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ESTIMATES FOR $\bar\partial$ AND HANKEL OPERATORS ON GENERALIZED FOCK SPACES ON $\mathbb{C}^n$

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Abstract. Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be a C^2 plurisubharmonic function on $\mathbb{C}^n$. Suppose that there exist $C_1, C_2 > 0$ such that $\sup_{\mathbb{C}^n} |\bar\partial\partial\varphi| < C_1$ and $H_\varphi(\xi,\xi)(z) \geq C_2|\xi|^2$ for $\xi \in \mathbb{R}^{2n}$ and $z \in \mathbb{C}^n$, where $H_\varphi(\xi,\xi)(z)$ is the real Hessian of φ at z. We prove $L^{p,\varphi}$ estimates for $\bar\partial$ on $\mathbb{C}^n$ for all $p \in [1,\infty]$. Moreover, by using the estimates for $\bar\partial$, we characterize boundedness and compactness of Hankel operators with anti-holomorphic symbols on generalized Fock spaces on $\mathbb{C}^n$.

1. INTRODUCTION

Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic function on $\mathbb{C}^n$. For any $0 < p \leq \infty$ we define the generalized Fock spaces $\mathcal{F}^{p,\varphi}$ as follows:

$$\mathcal{F}^{p,\varphi} = \left\{ f \in H(\mathbb{C}^n) : \|f\|_{p,\varphi} = \|fe^{-\varphi}\|_{L^p(dV)} < \infty \right\},$$

where dV denotes the volume measure in $\mathbb{C}^n$. Then it is known that $\mathcal{F}^{2,\varphi}$ is a closed linear subspace of $L^{2,\varphi}$ with the inner product

$$\langle f, g\rangle_\varphi = \int_{\mathbb{C}^n} f\bar g e^{-2\varphi}\, dV$$

where $f, g \in L^{2,\varphi}$. In fact, $\mathcal{F}^{2,\varphi}$ is a Hilbert space and the corresponding reproducing kernel $B(\zeta, z)$ induces the orthogonal projection $B : L^{2,\varphi} \to \mathcal{F}^{2,\varphi}$ which has the following integral representation

$$Bf(z) = \int_{\mathbb{C}^n} B(\zeta,z) f(\zeta) e^{-2\varphi(\zeta)}\, dV(\zeta), \quad z \in \mathbb{C}^n.$$

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Let $f = \sum'_{|J|=q} f_J \, d\bar{z}^J$, where the prime denotes summation over strictly increasing q-tuples J, and $d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$. Let $L^{p,\varphi}_q$ be the space of $(0,q)$-forms with coefficients in $L^{p,\varphi}$. That is,

$$L^{p,\varphi}_q = \left\{ f = \sum_{|J|=q}{}' f_J \, d\bar{z}^J : \|f\|_{p,\varphi} = \sum_{|J|=q}{}' \|f_J\|_{p,\varphi} < \infty \right\}.$$

If $\varphi(z) = \frac{1}{2}|z|^2$, $\mathcal{F}^{2,\varphi}$ is the classical Fock space. In [4], Boo constructed a solution operator K for the $\bar{\partial}$-equation in $\mathbb{C}^n$ that is canonical with respect to the space $L^{2,\varphi}_q$ with $\varphi(z) = \frac{1}{2}|z|^2$.

The quadratic form

$$H_\varphi(\xi, \xi)(z) = \sum_{j,k=1}^{2n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(z) \xi_j \xi_k,$$

defined for all $\xi \in \mathbb{R}^{2n}$ is called the real Hessian of φ at z, where $z_j = x_{2j-1} + \sqrt{-1} x_{2j}$.

We prove the $L^{p,\varphi}$ boundedness of a solution operator K for $\bar{\partial}$ in $L^{p,\varphi}_q$.

Theorem 1.1. *Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be a C^2 plurisubharmonic function on $\mathbb{C}^n$. Suppose that there exist $C_1, C_2 > 0$ such that $\sup_{\mathbb{C}^n} |\bar{\partial}\partial\varphi| < C_1$ and $H_\varphi(\xi, \xi)(z) \geq C_2 |\xi|^2$ for $\xi \in \mathbb{R}^{2n}$ and $z \in \mathbb{C}^n$. Let $1 \leq p \leq \infty$, and $q \geq 0$. Let $f \in L^{p,\varphi}_{q+1} \cap C^1(\mathbb{C}^n)$ be $\bar{\partial}$ closed. Then there exists a solution operator K_q for $\bar{\partial}$ on $L^{p,\varphi}_q$ such that*

