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PATH PROPERTIES OF l^∞-VALUED RANDOM FIELDS

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Abstract. In this paper we investigate path properties for strictly stationary and linearly positive quadrant dependent (LPQD) or linearly negative quadrant dependent (LNQD) random fields with multidimensional indices taking values in l^∞-space.

1. Introduction and Results

In the last years there has been growing interest in concepts of positive/negative dependence for families of random variables. Such concepts are of considerable use in deriving inequalities in probability and statistics. Recently, Csörgő et al. [3] and Choi and Csörgő [1, 2] studied asymptotic properties for l^∞(and l^p)-valued Gaussian random fields. In this paper we are interested in path properties for any positive or negative dependent random field with multidimensional indices taking values in l^∞-space.

For the aim of the present paper, we need to elaborate upon definitions and notations which will play a basic role in the present work. For a positive integer N, let $\mathbb{R}^N$ and $\mathbb{R}^N_+$, respectively, be N-dimensional and nonnegative N-dimensional Euclidean spaces with coordinatewise partial order $\leq$, and let $\mathbb{Z}^N_+$ be the N-dimensional lattice of all points in $\mathbb{R}^N_+$ having nonnegative integer coordinates. Let $\{X_i(\mathbf{t})\,;\,\mathbf{t}\in[0,\infty)^N\}_{i=1}^\infty$ be a sequence of random fields indexed by N-dimensional parameter $\mathbf{t}:=(t_1,\cdots,t_N)$ on the probability space $(\Omega,\mathfrak{F},P)$, which will be called *centered* if $E(X_k(\mathbf{t}))=0$.

Esary et al. [5] and Joag-Dev and Proschan [7] introduced definitions of positive and negative associations, respectively: Let $\mathcal{C}$ be a set of functions of the form $f:[0,\infty)^N\to\mathbb{R}$ which are coordinatewise nondecreasing. A random field $\{X_i(\mathbf{t})\,;\,\mathbf{t}\in[0,\infty)^N\}_{i=1}^\infty$ is said to be *positively associated* (PA, for short) if, for any $f,g\in\mathcal{C}$ and any finite subsets A and B of $\mathbb{Z}_+$,

$$\mathrm{Cov}\left(f(X_i(\mathbf{t});\ i\in A),\ g(X_j(\mathbf{t});\ j\in B)\right)\geq 0,$$

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while a random field $\{X_i(\mathbf{t})\}_{i=1}^{\infty}$ is said to be *negatively associated* (NA, for short) if, for any $f, g \in \mathcal{C}$ and any disjoint finite subsets A and B of $\mathbb{Z}_+$,

$$\mathrm{Cov}\big(f(X_i(\mathbf{t});\ i \in A),\ g(X_j(\mathbf{t});\ j \in B)\big) \leq 0.$$

Newman [12] introduced and discussed the following another concepts of positive or negative dependence. The random field $\{X_i(\mathbf{t})\,;\, \mathbf{t} \in [0,\infty)^N\}_{i=1}^{\infty}$ is said to be *linearly positive quadrant dependent* (LPQD) if, for any positive numbers λ_i and any disjoint finite subsets A, B of $\mathbb{Z}_+$, the inequality

$$\begin{aligned} &P\Big\{\sum_{i\in A}\lambda_i X_i(\mathbf{t}) \geq x,\ \sum_{j\in B}\lambda_j X_j(\mathbf{t}) \geq y\Big\} \\ &\geq P\Big\{\sum_{i\in A}\lambda_i X_i(\mathbf{t}) \geq x\Big\} P\Big\{\sum_{j\in B}\lambda_j X_j(\mathbf{t}) \geq y\Big\} \end{aligned} \tag{1.1}$$

holds for all real $x,\ y \in \mathbb{R}$, which is equivalent to the inequality (Lehmann [8], pp. 1137-1138)

$$\begin{aligned} &P\Big\{\sum_{i\in A}\lambda_i X_i(\mathbf{t}) \leq x,\ \sum_{j\in B}\lambda_j X_j(\mathbf{t}) \leq y\Big\} \\ &\geq P\Big\{\sum_{i\in A}\lambda_i X_i(\mathbf{t}) \leq x\Big\} P\Big\{\sum_{j\in B}\lambda_j X_j(\mathbf{t}) \leq y\Big\}, \end{aligned} \tag{1.2}$$

while the random field $\{X_i(\mathbf{t})\}_{i=1}^{\infty}$ is said to be *linearly negative quadrant dependent* (LNQD) if the inequalities in (1.1) and (1.2) are reversed. In general, two random variables X and Y have been called *positively* (resp. *negatively*) *quadrant dependent* (PQD) (resp. NQD) by Lehmann [8], if $P\big(X \geq x, Y \geq y\big) \geq$ (resp. $\leq$) $P\big(X \geq x\big)$ $P\big(Y \geq y\big)$ for all $x,\ y \in \mathbb{R}$.

From the definitions, it is obvious that PA or NA implies LPQD or LNQD (cf. [5, 7]), respectively, but the converse is not true (see e.g. Joag-Dev [6], pp. 1038-1039). The positive or negative dependence plays an important role in a wide variety of areas, including statistical mechanics, quantum field theory, percolation models, multinomial distribution, permutation distribution, reliability theory, mathematical physics and multivariate statistical analysis.

Since LPQD and LNQD is strictly weaker than PA and NA, respectively, studying the limit theorems for LPQD or LNQD random sequences is of interest in this field. The following is not necessarily an exhaustive list of papers for LPQD or LNQD random variables: Newman [12, 13], Li and Wang [9], Wang and Zhang [15], Zhang [16].

The objective of this paper is to establish a generalized uniform law of the iterated logarithm and investigate path properties for LPQD or LNQD random fields taking values in l^∞-space, whose description now follows. For two vectors $\mathbf{s} = (s_1, \cdots, s_N)$ and $\mathbf{t} = (t_1, \cdots, t_N)$ in N-dimensional parameter space $[0, \infty)^N$, denote

$$\begin{aligned}
&\mathbf{s} \pm \mathbf{t} = (s_1 \pm t_1, \cdots, s_N \pm t_N), \quad \mathbf{st} = (s_1 t_1, \cdots, s_N t_N), \\
&\mathbf{s} \leq \mathbf{t} \quad \text{if} \;\; s_m \leq t_m \;\; \text{for each} \;\; m = 1, 2, \cdots, N, \;\; \mathbf{0} = (0, \cdots, 0), \quad \mathbf{1} = (1, \cdots, 1), \\
&a\mathbf{t} = (at_1, \cdots, at_N) \;\; \text{for } a \in (-\infty, \infty), \;\; (\mathbf{s}, \mathbf{t}) = (s_1, \cdots, s_N, t_1, \cdots, t_N) \in [0, \infty)^{2N}.
\end{aligned}$$

Assume that $\{X_i(\mathbf{t})\,;\, \mathbf{t} \in [0, \infty)^N\}_{i=1}^{\infty}$ is a sequence of centered strictly stationary and LPQD (LNQD) random fields with $X_i(\mathbf{0}) = 0$ and stationary increments

$$\sigma_i(\|\mathbf{t}\|) := \sqrt{E\{X_i(\mathbf{s} + \mathbf{t}) - X_i(\mathbf{s})\}^2}, \quad i \geq 1,$$

where $\sigma_i(t)$ are nondecreasing continuous functions of $t > 0$, and $\|\cdot\|$ denotes the Euclidean norm such that $\|\mathbf{t}\| = \big(\sum_{m=1}^{N} t_m^2\big)^{1/2}$. Put

$$\sigma_*(t) = \sup_{i \geq 1} \sigma_i(t)$$

and assume that $\sigma_*(\cdot)$ is a regularly varying function with exponent $\alpha > 0$ at ∞. Recall that a positive function $\sigma(t)$ of $t > 0$ is said to be *regularly varying* with exponent $\alpha > 0$ at $b \geq 0$ if $\lim_{t \to b}\{\sigma(xt)/\sigma(t)\} = x^\alpha$ for $x > 0$.

