on $\mathbb{C}$ such that

$$\text{int} J_i = \{z \in \mathbb{C} : \ f_i(z) < 0\} \quad \text{and} \quad \text{bd} J_i = \{z \in \mathbb{C} : \ f_i(z) = 0\}. \tag{4.11}$$

Consequently, for each i, one has by [4, Corollary 1, p.56] that

$$N_{J_i}(z_0) = \text{cone}\, \partial f_i(z_0) \quad \text{for each } z_0 \in \text{bd} J_i. \tag{4.12}$$

Theorem 4.3. *Let $p > 0$, let G be a strict RS-set (resp. an RS-set) and let $\hat{x} \in \mathcal{F}$ (resp. $\hat{x} \in \mathcal{F}_0$) with $d_G(\hat{x}) > d_X(\hat{x})$. Suppose that, for each $i \in I_1$, there exists a convex function f_i satisfying* (4.11) *such that f_i is p-uniformly convex at each $z_0 \in \text{bd} J_i$. Then the best simultaneous approximation to $\hat{x}$ from G is strongly unique of order $\alpha = \max\{p, 2\}$.*

Proof. Let $g_0 \in P_G(\hat{x})$. Then, by Theorem 4.1, g_0 is the unique best simultaneous approximation to $\hat{x}$ from G as each J_i $(i \in I_1)$ is strictly convex by assumption. From Theorem 2.1, there exist $\{(\mathbf{a}^j, w^j)\}_{j=1}^k \subseteq M^+_{\hat{x}-g_0}$ with $k \geq 1$, $\{i_j\}_{j=1}^l \subseteq I(g_0)$, $\{\tau_{i_j}\}_{j=1}^l$ with each $\tau_{i_j} \in -N_{J_{i_j}}(\tilde{c}_{i_j}(g_0)) \setminus \{0\}$ and $\{\lambda_i\}_{i=1}^k \subseteq (0, +\infty)$ such that (2.16) holds. Hence, (2.17) holds. Without loss of generality, we may assume that $\sum_{j=1}^k \lambda_j = d_G(\hat{x})$. Then from Lemma 4.1, one has that

$$\begin{aligned} \|\hat{x} - g\|^2_{\mathcal{F}} \geq d_G(\hat{x})^2 &+ \frac{1}{d_G(\hat{x})} \sum_{j=1}^k \lambda_j |\phi_{g_0 - g}(\mathbf{a}^j, w^j)|^2 \\ &+2 \sum_{j=1}^k \lambda_j \text{Re}\phi_{g_0-g}(\mathbf{a}^j, w^j) \quad \text{for each } g \in G. \end{aligned} \tag{4.13}$$

Let

$$\gamma_\alpha(g) := \frac{\|\hat{x} - g\|^\alpha_{\mathcal{F}} - d_G(\hat{x})^\alpha}{\|g - g_0\|^\alpha} \quad \text{for each } g \in G \setminus \{g_0\}. \tag{4.14}$$

Below we will verify $\inf_{g \in G \setminus \{g_0\}} \gamma_\alpha(g) > 0$. Otherwise, there exists a sequence $\{g_n\} \subseteq G \setminus \{g_0\}$ such that $\lim_{n\to\infty} \gamma_\alpha(g_n) = 0$. This with (4.14) implies that $\{g_n\}$ is bounded; hence $\lim_{n\to\infty} \|\hat{x} - g_n\|_{\mathcal{F}} = d_G(\hat{x})$. Noting that $P_G(\hat{x}) = \{g_0\}$, we may assume that $\lim_{n\to\infty} g_n = g_0$ by the compactness. By (4.12), for each $j = 1, 2, \ldots, l$, there exist $\alpha_{i_j} \in \partial f_{i_j}(\tilde{c}_{i_j}(g_0))$ and $d_{i_j} > 0$ such that $\tau_{i_j} = -d_{i_j}\alpha_{i_j}$. It follows from (2.17) that

$$\sum_{j=1}^k \lambda_j \text{Re}\phi_{g_0-g_n}(\mathbf{a}^j, w^j) + \sum_{j=1}^l d_{i_j} \text{Re}\tilde{c}_{i_j}(g_n - g_0)\overline{\alpha}_{i_j} = 0 \quad \text{for each } n \in \mathbb{N}. \tag{4.15}$$