$$\bar{\partial}(K_q f) = f$$

and

$$\|K_q f\|_{p,\varphi} \leq C \|f\|_{p,\varphi}.$$

In [11], Ortega-Schuster-Varolin gave a series of sufficient geometric conditions that would guarantee that a smooth hypersurface in $\mathbb{C}^n$ is an interpolation or sampling hypersurface in $L^{p,\varphi}(\mathbb{C}^n)$ spaces under the condition such that $C^{-1} < \sup_{\mathbb{C}^n} |\bar{\partial}\partial\varphi| < C$ for some $C > 0$. However, strictly speaking, they proved the results only for the cases $2 \leq p \leq \infty$ and omitted the range $1 \leq p < 2$ because of the absence of a suitable reference for $L^{p,\varphi}$ estimates for solutions of $\bar{\partial}$ in this range. Theorem 1.1 in this paper can recover the gap for the range $1 \leq p < 2$ as it provides such estimates in the case where φ is a plurisubharmonic function that satisfies $\sup_{\mathbb{C}^n} |\bar{\partial}\partial\varphi| < C_1$ and $H_\varphi(\xi, \xi)(z) \geq C_2 |\xi|^2$ for $\xi \in \mathbb{R}^{2n}$ and $z \in \mathbb{C}^n$.

However, in one dimensional case $L^{p,\varphi}$ estimates for $\bar{\partial}$ have been proved in even greater generality such that φ is a subharmonic function with $\Delta\varphi$ a doubling measure (see [5], [9], and [10]).

Given $g \in C^1(\mathbb{C}^n)$ so that there exists a dense subset A of $\mathcal{F}^{2,\varphi}$ with $gf \in L^{2,\varphi}$ for $f \in A$, the big Hankel operator H_g with symbol g is densely defined by

$$H_g f = gf - B(gf), \quad f \in A,$$

where B is the orthogonal projection of $L^{2,\varphi}$ onto $\mathcal{F}^{2,\varphi}$.

We subsequently use the weighted $L^{p,\varphi}$ estimates for $\bar{\partial}$ (with the same restrictions on φ as above) to characterize boundedness and compactness of Hankel operators with anti-holomorphic symbols.

Theorem 1.2. *Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be a C^2 plurisubharmonic function on $\mathbb{C}^n$. Suppose that there exist $C_1, C_2 > 0$ such that $\sup_{\mathbb{C}^n} |\bar{\partial}\partial\varphi| < C_1$ and $H_\varphi(\xi,\xi)(z) \geq C_2|\xi|^2$ for $\xi \in \mathbb{R}^{2n}$ and $z \in \mathbb{C}^n$. Let $1 \leq p \leq \infty$. Let g be an entire function in $\mathbb{C}^n$. Then $H_{\bar{g}}$ extends to a bounded linear operator on $\mathcal{F}^{p,\varphi}$ if and only if g is a polynomial of degree less than or equal to one.*

Theorem 1.3. *Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be a C^2 plurisubharmonic function on $\mathbb{C}^n$. Suppose that there exist $C_1, C_2 > 0$ such that $\sup_{\mathbb{C}^n} |\bar{\partial}\partial\varphi| < C_1$ and $H_\varphi(\xi,\xi)(z) \geq C_2|\xi|^2$ for $\xi \in \mathbb{R}^{2n}$ and $z \in \mathbb{C}^n$. Let $1 \leq p \leq \infty$. Let g be an entire function in $\mathbb{C}^n$. Then $H_{\bar{g}}$ extends to a compact linear operator on $\mathcal{F}^{p,\varphi}$ if and only if g is constant.*

In dimension 1, Constantin and Ortega-Cerdà [6] characterized boundedness and compactness of Hankel operators for $\mathcal{F}^{2,\varphi}$, where φ is a subharmonic function with $\Delta\varphi$ a doubling measure.

In [3], Bommier-Hato and Youssfi characterized when the Hankel operator with anti-holomorphic symbol is in the Schatten class on some weighted Fock spaces. However, in our $\mathcal{F}^{p,\varphi}$ spaces, the Schatten class characterization is the same as Theorem 1.3 since $H_{\bar{g}} \equiv 0$ when g is constant.

Example 1.4. Let $\alpha \in \mathbb{R}$ and $T > 0$ with $|\alpha| < T$. Then

$$\varphi(z) = |z|^2 + \alpha \log(T + |z|^2)$$

is a C^∞ strictly convex function on $\mathbb{C}^n$. Moreover, we know that there exist $C_1, C_2 > 0$ such that $C_1|\xi|^2 \leq H_\varphi(\xi,\xi)(z) \leq C_2|\xi|^2$ for $\xi \in \mathbb{R}^{2n}$ and $z \in \mathbb{C}^n$.

2. Solution Operators for $\bar{\partial}$

In this section, we construct a solution operator for $\bar{\partial}$ on $\mathbb{C}^n$. The operator is well known, see for instance ([2], [4]).