Let $\{\mathbb{X}(\mathbf{t}) := (X_1(\mathbf{t}), X_2(\mathbf{t}), \cdots)\,;\, \mathbf{t} \in [0, \infty)^N\}$ be a centered strictly stationary and LPQD (LNQD) random field taking values in l^∞-space (i.e. l^∞-valued random field) with l^∞-norm $\|\cdot\|_\infty$ defined by $\|\mathbb{X}(\mathbf{t})\|_\infty = \sup_{i \geq 1} |X_i(\mathbf{t})|$.

For each $m = 1, 2, \cdots, N$, let $a_m(T)$ and $b_m(T)$ be positive nondecreasing functions of $T > 0$ such that $a_m(T) \leq b_m(T)$ and $\lim_{T \to \infty} b_m(T) = \infty$. Denote

$$\begin{aligned}
&\boldsymbol{a}_T = (a_1(T), \cdots, a_N(T)), \quad \mathbf{b}_T = (b_1(T), \cdots, b_N(T)), \\
&\gamma(T) = \sqrt{2\big\{\log\big(\|\mathbf{b}_T\|/\|\boldsymbol{a}_T\|\big) + \log\log\|\mathbf{b}_T\|\big\}},
\end{aligned}$$

where $\log z = \log(\max\{z, e\})$.

Note that the condition (i) in Theorem 1.1 below implies conditions (C2) and (I) in [9] and [15], respectively. The main results are as follows.

Theorem 1.1. *Let* $\{\mathbb{X}(\mathbf{t}) := (X_1(\mathbf{t}), X_2(\mathbf{t}), \cdots)\,;\, \mathbf{t} \in [0, \infty)^N\}$ *be a centered strictly stationary and LPQD* (*LNQD*) l^∞*-valued random field with* l^∞*-norm* $\|\cdot\|_\infty$ *and* $E|X_1(\mathbf{t})|^{2+\delta} < \infty$ *for some* $\delta \in (0, 1]$*, which satisfies conditions*

(i) $\displaystyle\sum_{j \geq k+1} |\operatorname{Cov}(X_i(\mathbf{1}), X_i(\mathbf{b}_j))| = O(\|\mathbf{b}_k\|^{-\lambda})$ for each $i, k \geq 1$ and some $\lambda > 2$,

(ii) $\displaystyle\inf_{x \geq 1} \sigma^2(x)/x > 0.$

where $u_k = O(v_k)$ *denotes* $\limsup_{k\to\infty} u_k/v_k < \infty$. *Then*

$$\limsup_{T\to\infty} \sup_{\|\mathbf{s}\|\le\|\mathbf{b}_T\|} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_T\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)\sqrt{2\log\log\|\mathbf{b}_T\|}}$$

$$= \limsup_{T\to\infty} \frac{\|\mathbb{X}(\mathbf{b}_T)\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)\sqrt{2\log\log\|\mathbf{b}_T\|}} = 1 \quad \text{a.s.} \tag{1.3}$$

The first result in (1.3) implies a *generalized uniform law of the iterated logarithm* (LIL) for LPQD or LNQD l^∞-valued random fields, but the second one in (1.3) is a standard form of the ordinary LIL for l^∞-valued random fields which is an extension of Theorem 1 in [3]. Since $\gamma(T) \ge \sqrt{2\log\log\|\mathbf{b}_T\|}$, it is natural that (cf. see (2.1) in the proof of Theorem 1.1)

$$\limsup_{T\to\infty} \sup_{\|\mathbf{s}\|\le\|\mathbf{b}_T\|} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_T\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)\gamma(T)} \le 1 \quad \text{a.s.} \tag{1.4}$$

In order to obtain a limit result, we consider the following condition (iii) of Theorem 1.2.

Theorem 1.2. *Let* $\{\mathbb{X}(\mathbf{t})\,;\,\mathbf{t}\in[0,\infty)^N\}$ *be as in* Theorem 1.1 *with conditions* (i)-(ii)*, and let*

(iii) $\displaystyle\lim_{T\to\infty} \frac{\log\big(\|\mathbf{b}_T\|/\|\boldsymbol{a}_T\|\big)}{\log\log\|\mathbf{b}_T\|} = \infty.$

Then we have

$$\lim_{T\to\infty} \sup_{\|\mathbf{s}\|\le\|\mathbf{b}_T\|} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_T\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)\gamma(T)}$$

$$= \lim_{T\to\infty} \sup_{\|\mathbf{s}\|\le\|\mathbf{b}_T\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{b}_T)-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)\gamma(T)} = 1 \quad \text{a.s.} \tag{1.5}$$

Theorem 1.2 for l^∞-valued random fields generalizes some limit results in [2, 3, 4, 10, 11]. Returning to our present exposition, we present an example for Gaussian random field.

Example 1.1. Let $\{\mathbb{X}(\mathbf{t})\,;\,\mathbf{t}\in[0,\infty)^N\}$ be a centered stationary and LPQD (LNQD) l^∞-valued Gaussian random field with conditions (i)-(ii) in Theorem 1.1. For each $i = 1, 2, \ldots, N$, let $a_i(T) = \sqrt{i}\,\log T$ and $b_i(T) = \sqrt{i}\,T$, where $T := 1/\varsigma$ for $0 < \varsigma < 1$. Then,

$\boldsymbol{a}_T := (a_1(T), \cdots, a_N(T)) = (1, \sqrt{2}, \cdots, \sqrt{N})\log T, \quad \mathbf{b}_T = (1, \sqrt{2}, \cdots, \sqrt{N})T,$

$\|\boldsymbol{a}_{1/\varsigma}\| = \sqrt{N(N+1)/2}\,\log(1/\varsigma), \qquad \|\mathbf{b}_{1/\varsigma}\| = \sqrt{N(N+1)/2}\,(1/\varsigma),$

$\gamma(1/\varsigma) \approx \sqrt{2\big(\log(1/\varsigma) - \log\log(1/\varsigma)\big)}$ for sufficiently small ς by (iii) in Theorem 1.2.

Hence, by Theorem 1.1, we have the *uniform law of the iterated logarithm*

$$\limsup_{\varsigma\downarrow 0} \sup_{\|\mathbf{s}\|\le\|\mathbf{b}_{1/\varsigma}\|} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{1/\varsigma}\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_{1/\varsigma}\|)\sqrt{2\log\log\|\mathbf{b}_{1/\varsigma}\|}} = 1 \qquad \text{a.s.}$$

On the other hand, adding condition (iii) of Theorem 1.2, we have the *modulus of continuity*

$$\lim_{\varsigma\downarrow 0} \sup_{\|\mathbf{s}\|\le\|\mathbf{b}_{1/\varsigma}\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{b}_{1/\varsigma})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_{1/\varsigma}\|)\sqrt{2\big(\log(1/\varsigma)-\log\log(1/\varsigma)\big)}} = 1 \qquad \text{a.s.}$$

2. Proofs

In this section, let c denote a positive constant which may take different values whenever they appear in different lines. We need the following properties.