Noting that $\tilde{c}_{i_j}(g_0) \in \text{bd} J_{i_j}$ and $\tilde{c}_{i_j}(g_n) \in J_{i_j}$ for each $j = 1, \ldots, l$, we obtain from (4.10) that there exists $\mu_p > 0$ such that

$$\text{Re}\tilde{c}_{i_j}(g_n - g_0)\overline{\alpha}_{i_j} + \mu_p |\tilde{c}_{i_j}(g_0 - g_n)|^p \leq 0 \quad \text{for each } j \text{ with } 1 \leq j \leq l \text{ and each } n \in \mathbb{N}.$$

This together with (4.13) and (4.15) implies that

$$\begin{aligned}\|\hat{x}-g_n\|_{\mathcal{F}}^2 \geq\ & d_G(\hat{x})^2+\frac{1}{d_G(\hat{x})}\sum_{j=1}^{k}\lambda_j|\phi_{g_0-g_n}(\mathbf{a}^j,w^j)|^2+2\sum_{j=1}^{k}\lambda_j\mathrm{Re}\phi_{g_0-g_n}(\mathbf{a}^j,w^j)\\ & +2\sum_{j=1}^{l}d_{i_j}\mathrm{Re}\tilde{c}_{i_j}(g_n-g_0)\overline{\alpha}_{i_j}+2\mu_p\sum_{j=1}^{l}d_{i_j}|\tilde{c}_{i_j}(g_0-g_n)|^p\\ =\ & d_G(\hat{x})^2+\frac{1}{d_G(\hat{x})}\sum_{j=1}^{k}\lambda_j|\phi_{g_0-g_n}(\mathbf{a}^j,w^j)|^2+2\mu_p\sum_{j=1}^{l}d_{i_j}|\tilde{c}_{i_j}(g_0-g_n)|^p.\end{aligned}$$

Define

$$\|g\|_\alpha := \left[\sum_{j=1}^{k}\lambda_j|\phi_g(\mathbf{a}^j,w^j)|^\alpha+\sum_{j=1}^{l}d_{i_j}|\tilde{c}_{i_j}(g)|^\alpha\right]^{\frac{1}{\alpha}}\quad \text{for each } g\in Y_n.$$

Then $\|\cdot\|_\alpha$ is a norm equivalent to the original norm so that there is $\gamma>0$ such that $\|g\|_\alpha\geq\gamma\|g\|$ for each $g\in Y_n$. Thus, for sufficiently large $n\in\mathbb{N}$, we have that

$$\begin{aligned}\|\hat{x}-g_n\|_{\mathcal{F}}^2 \geq\ & d_G(\hat{x})^2+\frac{1}{d_G(\hat{x})}\sum_{j=1}^{k}\lambda_j|\phi_{g_0-g_n}(\mathbf{a}^j,w^j)|^\alpha+2\mu_p\sum_{j=1}^{l}d_{i_j}|\tilde{c}_{i_j}(g_0-g_n)|^\alpha\\ \geq\ & d_G(\hat{x})^2+\min\left\{\frac{1}{d_G(\hat{x})},2\mu_p\right\}\|g_n-g_0\|_\alpha^\alpha\\ \geq\ & d_G(\hat{x})^2+\min\left\{\frac{1}{d_G(\hat{x})},2\mu_p\right\}\gamma^\alpha\|g_n-g_0\|^\alpha.\end{aligned}$$

It therefore follows from the Cauchy mean-valued theorem that,

$$\begin{aligned}\|\hat{x}-g_n\|_{\mathcal{F}}^\alpha-d_G(\hat{x})^\alpha \geq\ & \frac{\alpha}{2}d_G(\hat{x})^{\alpha-2}(\|\hat{x}-g\|^2-d_G(\hat{x})^2)\\ \geq\ & \frac{\alpha}{2}d_G(\hat{x})^{\alpha-2}\min\left\{\frac{1}{d_G(\hat{x})},2\mu_p\right\}\gamma^\alpha\|g_n-g_0\|^\alpha.\end{aligned}$$

This implies that

$$\gamma(g_n)=\frac{\alpha}{2}d_G(\hat{x})^{\alpha-2}\min\left\{\frac{1}{d_G(\hat{x})},2\mu_p\right\}\gamma^\alpha,$$

which contradicts that $\lim_{n\to\infty}\gamma(g_n)=0$. The proof is complete. ■

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