Let $\eta = \zeta - z$. Let $Q = (Q_1, \ldots, Q_n)$ and $S = (S_1, \ldots, S_n)$ be mappings from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}^n$. Define forms q and s by $q = \sum Q_j \, d\eta_j$ and $s = \sum S_j \, d\eta_j$. For $t \geq 0$ we let

$$P_t(\zeta, z) = C_n e^{(Q+tS)\cdot\eta}(d(q+ts))^n,$$

where $C_n^{-1} = (-1)^n n! \big(2\pi\sqrt{-1}\big)^n$, and $S \cdot \eta$ is defined by

$$S \cdot \eta(\zeta, z) = \sum S_j(\zeta, z)\eta_j(\zeta, z),$$

and so on. Define the kernel K by

$$K(\zeta, z) = \int_0^\infty P_t(\zeta, z).$$

Note that $d(q+ts) = dq + tds - s \wedge dt$, so $(d(q+ts))^n = A - n(dq+tds)^{n-1} \wedge s \wedge dt$, where A contains no differentials with respect to t. Hence

$$K(\zeta, z) = -C_n n \int_0^\infty e^{(Q+tS)\cdot\eta} s \wedge (dq + tds)^{n-1}\, dt.$$

Now

$$(dq + tds)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} (dq)^k \wedge (ds)^{n-1-k} t^{n-k-1}.$$

Thus

$$K(\zeta, z) = C_n\, e^{Q\cdot\eta} \sum_{k=0}^{n-1} \frac{n!}{k!} \frac{s \wedge (dq)^k \wedge (ds)^{n-1-k}}{(S\cdot\eta)^{n-k}}. \tag{2.1}$$

Before continuing let us note that since we are only interested in components of bidegree $= n$ in $d\zeta$ and dz together we can replace d by $\bar\partial$ everywhere in (2.1).

Let φ be a C^2 strictly convex function on $\mathbb{C}^n$ such that the real Hessian $H_\varphi(\xi, \xi)(z)$ of φ satisfies

$$H_\varphi(\xi, \xi)(z) \geq C|\xi|^2, \quad \xi \in \mathbb{R}^{2n}, z \in \mathbb{C}^n. \tag{2.2}$$

By the Taylor's theorem, we have

$$\begin{aligned}\varphi(z) =& \varphi(\zeta) + 2\,\mathrm{Re}\,[\partial\varphi(\zeta)\cdot(z-\zeta)] \\ &+ \frac{1}{2}\sum_{j,k=1}^{2n} \frac{\partial^2\varphi}{\partial x_j \partial x_k}(\zeta + \theta(z-\zeta))(x_j - \xi_j)(x_k - \xi_k)\end{aligned}$$

for some $\theta \in (0,1)$, where $z_j = x_{2j-1} + \sqrt{-1}x_{2j}$ and $\zeta_j = \xi_{2j-1} + \sqrt{-1}\xi_{2j}$. By (2.2), it follows that

$$\frac{1}{2}\sum_{j,k=1}^{2n} \frac{\partial^2\varphi}{\partial x_j \partial x_k}(\zeta + \theta(z-\zeta))(x_j - \xi_j)(x_k - \xi_k) \geq C|z-\zeta|^2, \quad z, \zeta \in \mathbb{C}^n.$$

Hence we get the following inequality:

$$2\,\mathrm{Re}\,[\partial\varphi(\zeta)\cdot(\zeta - z)] \geq \varphi(\zeta) - \varphi(z) + C|z-\zeta|^2, \quad z,\zeta \in \mathbb{C}^n. \tag{2.3}$$

To let the operator fit into our situation, choose $Q(\zeta, z) = -2\,\partial\varphi(\zeta)$ and $S(\zeta, z) = \bar{\eta}$. Then

$$\begin{aligned} &K(\zeta, z) \\ &= C_n e^{-2\partial\varphi(\zeta)\cdot(\zeta-z)} \sum_{k=0}^{n-1} \frac{n!}{k!} \frac{\partial|\zeta-z|^2 \wedge (-2\,\bar{\partial}\partial\varphi(\zeta))^k \wedge (\bar{\partial}\partial|\zeta- z|^2)^{n-1-k}}{|\zeta-z|^{2n-2k}}. \end{aligned} \tag{2.4}$$

The kernel K is of total bidegree $(n, n-1)$. Denote by K_q the component of K which is of bidegree $(0, q)$ in z, and hence $(n, n-q-1)$ in ζ. Then we have

$$K(\zeta, z) = \sum_{q=0}^{n-1} K_q(\zeta, z).$$

Let K and P denote the operators associated to the kernels $K(\zeta, z)$ and $P_0(\zeta, z)$; $Kf(z) = \int K(\zeta, z) \wedge f(\zeta)$ and similarly for P. Also note that

$$P_0(\zeta, z) = C_n e^{-2\partial\varphi(\zeta)\cdot(\zeta-z)}(-2\,\bar{\partial}\partial\varphi(\zeta))^n.$$