(P_1) Two random variables X and Y are PQD (resp. NQD) if and only if Cov $\big(f(X),\, g(Y)\big) \ge$ (resp. $\le$) 0 for all real-valued nondecreasing functions f and g (such that $f(X)$ and $g(Y)$ have finite variances) (see Lehmann [8]);

(P_2) (*Hoeffding equality*): For any absolutely continuous functions f and g on the real line and for any random variables X and Y satisfying $Ef^2(X)+Eg^2(Y)<\infty$, we have

$$\begin{aligned}&\mathrm{Cov}\,\big(f(X), g(Y)\big)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f'(x)g'(y)\Big\{P(X\ge x,\ Y\ge y)-P(X\ge x)P(Y\ge y)\Big\}dxdy.\end{aligned}$$

The main ingredients of the proofs of Theorems 1.1-1.2 are Propositions 2.1-2.3 below. Note that the conditions (i)-(ii) in Theorem 1.1 imply conditions (C2) and (I)-(II) in [9] and [15], respectively. Moreover, $\|\mathbb{X}(\mathbf{t})\|_\infty/\sigma_*(\|\mathbf{t}\|)$ is a standardized random variable. Thus Lemma 2 in [9] and Corollary 2.1 in [15] are easily changed to the following Berry-Esseen type theorem.

Proposition 2.1. (Berry-Esseen type theorem). *Let* $\{\mathbb{X}(\mathbf{t})\,;\,\mathbf{t}\in[0,\infty)^N\}$ *be as in* Theorem 1.1 *with conditions* (i)-(ii). *Then*

$$\sup_z \Big| P\Big\{\tfrac{\|\mathbb{X}(\mathbf{b}_T)\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)} \le z\Big\} - \Phi(z)\Big| = O\big(\|\mathbf{b}_T\|^{-1/5}\big), \quad T\to\infty,$$

where $\Phi(\cdot)$ *is a standard normal distribution function and* $\|\mathbf{b}_T\|\to\infty$ *as* $T\to\infty$.

Denote $\mathbf{b}_k=\mathbf{b}_{T_k}$ for a nonnegative increasing sequence $\{T_k\}_{k=1}^\infty$ in $\mathbb{R}_+$. Using Proposition 2.1, the following proposition is immediate from the proof of Lemma 9 in Petrov [14, p. 311].

Proposition 2.2. *Let* $\{\mathbb{X}(\mathbf{t})\}$ *be as in* Proposition 2.1. *Assume that* $g(x)$ *is a positive nondecreasing function of* $x > 0$ *and that* $\{\|\mathbf{b}_k\|\,;\, k \geq 1\}$ *is a positive nondecreasing sequence such that* $\sum_{k=1}^{\infty} \|\mathbf{b}_k\|^{-1/5} < \infty$. *Then the following statements are equivalent.*

$$\text{(A)} \qquad \sum_{k=1}^{\infty} P\Big\{ \frac{\|\mathbb{X}(\mathbf{b}_k)\|_\infty}{\sigma_*(\|\mathbf{b}_k\|)} > g(\|\mathbf{b}_k\|) \Big\} < \infty,$$

$$\text{(B)} \qquad \sum_{k=1}^{\infty} \frac{1}{g(\|\mathbf{b}_k\|)} \exp\Big(-\frac{1}{2} g^2(\|\mathbf{b}_k\|) \Big) < \infty.$$

The following proposition on the large deviation probability is proved in Section 3.

Proposition 2.3. *Let* $\{\mathbb{X}(\mathbf{t})\,;\, \mathbf{t} \in [0,\infty)^N\}$ *be a centered strictly stationary* l^∞*-valued random field. Then, for any* $\varepsilon > 0$ *there exists a constant* $c_\varepsilon > 0$ *such that, for* $v > 1$,

$$\begin{aligned} & P\Big\{ \sup_{\|\mathbf{s}\| \leq \|\mathbf{b}_T\|} \sup_{\|\mathbf{t}\| \leq \|\mathbf{b}_T\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t}) - \mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)} \geq v \Big\} \\ & \leq c_\varepsilon \bigg(P\Big\{ \frac{\|\mathbb{X}(\mathbf{b}_T)\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)} \geq \frac{v}{1+\varepsilon} \Big\} \\ & \qquad + \sum_{n=1}^{\infty} 2^{2N2^n} P\Big\{ \frac{\|\mathbb{X}(\mathbf{b}_T)\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)} \geq \frac{v}{1+\varepsilon} \sqrt{1 + 2N \log 3 \cdot 2^{n/2}} \Big\} \bigg). \end{aligned}$$

Proof of Theorem 1.1. Let us first prove

$$\text{(2.1)} \qquad \limsup_{T\to\infty} \sup_{\|\mathbf{s}\| \leq \|\mathbf{b}_T\|} \sup_{\|\mathbf{t}\| \leq \|\mathbf{b}_T\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t}) - \mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)\sqrt{2\log\log\|\mathbf{b}_T\|}} \leq 1 \qquad \text{a.s.}$$

For $\theta > 1$, set $A_k = \big\{T;\, \theta^{k-1} \leq \|\mathbf{b}_T\| \leq \theta^k\big\},\ k \geq 1$. Note that $\sqrt{2\log\log\theta^{k-1}} \geq \theta^{-1}\sqrt{2\log\log\theta^k}$ since $(\log u)/u$ is decreasing for $u > e^e$. By the regularity of $\sigma_*(\cdot)$, we get $\sigma_*(\|\mathbf{b}_T\|)/\sigma_*(\theta^k) \geq \theta^{-2\alpha}$ as $k \to \infty$, and hence

$$\text{(2.2)} \qquad \begin{aligned} & \limsup_{T\to\infty} \sup_{\|\mathbf{s}\| \leq \|\mathbf{b}_T\|} \sup_{\|\mathbf{t}\| \leq \|\mathbf{b}_T\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t}) - \mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)\sqrt{2\log\log\|\mathbf{b}_T\|}} \\ & \leq \limsup_{k\to\infty} \sup_{T \in A_k} \sup_{\|\mathbf{s}\| \leq \|\mathbf{b}_T\|} \sup_{\|\mathbf{t}\| \leq \|\mathbf{b}_T\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t}) - \mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)\sqrt{2\log\log\theta^{k-1}}} \\ & \leq \theta^{1+2\alpha} \limsup_{k\to\infty} \sup_{\|\mathbf{s}\| \leq \theta^k} \sup_{\|\mathbf{t}\| \leq \theta^k} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t}) - \mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\theta^k)\sqrt{2\log\log\theta^k}}. \end{aligned}$$

For convenience, let $\|\mathbf{b}_k\| = \theta^k$, where $\mathbf{b}_k := \mathbf{b}_{T_k}$ for a nonnegative increasing sequence $\{T_k\}_{k=1}^{\infty}$. Using Proposition 2.3, it follows that for any $\varepsilon > 0$ there exists a positive constant c_ε such that

$$P\Big\{\sup_{\|\mathbf{s}\|\le\theta^k}\sup_{\|\mathbf{t}\|\le\theta^k}\frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\theta^k)\sqrt{2\log\log\theta^k}}>1+2\varepsilon\Big\}$$

$$\le c_\varepsilon\Bigg(P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_k)\|_\infty}{\sigma_*(\theta^k)}\ge\frac{(1+2\varepsilon)\sqrt{2\log\log\theta^k}}{1+\varepsilon}\Big\} \tag{2.3}$$

$$+\sum_{n=1}^\infty 2^{2N2^n}P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_k)\|_\infty}{\sigma_*(\theta^k)}\ge\frac{(1+2\varepsilon)\sqrt{2\log\log\theta^k}}{1+\varepsilon}\sqrt{1+2N\log 3}\cdot 2^{n/2}\Big\}\Bigg).$$

Now let us apply Proposition 2.2 with $\|\mathbf{b}_k\|=\theta^k$ and

$$g(\theta^k):=\frac{(1+2\varepsilon)\sqrt{2\log\log\theta^k}}{1+\varepsilon}\quad\Big(\text{or}\quad\frac{(1+2\varepsilon)\sqrt{2\log\log\theta^k}}{1+\varepsilon}\sqrt{1+2N\log 3}\cdot 2^{n/2}\Big).$$