Then we have the homotopy formula (see [4])

$$\bar{\partial}K + K\bar{\partial} = I - P, \tag{2.5}$$

that a priori is valid only for, say, C^1-forms with compact support. Moreover, completeness of the metric and $L^{2,\varphi}$-boundedness of K (we will see in Theorem 3.1) guarantee that the homotopy formula holds not just for C^1-forms with compact support but also for forms in $L^{2,\varphi}$ (see Remark 2 in [4]). Let f be a $(0, q+1)$-form. Then

$$\begin{aligned} Kf(z) &= \int_{\mathbb{C}^n} K(\zeta, z) \wedge f(\zeta) \\ &= \int_{\mathbb{C}^n} K_q(\zeta, z) \wedge f(\zeta) = K_q f(z). \end{aligned}$$

Thus we have the following Koppelman's formula.

Theorem 2.1. *Let $q \geq 0$. Let $f \in L^{2,\varphi}_{q+1} \cap C^1(\mathbb{C}^n)$ be $\bar{\partial}$-closed, then we have*

$$f = \bar{\partial}(K_q f).$$

We are interested in addressing what happens when f is a function, since it gives a motivation to construct a peak function for $\mathcal{F}^{2,\varphi}$ in Section 4.

We choose a function $\mathcal{X} \in C_0^\infty(\mathbb{C}^n)$ such that $\mathcal{X} \equiv 1$ for $|\zeta| < 1$ and $\mathcal{X} \equiv 0$ when $|\zeta| > 2$. Put $\mathcal{X}_R(\zeta) = \mathcal{X}\big(\frac{\zeta}{R}\big)$. Let $f \in C^1(\mathbb{C}^n)$. By using Andersson-Berndtsson's formula (for functions) for the $2R$-ball, we have

$$\begin{aligned} \mathcal{X}_R f &= -\int_{|\zeta|=2R} (\mathcal{X}_R f) K_0 + \int_{|\zeta|<2R} \bar\partial(\mathcal{X}_R f) \wedge K_0 + \int_{|\zeta|<2R} (\mathcal{X}_R f) P_0 \\ &= \int_{|\zeta|<2R} (\bar\partial \mathcal{X}_R) f \wedge K_0 + \int_{|\zeta|<2R} \mathcal{X}_R (\bar\partial f) \wedge K_0 + \int_{|\zeta|<2R} (\mathcal{X}_R f) P_0. \end{aligned} \tag{2.6}$$

By (2.3), (2.4) and the fact that $\sup |\bar\partial\partial\varphi| \le C$, we have

$$\begin{aligned} |K_0(\zeta, z)| &\lesssim \frac{e^{-2\,\mathrm{Re}\,[\partial\varphi(\zeta)\cdot(\zeta - z)]}}{|\zeta - z|^{2n-1}} \\ &\lesssim \frac{e^{-\varphi(\zeta)+\varphi(z)-C|z-\zeta|^2}}{|\zeta - z|^{2n-1}}. \end{aligned} \tag{2.7}$$

Since $|\bar\partial \mathcal{X}_R| \lesssim 1/R$, if we suppose that $f, \bar\partial f \in L^{2,\varphi}(\mathbb{C}^n)$, by the estimate (2.7), we know that $\mathcal{X}_R f$ and the first two integrals in (2.6) converge uniformly to f, 0, and $\int_{\mathbb{C}^n} \bar\partial f \wedge K_0$, respectively when z belongs to a compact set. Hence, in the distribution sense, we have

$$f(z) = \int_{\mathbb{C}^n} \bar\partial f(\zeta) \wedge K_0(\zeta, z) + \int_{\mathbb{C}^n} f(\zeta) P_0(\zeta, z). \tag{2.8}$$

3. $L^{p,\varphi}$ Estimates for $\bar\partial$

We will prove that the operator K is $L^{p,\varphi}$ -bounded for $1 \le p \le \infty$.

Since

$$|K(\zeta, z)| \lesssim \frac{e^{-\varphi(\zeta)+\varphi(z)-C|z-\zeta|^2}}{|\zeta - z|^{2n-1}},$$

we have

$$|Kf(z)| \le \int_{\mathbb{C}^n} |f(\zeta)||k(\zeta, z)| e^{-2\varphi(\zeta)}\, dV(\zeta),$$

where $k(\zeta, z)$ has the estimate

$$|k(\zeta, z)| \lesssim \frac{e^{\varphi(z)+\varphi(\zeta)-C|z-\zeta|^2}}{|\zeta - z|^{2n-1}}. \tag{3.1}$$

Theorem 3.1. *Let* $1 \le p \le \infty$. *Then*

$$\|Kf\|_{p,\varphi} \le C\|f\|_{p,\varphi}.$$

Proof. First we consider the case $p = \infty$. We have

$$\begin{aligned} |Kf(z)| &\le \int_{\mathbb{C}^n} |f(\zeta)||k(\zeta,z)|e^{-2\varphi(\zeta)}\,dV(\zeta) \\ &\lesssim \sup\left[|f(\zeta)|e^{-\varphi(\zeta)}\right]\int_{\mathbb{C}^n} |k(\zeta,z)|e^{-\varphi(\zeta)}\,dV(\zeta). \end{aligned}$$