Considering the right hand side of (2.3) and (B) of Proposition 2.2, we have

$$\sum_{k=1}^\infty\frac{1+\varepsilon}{(1+2\varepsilon)\sqrt{2\log\log\theta^k}}\exp\Big(-\frac12\Big(\frac{1+2\varepsilon}{1+\varepsilon}\Big)^2 2\log\log\theta^k\Big)$$

$$\le c\sum_{k=1}^\infty\big(\log\theta^k\big)^{-1-\varepsilon'}<\infty,$$

where $\varepsilon'=\varepsilon/(1+\varepsilon)$, and also

$$\sum_{k=1}^\infty\sum_{n=1}^\infty 2^{2N2^n}\exp\Big(-\frac12\Big(\frac{1+2\varepsilon}{1+\varepsilon}\Big)^2(2\log\log\theta^k)(1+2N\log 3)2^n\Big)$$

$$\le\sum_{k=1}^\infty\sum_{n=1}^\infty 2^{2N2^n}(\log\theta^k)^{-(1+\varepsilon')(1+2N\log 3)2^n}$$

$$\le c\sum_{k=1}^\infty\sum_{n=1}^\infty 2^{2N2^n\big(1-(\log_2 k)\log 3\big)}\cdot 2^{-(\log_2 k)2^n}$$

$$\le c\sum_{k=1}^\infty\sum_{n=1}^\infty 2^{-2^n\log_2 k}\cdot 2^{-n}\le c\sum_{k=1}^\infty\sum_{n=1}^\infty k^{-2}\cdot 2^{-n}<\infty.$$

It follows from (2.3) and Proposition 2.2 that

$$\sum_{k=1}^\infty P\Big\{\sup_{\|\mathbf{s}\|\le\theta^k}\sup_{\|\mathbf{t}\|\le\theta^k}\frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\theta^k)\sqrt{2\log\log\theta^k}}>1+2\varepsilon\Big\}<\infty.$$

Thus the Borel-Cantelli lemma yields

$$\limsup_{k\to\infty}\sup_{\|\mathbf{s}\|\le\theta^k}\sup_{\|\mathbf{t}\|\le\theta^k}\frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\theta^k)\sqrt{2\log\log\theta^k}}\le 1+2\varepsilon\qquad\text{a.s.}$$

Combining this with (2.2) implies (2.1) since ε and θ are arbitrary.

By virtue of (2.1), the proof of (1.3) is completed if we show that

$$\limsup_{T\to\infty} \frac{\| \mathbb{X}(\mathbf{b}_T) \|_\infty}{\sigma_*(\|\mathbf{b}_T\|)\sqrt{2\log\log\|\mathbf{b}_T\|}} \geq 1 \qquad \text{a.s.} \tag{2.4}$$

Set $\mathbf{b}_k = \mathbf{b}_{T_k}$ for an increasing sequence $\{T_k\}_{k=1}^\infty$ in $\mathbb{R}_+$, and let $i_0 \geq 1$ be an integer such that $\sigma_{i_0}(\|\mathbf{b}_k\|) = \sigma_*(\|\mathbf{b}_k\|)$. Then

$$\limsup_{k\to\infty} \frac{\| \mathbb{X}(\mathbf{b}_k) \|_\infty}{\sigma_*(\|\mathbf{b}_k\|)\sqrt{2\log\log\|\mathbf{b}_k\|}} \geq \limsup_{k\to\infty} \frac{X_{i_0}(\mathbf{b}_k)}{\sigma_{i_0}(\|\mathbf{b}_k\|)\sqrt{2\log\log\|\mathbf{b}_k\|}} \tag{2.5}$$

and the inequality (2.4) is immediate from (2.5) if we prove

$$\limsup_{k\to\infty} \frac{X_{i_0}(\mathbf{b}_k)}{\sigma_{i_0}(\|\mathbf{b}_k\|)\sqrt{2\log\log\|\mathbf{b}_k\|}} > 1-4\varepsilon \qquad \text{a.s.} \tag{2.6}$$

for any small $\varepsilon > 0$. Let

$$B_k = \left\{ \frac{X_{i_0}(\mathbf{b}_k) - X_{i_0}(\mathbf{b}_{k/2})}{\sigma_{i_0}(\| \mathbf{b}_k - \mathbf{b}_{k/2} \|)} > (1-2\varepsilon)\sqrt{2\log\log \| \mathbf{b}_k - \mathbf{b}_{k/2} \|} \right\}.$$

Note that

$$U_k := \frac{X_{i_0}(\mathbf{b}_k) - X_{i_0}(\mathbf{b}_{k/2})}{\sigma_{i_0}(\| \mathbf{b}_k - \mathbf{b}_{k/2} \|)}, \quad k \geq 1,$$

is a standardized random variable. For $\theta > 1$, set $\|\mathbf{b}_k\| = \theta^k$. Then $\| \mathbf{b}_k - \mathbf{b}_{k/2} \| \approx \theta^k$ for sufficiently large k since $\| \mathbf{b}_k\| - \|\mathbf{b}_{k/2} \| \leq \| \mathbf{b}_k - \mathbf{b}_{k/2} \| \leq \| \mathbf{b}_k \| + \|\mathbf{b}_{k/2} \|$. To apply Proposition 2.2 with $\| \mathbf{b}_k - \mathbf{b}_{k/2} \|$, let $g(\| \mathbf{b}_k - \mathbf{b}_{k/2} \|) = (1-2\varepsilon) \sqrt{2\log\log \| \mathbf{b}_k - \mathbf{b}_{k/2} \|}$. Then

$$\begin{aligned}\sum_{k=1}^\infty \frac{1}{g\big(\| \mathbf{b}_k - \mathbf{b}_{k/2} \|\big)} \exp\Big(-\frac{1}{2} g^2\big(\| \mathbf{b}_k - \mathbf{b}_{k/2} \|\big)\Big) &\geq c\sum_{k=1}^\infty \exp\Big(-(1-\varepsilon)\log\log\theta^k\Big) \\ &\geq c\sum_{k=1}^\infty k^{-1+\varepsilon} = \infty.\end{aligned}$$

Consequently, applying Proposition 2.2 implies

$$\sum_{k=1}^\infty P(B_k) = \infty. \tag{2.7}$$

Next, let

$$B_k' = \Big\{ U_k > (1-3\varepsilon)\sqrt{2\log\log \| \mathbf{b}_k - \mathbf{b}_{k/2} \|} \Big\}.$$

We will show that

$$P\big(B_k',\ i.o.\big) = 1. \tag{2.8}$$

Choose a differential function $f(x)$ on $\mathbb{R}$ such that $|f'(x)| \le \kappa$ for some $0 < \kappa < \infty$ and

$$\begin{aligned} 0 &\le I\Big\{x > (1-2\varepsilon)\sqrt{2\log\log \| \mathbf{b}_k - \mathbf{b}_{k/2} \|}\Big\} \\ &\le f(x) \le I\Big\{x > (1-3\varepsilon)\sqrt{2\log\log \| \mathbf{b}_k - \mathbf{b}_{k/2} \|}\Big\} \le 1, \end{aligned} \tag{2.9}$$

where $I\{\cdot\}$ is an indicator function. In order to prove (2.8), it is enough to show that

$$\sum_{k=1}^{\infty} f(U_k) = \infty \qquad \text{a.s.} \tag{2.10}$$

From (2.7) and (2.9), we get

$$\sum_{k=1}^{\infty} Ef(U_k) \ge \sum_{k=1}^{\infty} P(B_k) = \infty. \tag{2.11}$$