Note that

$$\begin{aligned} \int_{\mathbb{C}^n} |k(\zeta,z)|e^{-\varphi(\zeta)}\,dV(\zeta) &\lesssim e^{\varphi(z)}\int_{\mathbb{C}^n} \frac{e^{-C|z-\zeta|^2}}{|z-\zeta|^{2n-1}}\,dV(\zeta) \\ &\lesssim e^{\varphi(z)}, \end{aligned}$$

where we use the inequality

$$\begin{aligned} \int_{\mathbb{C}^n} \frac{e^{-C|z-\zeta|^2}}{|z-\zeta|^{2n-1}}\,dV(\zeta) &= \int_{\mathbb{C}^n} \frac{e^{-C|\zeta|^2}}{|\zeta|^{2n-1}}\,dV(\zeta) \\ &\lesssim \int_{|\zeta|\le 1} \frac{1}{|\zeta|^{2n-1}}\,dV(\zeta) + \int_{|\zeta|\ge 1} \frac{e^{-C|\zeta|^2}}{|\zeta|^{2n-1}}\,dV(\zeta) \lesssim 1. \end{aligned}$$

Thus we have

$$\sup\left[|Kf(z)|e^{-\varphi(z)}\right] \lesssim \sup\left[|f(\zeta)|e^{-\varphi(\zeta)}\right].$$

Now we consider the case $p = 1$. By Fubini's theorem, we have

$$\begin{aligned} \|Kf\|_{1,\varphi} &\lesssim \int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n} |f(\zeta)||k(\zeta,z)|e^{-2\varphi(\zeta)}\,dV(\zeta)\right)e^{-\varphi(z)}\,dV(z) \\ &\lesssim \int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n} |k(\zeta,z)|e^{-\varphi(z)}\,dV(z)\right)|f(\zeta)|e^{-2\varphi(\zeta)}\,dV(\zeta). \end{aligned}$$

Now

$$\begin{aligned} \int_{\mathbb{C}^n} |k(\zeta,z)|e^{-\varphi(z)}\,dV(z) &\lesssim e^{\varphi(\zeta)}\int_{\mathbb{C}^n} \frac{e^{-C|z-\zeta|^2}}{|z-\zeta|^{2n-1}}\,dV(z) \\ &\lesssim e^{\varphi(\zeta)}. \end{aligned}$$

Thus we have

$$\|Kf\|_{1,\varphi} \lesssim \int_{\mathbb{C}^n} e^{\varphi(\zeta)}|f(\zeta)|e^{-2\varphi(\zeta)}\,dV(\zeta) = \|f\|_{1,\varphi}.$$

We define

$$T_\varphi(g)(z) = e^{-\varphi(z)} K[ge^\varphi].$$

Clearly if we denote $g(z) = f(z)e^{-\varphi(z)}$, then $\|g\|_{L^p(dV)} = \|f\|_{p,\varphi}$ and the estimate $\|K(f)\|_{p,\varphi} \le C\|f\|_{p,\varphi}$ is equivalent to $\|T_\varphi(g)\|_{L^p(dV)} \le C\|g\|_{L^p(dV)}$. Since the cases $p = 1, \infty$ of this estimate are proved, the others follow by the Riesz-Thorin interpolation theorem because T_φ is linear. ■

4. Stimates for the Reproducing Kernel

We need the following Cauchy-type estimates for functions in $\mathcal{F}^{p,\varphi}$.

Lemma 4.1. ([8]). *Let $p > 0$. For any $r > 0$ there exists $C = C(r) > 0$ such that for any $f \in H(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$*

(a) $|f(z)e^{-\varphi(z)}|^p \le C \int_{B(z,r)} |f(w)e^{-\varphi(w)}|^p \, dV(w),$

(b) $|\nabla(|f|e^{-\varphi})(z)|^p \le C \int_{B(z,r)} |f(w)e^{-\varphi(w)}|^p \, dV(w).$