By Markov inequality, we have

$$\begin{aligned} &P\left\{\sum_{k=1}^{\infty} f(U_k) < \frac{1}{2}\sum_{k=1}^{n} Ef(U_k)\right\} \\ &\le P\left\{\left|\sum_{k=1}^{n} f(U_k) - \sum_{k=1}^{n} Ef(U_k)\right| > \frac{1}{2}\sum_{k=1}^{n} Ef(U_k)\right\} \\ &\le 4\,\mathrm{Var}\left(\sum_{k=1}^{n} f(U_k)\right) \bigg/ \left(\sum_{k=1}^{n} Ef(U_k)\right)^2 \\ &\le \frac{4}{\sum_{k=1}^{n} Ef(U_k)} + \frac{8\sum_{k=1}^{\infty}\sum_{j=k+1}^{\infty} \left|\,\mathrm{Cov}\big(f(U_k), f(U_j)\big)\right|}{\Big(\sum_{k=1}^{n} Ef(U_k)\Big)^2}. \end{aligned} \tag{2.12}$$

Noting that U_k and U_j are LPQD (resp. LNQD) from the definition of LPQD (resp. LNQD), it follows from (i), (P_1), (P_2) and the regularity of $\sigma_*(\cdot)$ that

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{j=k+1}^{\infty} \left| \operatorname{Cov}\big(f(U_k), f(U_j)\big) \right| \\
&\le \sum_{k=1}^{\infty}\sum_{j=k+1}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f'(x)|\,|f'(y)| \Big| P\{U_k \ge x, U_j \ge y\} \\
&\quad -P\{U_k \ge x\}P\{U_j \ge y\}\Big| dxdy \\
&\le \kappa^2 \sum_{k=1}^{\infty}\sum_{j=k+1}^{\infty} \Big| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(P\{U_k \ge x, U_j \ge y\} \\
&\quad -P\{U_k \ge x\}P\{U_j \ge y\}\Big) dxdy \Big| \\
&= \kappa^2 \sum_{k=1}^{\infty}\sum_{j=k+1}^{\infty} \Big| \operatorname{Cov}(U_k, U_j)\Big| \\
&\le c\sum_{k=1}^{\infty} \frac{1}{\sigma_{i_0}^2(\|\mathbf{b}_k - \mathbf{b}_{k/2}\|)} \\
&\quad \sum_{j=k+1}^{\infty} \Big| \operatorname{Cov}\big(X_{i_0}(\mathbf{b}_k) - X_{i_0}(\mathbf{b}_{k/2}),\, X_{i_0}(\mathbf{b}_j) - X_{i_0}(\mathbf{b}_{j/2})\big)\Big| \\
&\le c\sum_{k=1}^{\infty} \frac{\|\mathbf{b}_k - \mathbf{b}_{k/2}\|}{\sigma_{i_0}^2(\|\mathbf{b}_k - \mathbf{b}_{k/2}\|)} \sum_{j=k+1}^{\infty} \Big| \operatorname{Cov}\big(X_{i_0}(\mathbf{1}),\, X_{i_0}(\mathbf{b}_j) - X_{i_0}(\mathbf{b}_{j/2})\big)\Big| \\
&\le c\sum_{k=1}^{\infty} (\theta^k)^{1-2\alpha} \|\mathbf{b}_{(k+1)/2}\| \sum_{j \ge k+1} \Big| \operatorname{Cov}\big(X_{i_0}(\mathbf{1}),\, X_{i_0}(\mathbf{b}_j)\big)\Big| \\
&\le c\sum_{k=1}^{\infty} \theta^{k(1-2\alpha)}\theta^{(k+1)/2} \|\mathbf{b}_k\|^{-\lambda} \le c\sum_{k=1}^{\infty} \theta^{-(\lambda-2+2\alpha)k} < \infty
\end{aligned} \tag{2.13}
$$

for $\alpha > 0$ and $\lambda > 2$. Combining (2.11)-(2.13) and letting $n \to \infty$ yields

$$
P\left\{ \sum_{k=1}^{\infty} f(U_k) < \infty \right\} = 0.
$$

This proves (2.10) and consequently (2.8). Let

$$
C_k = \left\{ \frac{X_{i_0}(\mathbf{b}_{k/2})}{\sigma_{i_0}(\|\mathbf{b}_{k/2}\|)} \ge -2\sqrt{2\log\log \|\mathbf{b}_{k/2}\|} \right\}.
$$

It follows from (2.1) and (2.8) that $P(B_k' \cap C_k,\ i.o.) = 1$. It is easy to see that

$$
\begin{aligned}
&P\Big\{ \frac{X_{i_0}(\mathbf{b}_k)}{\sigma_{i_0}(\|\mathbf{b}_k\|)} > (1-4\varepsilon)\sqrt{2\log\log \|\mathbf{b}_k\|}, \quad i.o.\Big\} \\
&\ge P\Big\{ \frac{X_{i_0}(\mathbf{b}_k)}{\sigma_{i_0}(\|\mathbf{b}_k\|)} > (1-3\varepsilon)\sqrt{2\log\log \|\mathbf{b}_k - \mathbf{b}_{k/2}\|} - 2\sqrt{2\log\log \|\mathbf{b}_{k/2}\|}, \quad i.o.\Big\} \\
&\ge P\Big\{ B_k' \cap C_k, \quad i.o.\Big\} = 1
\end{aligned}
$$

for k large enough. This implies (2.6).

Proof of Theorem 1.2. Let $\theta = 1+\varepsilon$ for $0<\varepsilon<1$. Let k_i and $l_i\,(1\le i\le N)$ be positive integers, and set $\theta^{\mathbf{k}} = (\theta^{k_1},\cdots,\theta^{k_N})$, $\theta^{a\boldsymbol{\ell}} = (\theta^{al_1},\cdots,\theta^{al_N}) \in \mathbb{R}_+^N$ for $-\infty<a<\infty$, $k=\frac{1}{N}\sum_{i=1}^N k_i$ and $l=\frac{1}{N}\sum_{i=1}^N l_i$. Define

$$B_{\boldsymbol{\ell},\mathbf{k}} = \Big\{T\,;\,\theta^{l_i-1}\le a_i(T)\le\theta^{l_i},\quad \theta^{k_i-1}\le b_i(T)\le\theta^{k_i},\ 1\le i\le N\Big\}.$$

In the sequel, we always consider $\boldsymbol{\ell}$ and $\mathbf{k}$ such that $B_{\boldsymbol{\ell},\mathbf{k}}\ne\varnothing$. Note that $\theta^{l-1}\le \|\boldsymbol{a}_T\|\le\theta^{Nl}$ and $\theta^{k-1}\le\|\mathbf{b}_T\|\le\theta^{Nk}$ for $T\in B_{\boldsymbol{\ell},\mathbf{k}}$. The condition (iii) implies that

$$l < Nk - 2(\log\log\theta^k)/(\log\theta)^2 =: K \qquad \text{and}$$

$$\gamma(T) = \Big(2\log\big(\|\mathbf{b}_T\|/\|\boldsymbol{a}_T\|\big)\Big\{1+\frac{\log\log\|\mathbf{b}_T\|}{\log\big(\|\mathbf{b}_T\|/\|\boldsymbol{a}_T\|\big)}\Big\}\Big)^{1/2} \approx \sqrt{2\log\big(\|\mathbf{b}_T\|/\|\boldsymbol{a}_T\|\big)}$$

for sufficiently large k (or T). Thus (1.5) is immediate from (1.4) if we show that

$$\liminf_{T\to\infty}\sup_{\|\mathbf{s}\|\le\|\mathbf{b}_T\|}\frac{\|\mathbb{X}(\mathbf{s}+\mathbf{b}_T)-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)\sqrt{2\log\big(\|\mathbf{b}_T\|/\|\boldsymbol{a}_T\|\big)}}\ge 1 \qquad \text{a.s.} \tag{2.14}$$

By the definition of $\sigma_*(\cdot)$, there exists an integer $\iota\ge 1$ such that $\sigma_\iota(\|\theta^{\mathbf{k}}\|) = \sigma_*(\|\theta^{\mathbf{k}}\|)$, $\mathbf{k}>\mathbf{1}$. Put $\beta_{k,\boldsymbol{\ell}} = \frac{1}{\sqrt{N}}(\theta^{k-1}\theta^{-l_1/N},\cdots,\theta^{k-1}\theta^{-l_N/N})$. Then we can write

$$\begin{aligned}
&\liminf_{T\to\infty}\sup_{\|\mathbf{s}\|\le\|\mathbf{b}_T\|}\frac{\|\mathbb{X}(\mathbf{s}+\mathbf{b}_T)-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)\sqrt{2\log\big(\|\mathbf{b}_T\|/\|\boldsymbol{a}_T\|\big)}}\\
&\ge \liminf_{k\to\infty}\inf_{T\in B_{\boldsymbol{\ell},\mathbf{k}}}\sup_{\|\mathbf{s}\|\le\|\mathbf{b}_T\|}\frac{\|\mathbb{X}(\mathbf{s}+\theta^{\mathbf{k}})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\theta^{\mathbf{k}}\|)\sqrt{2\log\theta^{Nk-l+1}}}\\
&\quad -\limsup_{k\to\infty}\sup_{T\in B_{\boldsymbol{\ell},\mathbf{k}}}\sup_{\|\mathbf{s}\|\le\|\mathbf{b}_T\|}\frac{\|\mathbb{X}(\mathbf{s}+\theta^{\mathbf{k}})-\mathbb{X}(\mathbf{s}+\mathbf{b}_T)\|_\infty}{\sigma_*(\|\theta^{\mathbf{k-1}}\|)\sqrt{2\log\theta^{Nk-l+1}}}\\
&\ge \liminf_{k\to\infty}\inf_{l<K}\sup_{\|\mathbf{s}\|\le\theta^{k-1}}\frac{\|\mathbb{X}(\mathbf{s}+\theta^{\mathbf{k}})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\theta^{\mathbf{k}}\|)\sqrt{2\log\theta^{Nk-l+1}}} \qquad (2.15)\\
&\quad -\limsup_{k\to\infty}\sup_{l<K}\sup_{\|\mathbf{s}\|\le\|\theta^{\mathbf{k}}\|}\sup_{\|\theta^{\mathbf{k-1}}\|\le\|\mathbf{b}_T\|\le\|\theta^{\mathbf{k}}\|}\\
&\qquad\frac{\|\mathbb{X}(\mathbf{s}+\theta^{\mathbf{k}})-\mathbb{X}(\mathbf{s}+\mathbf{b}_T)\|_\infty}{\sigma_*(\|\theta^{\mathbf{k}}-\theta^{\mathbf{k-1}}\|)\sqrt{2\log\theta^{Nk-l+1}}}\frac{\sigma_*(\|\theta^{\mathbf{k}}-\theta^{\mathbf{k-1}}\|)}{\sigma_*(\|\theta^{\mathbf{k-1}}\|)}\\
&\ge \liminf_{k\to\infty}\inf_{l<K}\max_{\mathbf{1}\le\mathbf{i}\le\beta_{k,\boldsymbol{\ell}}}\frac{X_\iota(\mathbf{i}\,\theta^{\boldsymbol{\ell}/N}+\theta^{\mathbf{k}})-X_\iota(\mathbf{i}\,\theta^{\boldsymbol{\ell}/N})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)\sqrt{2\log\theta^{Nk-l+1}}}
\end{aligned}$$

$$
\begin{aligned}
&- \limsup_{k\to\infty} \sup_{l<K} \sup_{\|\mathbf{s}\|\le\|\theta^{\mathbf{k}}\|} \sup_{\|\theta^{\mathbf{k-1}}\|\le\|\mathbf{t}\|\le\|\theta^{\mathbf{k}}\|} \\
&\frac{\|\mathbb{X}(\mathbf{s}+\theta^{\mathbf{k}})-\mathbb{X}(\mathbf{s}+\mathbf{t})\|_\infty}{\sigma_*(\|\theta^{\mathbf{k}}-\theta^{\mathbf{k-1}}\|)\sqrt{2\log\theta^{Nk-l+1}}} \frac{\sigma_*(\|\theta^{\mathbf{k}}-\theta^{\mathbf{k-1}}\|)}{\sigma_*(\|\theta^{\mathbf{k-1}}\|)} \\
&=: J_1 - J_2,
\end{aligned}
$$

where $\mathbf{i} := (i_1, \cdots, i_N) \in \mathbb{Z}_+^N$. First, we claim that

$$
J_1 \ge 1 \qquad \text{a.s.} \tag{2.16}
$$

Let $\{X_i(\mathbf{t}) \,;\, \mathbf{t} \in [0,\infty)^N\}_{i=1}^\infty$ be a centered strictly stationary and LNQD random field taking values in l^∞-space, then it follows that

$$
\begin{aligned}
&P\Big\{ \inf_{l<K} \max_{\mathbf{1}\le\mathbf{i}\le\beta_{k,\boldsymbol{\ell}}} \frac{X_\iota(\mathbf{i}\,\theta^{\boldsymbol{\ell}/N}+\theta^{\mathbf{k}})-X_\iota(\mathbf{i}\,\theta^{\boldsymbol{\ell}/N})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)\sqrt{2\log\theta^{Nk-l+1}}} \le \sqrt{1-\varepsilon}\Big\} \\
&\le \sum_{l<K} \Big(P\Big\{ \frac{X_\iota(\theta^{\mathbf{k}})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)} \le \sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}} \Big\}\Big)^{\theta^{Nk-l}}.
\end{aligned} \tag{2.17}
$$

Similarly, if $\{X_i(\mathbf{t})\}_{i=1}^\infty$ is a centered strictly stationary and LPQD l^∞-valued random field, then we have

$$
\begin{aligned}
&P\Big\{ \inf_{l<K} \max_{\mathbf{1}\le\mathbf{i}\le\beta_{k,\boldsymbol{\ell}}} \frac{X_\iota(\mathbf{i}\,\theta^{\boldsymbol{\ell}/N}+\theta^{\mathbf{k}})-X_\iota(\mathbf{i}\,\theta^{\boldsymbol{\ell}/N})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)\sqrt{2\log\theta^{Nk-l+1}}} \le \sqrt{1-\varepsilon}\Big\} \\
&\le \sum_{l<K} \Big(1 - P\Big\{ \max_{\mathbf{1}\le\mathbf{i}\le\beta_{k,\boldsymbol{\ell}}} \frac{X_\iota(\mathbf{i}\,\theta^{\boldsymbol{\ell}/N}+\theta^{\mathbf{k}})-X_\iota(\mathbf{i}\,\theta^{\boldsymbol{\ell}/N})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)} > \sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}}\Big\}\Big) \\
&\le \sum_{l<K} \Big(1 - P\Big\{ \min_{\mathbf{1}\le\mathbf{i}\le\beta_{k,\boldsymbol{\ell}}} \frac{X_\iota(\mathbf{i}\,\theta^{\boldsymbol{\ell}/N}+\theta^{\mathbf{k}})-X_\iota(\mathbf{i}\,\theta^{\boldsymbol{\ell}/N})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)} > \sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}}\Big\}\Big) \\
&\le \sum_{l<K} \Big(1 - P\Big\{ \frac{X_\iota(\theta^{\boldsymbol{\ell}/N}+\theta^{\mathbf{k}})-X_\iota(\theta^{\boldsymbol{\ell}/N})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)} > \sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}}, \cdots \\
&\qquad \cdots, \frac{X_\iota(\beta_{k,\boldsymbol{\ell}}\,\theta^{\boldsymbol{\ell}/N}+\theta^{\mathbf{k}})-X_\iota(\beta_{k,\boldsymbol{\ell}}\,\theta^{\boldsymbol{\ell}/N})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)} > \sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}}\Big\}\Big) \\
&\le \sum_{l<K} \Big(1 - \Big(P\Big\{ \frac{X_\iota(\theta^{\mathbf{k}})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)} > \sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}}\Big\}\Big)^{\theta^{Nk-l}}\Big) \\
&= \sum_{l<K} \Big(1 - \Big(1 - P\Big\{ \frac{X_\iota(\theta^{\mathbf{k}})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)} \le \sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}}\Big\}\Big)^{\theta^{Nk-l}}\Big) \\
&\le c \sum_{l<K} \Big(P\Big\{ \frac{X_\iota(\theta^{\mathbf{k}})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)} \le \sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}}\Big\}\Big)^{\theta^{Nk-l}}
\end{aligned} \tag{2.18}
$$