We note that

$$\begin{aligned} P_0(\zeta, z) &= C_n e^{-2\partial\varphi(\zeta)\cdot(\zeta-z)}(-2\,\bar\partial\partial\varphi(\zeta))^n \\ &= N(\zeta)e^{-2\partial\varphi(\zeta)\cdot(\zeta-z)} dV(\zeta) \end{aligned}$$

for some function $N \in C(\mathbb{C}^n)$. By assumption of φ, there exist $C_1, C_2 > 0$ such that $C_1 < |N(\zeta)| < C_2$ for $\zeta \in \mathbb{C}^n$. Since $*_\zeta P(\zeta, z) = N(\zeta)e^{-2\varphi(\zeta)\cdot(\zeta-z)}$, by (2.8), for $f \in F^{2,\varphi}$ we get

$$f(z) = \int_{\mathbb{C}^n} *_\zeta P_0(\zeta, z) f(\zeta) \, dV(\zeta),$$

where $*$ is the Hodge $*$-operator (see [12]). Let $\widetilde{P}_z(\zeta) = \frac{1}{N(z)} *_\zeta P_0(\zeta, z)$. Then we have $\widetilde{P}_z(z) = 1$ and

$$|\widetilde{P}_z(\zeta)| \lesssim e^{-\varphi(\zeta)+\varphi(z)-C|z-\zeta|^2}, \quad z, \zeta \in \mathbb{C}^n.$$

However, $\widetilde{P}_z$ is not an entire function. Thus we take

$$P_z(\zeta) = e^{-2\partial\varphi(z)\cdot(z-\zeta)}.$$

Then P_z is an entire function such that $P_z(z) = 1$ and

$$|P_z(\zeta)| \le e^{\varphi(\zeta)-\varphi(z)-C|z-\zeta|^2}, \quad z, \zeta \in \mathbb{C}^n.$$

Hence P_z is a peak function for $\mathcal{F}^{2,\varphi}$. By using a peak function, we can derive some lower estimates for the reproducing kernel on the diagonal.

Proposition 4.2. *There exists* $C > 0$ *such that*

$$C^{-1}e^{2\varphi(z)} \le B(z,z) \le Ce^{2\varphi(z)}.$$

Proof. By (a) of Lemma 4.1, for $f \in F^{2,\varphi}$ we have

$$|f(z)|^2 e^{-2\varphi(z)} \lesssim \|f\|_{2,\varphi}^2.$$

Hence

$$B(z,z) \le Ce^{2\varphi(z)}.$$

For some $c_0 > 0$ (to be determined) we define the entire function

$$f_z(\zeta) = c_0 e^{\varphi(z)} P_z(\zeta).$$

Then

$$\int_{\mathbb{C}^n} |f_z(\zeta)|^2 e^{-2\varphi(\zeta)}\, dV(\zeta) \le c_0^2 \int_{\mathbb{C}^n} e^{-2C|z-\zeta|^2}\, dV(\zeta) \le 1$$

for c_0 small enough. For such a fixed c_0 we have $f_z(z) = c_0 e^{\varphi(z)}$ and therefore

$$B(z,z) = \sup\{|f(z)|^2 : f \in F^{2,\varphi}, \|f\|_{2,\varphi} \le 1\} \gtrsim e^{2\varphi(z)}. \qquad \blacksquare$$

Proposition 4.3. *There exists* $C > 0$ *such that for any* $\zeta, z \in \mathbb{C}^n$

$$|B(\zeta,z)| \le Ce^{\varphi(\zeta)+\varphi(z)}.$$

Moreover there is an $r > 0$ *such that*

$$|B(\zeta,z)| \gtrsim e^{\varphi(\zeta)+\varphi(z)}, \quad \zeta \in B(z,r).$$

Proof. Applying (a) in Lemma 4.1 to the reproducing kernel $B(\zeta,z)$, we have

$$\begin{aligned}|B(\zeta,z)|^2 e^{-2\varphi(\zeta)} &\lesssim \int_{B(\zeta,r)} |B(w,z)|^2 e^{-2\varphi(w)}\, dV(w)\\ &\lesssim \int_{\mathbb{C}^n} |B(w,z)|^2 e^{-2\varphi(w)}\, dV(w)\\ &= B(z,z) \lesssim e^{2\varphi(z)}.\end{aligned}$$

Moreover, Lemma 4.1 (b) implies that for all $\zeta \in B(z,r)$,

$$\begin{aligned}\Big||B(\zeta,z)|e^{-\varphi(\zeta)} - |B(z,z)|e^{-\varphi(z)}\Big| &\lesssim |\zeta - z| \left[\int_{\mathbb{C}^n} |B(w,z)|^2 e^{-2\varphi(w)}\, dV(w)\right]^{1/2}\\ &\lesssim |\zeta - z| B(z,z)^{1/2} \lesssim re^{\varphi(z)}.\end{aligned}$$

Thus Proposition 4.2 implies that

$$\begin{aligned}|B(\zeta,z)|e^{-\varphi(\zeta)} &\gtrsim |B(z,z)|e^{-\varphi(z)} - re^{\varphi(z)}\\ &\gtrsim (1-r)e^{\varphi(z)}.\end{aligned}$$

If we choose r small enough, then we get the required result. ■

In fact, Delin and Lindholm get the more refined upper estimates for $B(\zeta, z)$.