for k large enough and some constant $c > 0$. On the other hand, if Z is a standard normal random variable, then we have

$$\begin{aligned}\Phi(x) &:= P\{Z \le x\} = 1 - P\{Z > x\} \le \exp\big(-P\{Z > x\}\big)\\ &\le \exp\Big(-\frac{1}{\sqrt{2\pi}x^2}e^{-x^2/2}\Big) \le \exp\Big(-c\,e^{-x^2/2}\Big)\end{aligned}$$

for all large $x > 0$. Noting that $Y_{\mathbf{k}} := X_\iota(\theta^{\mathbf{k}})/\sigma_\iota(\|\theta^{\mathbf{k}}\|)$ is a standardized random variable and applying Proposition 2.1 to $Y_{\mathbf{k}}$ in place of $\|\mathbb{X}(\mathbf{b}_T)\|_\infty/\sigma_*(\|\mathbf{b}_T\|)$, it follows from (2.17) and (2.18) that

$$\begin{aligned}
&P\Big\{\inf_{l<K}\max_{\mathbf{1}\le\mathbf{i}\le\beta_{k,\boldsymbol{\ell}}}\frac{X_\iota(\mathbf{i}\,\theta^{\boldsymbol{\ell}/N}+\theta^{\mathbf{k}})-X_\iota(\mathbf{i}\,\theta^{\boldsymbol{\ell}/N})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)\sqrt{2\log\theta^{Nk-l+1}}}\le\sqrt{1-\varepsilon}\Big\}\\
&\le c\sum_{l<K}\Big(P\Big\{\frac{X_\iota(\theta^{\mathbf{k}})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)}\le\sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}}\Big\}\Big)^{\theta^{Nk-l}}\\
&\le c\sum_{l<K}\Big(\Big|P\Big\{\frac{X_\iota(\theta^{\mathbf{k}})}{\sigma_\iota(\|\theta^{\mathbf{k}}\|)}\le\sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}}\Big\}-\Phi\Big(\sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}}\Big)\Big|\\
&\qquad\qquad+\Phi\Big(\sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}}\Big)\Big)^{\theta^{Nk-l}}\\
&\le c\sum_{l<K}\Big((\theta^{k-1})^{-1/5}+\Phi\Big(\sqrt{2(1-\varepsilon)\log\theta^{Nk-l+1}}\Big)\Big)^{\theta^{Nk-l}}\\
&\le c\sum_{l<K}\Big\{\theta^{-(k-1)/5}+\exp\Big(-c\,e^{-(1-\varepsilon)\log\theta^{Nk-l+1}}\Big)\Big\}^{\theta^{Nk-l}}\\
&\le c\sum_{l<K}\Big\{\exp\Big(-c\,\theta^{-(1-\varepsilon)(Nk-l+1)}\Big)\Big\}^{\theta^{Nk-l}}\\
&\le c\sum_{l<K}\exp\Big(-c\,\theta^{(Nk-l)\varepsilon}\Big)\le c\,\exp\Big(-c\,\big(k\log\theta\big)^{2\varepsilon/\log\theta}\Big)\\
&\le c\,\exp\big(-c_\varepsilon k^{2\varepsilon/\log\theta}\big)\le c\,\exp(-c_\varepsilon k)
\end{aligned}$$

for all large k, since $2\varepsilon > \log\theta$ for any small $\varepsilon > 0$. The Borel-Cantelli lemma implies (2.16).

Next, we show that

$$J_2 \le c\,\varepsilon^\alpha \qquad \text{a.s.} \tag{2.19}$$

for any small $\varepsilon > 0$, where $c > 0$ is a constant. Since $\sigma_*(\cdot)$ is a regularly varying function with exponent $\alpha > 0$ at ∞, we have

$$\frac{\sigma_*(\|\theta^{\mathbf{k}} - \theta^{\mathbf{k-1}}\|)}{\sigma_*(\|\theta^{\mathbf{k-1}}\|)} = \frac{\sigma_*((\theta - 1)\|\theta^{\mathbf{k-1}}\|)}{\sigma_*(\|\theta^{\mathbf{k-1}}\|)} \le c\,\varepsilon^{\alpha}.$$

Therefore, (2.19) is proved if we show that

$$\limsup_{k\to\infty} \sup_{l<K} \sup_{\|\mathbf{s}\|\le\|\theta^{\mathbf{k}}\|} \sup_{\|\theta^{\mathbf{k-1}}\|\le\|\mathbf{t}\|\le\|\theta^{\mathbf{k}}\|} \frac{\|\mathbb{X}(\mathbf{s}+\theta^{\mathbf{k}}) - \mathbb{X}(\mathbf{s}+\mathbf{t})\|_\infty}{\sigma_*(\|\theta^{\mathbf{k}} - \theta^{\mathbf{k-1}}\|)\sqrt{2\log\theta^{Nk-l+1}}} \le 1 \qquad \text{a.s.} \tag{2.20}$$

Similarly to the proof of Proposition 2.3, it follows by the stationary of $\{X_i(\mathbf{t})\}_{i=1}^{\infty}$ that