Theorem 4.4. ([7], [8]). *Let φ be a plurisubharmonic function in $\mathbb{C}^n$ such that*

$$C^{-1}\sqrt{-1}\partial\bar{\partial}|z|^2 \leq \sqrt{-1}\partial\bar{\partial}\varphi \leq C\sqrt{-1}\partial\bar{\partial}|z|^2$$

as positive currents, for some constant $C > 0$. Then

$$|B(\zeta,z)| \leq Ce^{\varphi(\zeta)+\varphi(z)-T|z-\zeta|},$$

where $T > 0$ is a constant proportional to the lower bound of $\sqrt{-1}\bar{\partial}\partial\varphi$ and C depends on the upper bound.

By using the upper estimates for $B(\zeta, z)$ in Theorem 4.4, Lindholm proved that the orthogonal projection B projects $L^{p,\varphi}$ boundedly onto $\mathcal{F}^{p,\varphi}$ for $1 \leq p \leq \infty$.

5. Hankel Operators on $\mathcal{F}^{p,\varphi}$

Let g be an entire function in $\mathbb{C}^n$ such that

$$\bar{g}B(\zeta,\cdot) \in L^{2,\varphi} \quad \text{for all} \quad \zeta \in \mathbb{C}^n. \tag{5.1}$$

Let $A := \mathrm{Span}\{B(\zeta,\cdot) : \zeta \in \mathbb{C}^n\}$. Then A is dense in $\mathcal{F}^{2,\varphi}$. Thus the big Hankel operator $H_{\bar{g}}$ is densely defined if g satisfies the condition (5.1). We know that if g is polynomial, then it satisfies the condition (5.1) from Theorem 4.4.

Notice that if g is an entire function, then $H_{\bar{g}}f$ is the minimal $L^{2,\varphi}$-norm solution of the $\bar{\partial}$-equation

$$\bar{\partial}u = f\bar{\partial}\bar{g}. \tag{5.2}$$

Hence, $H_{\bar{g}}f = (I - B)u$ for some solution u of the $\bar{\partial}$-equation (5.2).

Remark 5.1. If $n = 1$, the canonical solution operator S to $\bar{\partial}$ is densely defined on $L^{2,\varphi}$ by

$$\frac{\partial}{\partial\bar{z}}(Sf) = f \quad \text{and} \quad Sf \perp \mathcal{F}^{2,\varphi}.$$

Let us consider the restriction of S to $\mathcal{F}^{2,\varphi}$. Notice that if $f \in A = \mathrm{Span}\{B(\zeta,\cdot) : \zeta \in \mathbb{C}\}$, then $\bar{z}f \in L^{2,\varphi}$ and

$$Sf = (I - B)(\bar{z}f) = H_{\bar{z}}f.$$

That is, the canonical solution operator coincides with the big Hankel operator acting on $\mathcal{F}^{2,\varphi}$ with symbol $\bar{z}$. Motivated by this fact, we now consider Hankel operators with anti-holomorphic symbols on $\mathcal{F}^{p,\varphi}$. ■

Let g be an entire function in $\mathbb{C}^n$ satisfying the condition (5.1). Let

$$b_\zeta(z) = \frac{B(\zeta, z)}{\sqrt{B(\zeta,\zeta)}}, \quad \zeta, z \in \mathbb{C}^n.$$

By the reproducing formula in $\mathcal{F}^{2,\varphi}$ we get

$$H_{\bar{g}} b_\zeta(z) = (\overline{g(z)} - \overline{g(\zeta)}) b_\zeta(z), \quad \zeta, z \in \mathbb{C}^n. \tag{5.3}$$

We consider the boundedness and compactness of $H_{\bar{g}}$.

Theorem 5.2. *Let $1 \le p \le \infty$. Let g be an entire function in $\mathbb{C}^n$. Then $H_{\bar{g}}$ extends to a bounded linear operator on $\mathcal{F}^{p,\varphi}$ if and only if g is a polynomial of degree less than or equal to one.*

Proof. Assume first that g is a polynomial of degree less than or equal to one. Then $\sup|\partial g| < \infty$. Since $H_{\bar{g}}f$ is the minimal $L^{2,\varphi}$-norm solution of the $\bar{\partial}$-equation, we have $H_{\bar{g}}f = (I - B)[K_0(f\bar{\partial}\bar{g})]$, where K_0 is the solution operator of the equation (5.2) constructed in Section 2. In [8], Lindholm proved that the orthogonal projection B projects $L^{p,\varphi}$ boundedly onto $\mathcal{F}^{p,\varphi}$ for $1 \le p \le \infty$. By Theorem 1.1, K_0 is bounded on $L^{p,\varphi}$. Thus we have

$$\begin{aligned}\|H_{\bar{g}}f\|_{p,\varphi} &= \|(I - B)[K_0(f\bar{\partial}\bar{g})]\|_{p,\varphi} \\ &\lesssim \|K_0(f\bar{\partial}\bar{g})\|_{p,\varphi} \\ &\lesssim \|f\bar{\partial}\bar{g}\|_{p,\varphi} \\ &\lesssim \sup|\partial g| \|f\|_{p,\varphi},\end{aligned}$$

which shows that $H_{\bar{g}}$ can be extended to a bounded linear operator on $\mathcal{F}^{p,\varphi}$.

Conversely, assume that $H_{\bar{g}}$ is bounded on $\mathcal{F}^{p,\varphi}$. Then we have $\|H_{\bar{g}}b_\zeta\|_{p,\varphi} < M$ for $\zeta \in \mathbb{C}^n$. Using Proposition 4.2 and Proposition 4.3, there exists $r > 0$ such that

$$\begin{aligned}|b_\zeta(z)| &= \frac{|B(\zeta, z)|}{\sqrt{B(\zeta,\zeta)}} \\ &\gtrsim e^{\varphi(z)} \quad \text{on} \quad z \in B(\zeta, r).\end{aligned}$$

Hence we have

$$\begin{aligned} M^p > \|H_{\bar g} b_\zeta\|^p_{p,\varphi} &= \int_{\mathbb{C}^n} |g(z)-g(\zeta)|^p |b_\zeta(z)|^p e^{-p\varphi(z)}\, dV(z) \\ &\geq \int_{B(\zeta,r)} |g(z)-g(\zeta)|^p |b_\zeta(z)|^p e^{-p\varphi(z)}\, dV(z) \\ &\gtrsim \int_{B(\zeta,r)} |g(z)-g(\zeta)|^p \, dV(z). \end{aligned}$$

Since g is an entire function, by the Cauchy estimates applied to $g_\zeta(z) := g(z) - g(\zeta)$, we can now conclude

$$|\partial g(\zeta)|^p \lesssim \int_{B(\zeta,r)} |g(z)-g(\zeta)|^p \, dV(\zeta) \lesssim M^p, \quad \zeta \in \mathbb{C}^n.$$

Thus g is a polynomial of degree less than or equal to one. ■

Theorem 5.3. *Let $1 \leq p \leq \infty$. Let g be an entire function in $\mathbb{C}^n$. Then $H_{\bar g}$ extends to a compact linear operator on $\mathcal{F}^{p,\varphi}$ if and only if g is constant.*

Proof. Assume first that g is constant. Then $H_{\bar g} \equiv 0$ and so it is compact.

Suppose now $H_{\bar g}$ is compact. Since $H_{\bar g}$ is bounded, g is a polynomial of degree less than or equal to one. By Theorem 4.4, we have

$$\begin{aligned} \|b_\zeta\|^p_{p,\varphi} &= \int_{\mathbb{C}^n} \frac{|B(\zeta,z)|^p}{B(\zeta,\zeta)^{p/2}} e^{-p\varphi(z)}\, dV(z) \\ &\lesssim \int_{\mathbb{C}^n} e^{-pT|\zeta - z|}\, dV(z) \lesssim 1, \end{aligned}$$

uniformly in $\zeta \in \mathbb{C}^n$. Thus the set $\{b_\zeta\}_{\zeta\in\mathbb{C}^n}$ is bounded in $\mathcal{F}^{p,\varphi}$. By compactness it follows that the set $\{H_{\bar g} b_\zeta\}_{\zeta\in\mathbb{C}^n}$ is relatively compact in $L^{p,\varphi}$. Then by Riesz-Tamarkin compactness theorem [1] we have

$$\lim_{R\to\infty} \int_{|z|>R} |H_{\bar g} b_\zeta(z)|^p \, e^{-p\varphi(z)}\, dV(z) = 0,$$

uniformly in $\zeta \in \mathbb{C}^n$. We choose $r > 0$ so that

$$|b_\zeta(z)| \gtrsim e^{\varphi(z)} \quad \text{on} \quad B(\zeta, r).$$

For $|\zeta| > R + r$, the inclusion $B(\zeta, r) \subset \{|z| > R\}$ holds, and

$$\begin{aligned}\int_{|z|>R} |H_{\bar{g}} b_\zeta(z)|^p\, e^{-p\varphi(z)}\, dV(z) &= \int_{|z|>R} |g(z)-g(\zeta)|^p |b_\zeta(z)|^p e^{-p\varphi(z)}\, dV(z)\\ &\gtrsim \int_{B(\zeta,r)} |g(z)-g(\zeta)|^p |b_\zeta(z)|^p e^{-p\varphi(z)}\, dV(z)\\ &\gtrsim \int_{B(\zeta,r)} |g(z)-g(\zeta)|^p\, dV(z)\\ &\gtrsim |\partial g(\zeta)|^p.\end{aligned}$$

This implies that

$$\lim_{|\zeta|\to\infty} |\partial g(\zeta)| = 0,$$

which shows that g is constant. ∎

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