$$\begin{aligned}
&P\Big\{\sup_{l<K} \sup_{\|\mathbf{s}\|\le\|\theta^{\mathbf{k}}\|} \sup_{\|\theta^{\mathbf{k-1}}\|\le\|\mathbf{t}\|\le\|\theta^{\mathbf{k}}\|} \frac{\|\mathbb{X}(\mathbf{s}+\theta^{\mathbf{k}}) - \mathbb{X}(\mathbf{s}+\mathbf{t})\|_\infty}{\sigma_*(\|\theta^{\mathbf{k}} - \theta^{\mathbf{k-1}}\|)\sqrt{2\log\theta^{Nk-l+1}}} > 1+2\varepsilon\Big\}\\
&\le \sum_{l<K} P\Big\{\sup_{\|\mathbf{s}\|\le\|\theta^{\mathbf{k}}\|} \sup_{\|\theta^{\mathbf{k-1}}\|\le\|\mathbf{t}\|\le\|\theta^{\mathbf{k}}\|} \frac{\|\mathbb{X}(\mathbf{s}+\theta^{\mathbf{k}}) - \mathbb{X}(\mathbf{s}+\mathbf{t})\|_\infty}{\sigma_*(\|\theta^{\mathbf{k}} - \theta^{\mathbf{k-1}}\|)}\\
&\qquad > (1+2\varepsilon)\sqrt{2\log\theta^{Nk-l+1}}\Big\}\\
&\le \sum_{l<K} c_\varepsilon\Big(P\Big\{\frac{\|\mathbb{X}(\theta^{\mathbf{k}} - \theta^{\mathbf{k-1}})\|_\infty}{\sigma_*(\|\theta^{\mathbf{k}} - \theta^{\mathbf{k-1}}\|)} > \frac{1+2\varepsilon}{1+\varepsilon}\sqrt{2\log\theta^{Nk-l+1}}\Big\}\\
&\qquad + \sum_{n=1}^{\infty} 2^{2N2^n} P\Big\{\frac{\|\mathbb{X}(\theta^{\mathbf{k}} - \theta^{\mathbf{k-1}})\|_\infty}{\sigma_*(\|\theta^{\mathbf{k}} - \theta^{\mathbf{k-1}}\|)}\\
&\qquad > \frac{1+2\varepsilon}{1+\varepsilon}\sqrt{2\log\theta^{Nk-l+1}}\sqrt{1+2N\log 3}\cdot 2^{n/2}\Big\}\Big)\\
&\le c_\varepsilon \sum_{l<K}\Big(P\Big\{\frac{\|\mathbb{X}(\theta^{\mathbf{k}})\|_\infty}{\sigma_*(\|\theta^{\mathbf{k}}\|)} > \frac{1+2\varepsilon}{1+\varepsilon}\sqrt{2\log\theta^{Nk-l+1}}\Big\}\\
&\qquad + \sum_{n=1}^{\infty} 2^{2N2^n} P\Big\{\frac{\|\mathbb{X}(\theta^{\mathbf{k}})\|_\infty}{\sigma_*(\|\theta^{\mathbf{k}}\|)} > \frac{1+2\varepsilon}{1+\varepsilon}\sqrt{2\log\theta^{Nk-l+1}}\sqrt{1+2N\log 3}\cdot 2^{n/2}\Big\}\Big).
\end{aligned} \tag{2.21}$$

Set $\mathbf{b}_k = \mathbf{b}_{T_k}$ for an increasing subsequence $\{T_k\}_{k=1}^{\infty}$ of $\{T;\, T>0\}$. Let us apply Proposition 2.2 with $\|\mathbf{b}_k\| := \|\theta^{\mathbf{k}}\|$ and

$$g(\|\mathbf{b}_k\|) = \frac{1+2\varepsilon}{1+\varepsilon}\sqrt{2\log\theta^{Nk-l+1}} \quad \Big(\text{or } \frac{1+2\varepsilon}{1+\varepsilon}\sqrt{2\log\theta^{Nk-l+1}}\sqrt{1+2N\log 3}\cdot 2^{n/2}\Big).$$

Considering the right hand side of (2.21) and (B) of Proposition 2.2, it follows by the same way as in the proof of (2.1) that

$$\sum_{k=1}^{\infty}\sum_{l<K}\frac{1+\varepsilon}{(1+2\varepsilon)\sqrt{2\log\theta^{Nk-l+1}}}\exp\Big(-\frac{1}{2}\Big(\frac{1+2\varepsilon}{1+\varepsilon}\Big)^2 2\log\theta^{Nk-l+1}\Big)$$
$$\le c\sum_{k=1}^{\infty}\exp\big(-(1+\varepsilon')\log\theta^{2(\log\log\theta^k)/(\log\theta)^2}\big)$$

$$= c\sum_{k=1}^{\infty}\exp\big(\log(\log\theta^k)^{-2(1+\varepsilon')/\log\theta}\big)$$
$$\le c\sum_{k=1}^{\infty}k^{-1-\varepsilon'}<\infty,$$

where $\varepsilon'=\varepsilon/(1+\varepsilon)$ and

$$\sum_{k=1}^{\infty}\sum_{l<K}\sum_{n=1}^{\infty}2^{2N2^n}\exp\Big(-\frac{1}{2}\Big(\frac{1+2\varepsilon}{1+\varepsilon}\Big)^2 2\,\log\theta^{Nk-l+1}(1+2N\log 3)\,2^n\Big)$$
$$\le\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}2^{2N2^n}\exp\Big(-(1+\varepsilon')\log\theta^{2(\log\log\theta^k)/(\log\theta)^2}\cdot(1+2N\log 3)\,2^n\Big)$$
$$\le c\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}2^{2N2^n}2^{-(\log_2 k)(1+2N)2^n}$$
$$= c\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}2^{-(\log_2 k)2^n}\cdot 2^{2N2^n(1-\log_2 k)}$$
$$\le c\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}k^{-1-\varepsilon'}2^{-n}<\infty.$$

Therefore, by (2.21) and Proposition 2.2, we obtain

$$\sum_{k=1}^{\infty}P\Big\{\sup_{l<K}\sup_{\|\mathbf{s}\|\le\|\theta^{\mathbf{k}}\|}\sup_{\|\theta^{\mathbf{k-1}}\|\le\|\mathbf{t}\|\le\|\theta^{\mathbf{k}}\|}\frac{\|\mathbb{X}(\mathbf{s}+\theta^{\mathbf{k}})-\mathbb{X}(\mathbf{s}+\mathbf{t})\|_\infty}{\sigma_*(\|\theta^{\mathbf{k}}-\theta^{\mathbf{k-1}}\|)\sqrt{2\log\theta^{Nk-l+1}}}>1+2\varepsilon\Big\}<\infty,$$

which gives (2.20). Combining (2.16) with (2.19) yields (2.14) via (2.15) since ε is arbitrary.

3. Appendix: Proof of Proposition 2.3

First we prove an auxiliary result for the strictly stationary l^∞-valued random field of Proposition 2.3.

Lemma 3.1. *Let $\mathbb{D}$ be a compact subset of $\mathbb{R}^N$ with Euclidean norm $\|\cdot\|$ and let $\{Z_k(\mathbf{t});\, \mathbf{t}\in\mathbb{D}\}_{k=1}^{\infty}$ be a sequence of separable and centered strictly stationary random fields with $Z_k(\mathbf{0})=0$. Assume that $\{\mathbf{U}(\mathbf{t}) := (Z_1(\mathbf{t}), Z_2(\mathbf{t}),\cdots)\,;\, \mathbf{t}\in\mathbb{D}\}$ is a centered strictly stationary l^{∞}-valued random field with l^{∞}-norm $\|\cdot\|_{\infty}$. Suppose that*

$$\begin{aligned} 0<\ \Gamma_k &:= \sup_{\mathbf{t}\in\mathbb{D}} \sqrt{E(Z_k(\mathbf{t}))^2} < \infty, \quad \Gamma := \sup_{k\ge 1}\Gamma_k, \\ \sigma_k^2(\|\mathbf{t}-\mathbf{s}\|) &:= E\{Z_k(\mathbf{t})-Z_k(\mathbf{s})\}^2 \\ &\le \varphi_k^2(\|\mathbf{t}-\mathbf{s}\|), \quad \sigma_*(t)=\sup_{k\ge1}\sigma_k(t), \varphi_*(t)=\sup_{k\ge1}\varphi_k(t), \end{aligned}$$

where $\sigma_k(t)$ and $\varphi_k(t)$ are positive nondecreasing and continuous functions of $t>0$. Then, for $\lambda>0$ and $K>(2\sqrt{2}+2)\sqrt{1+N\log 3}$, there exists a positive constant c such that

$$\begin{aligned} &P\Big\{\sup_{\mathbf{t}\in\mathbb{D}} \|\mathbf{U}(\mathbf{t})\|_{\infty} \ge x\Big(\Gamma + K\textstyle\int_0^{\infty}\varphi_*(\lambda 2^{-y^2})dy\Big)\Big\} \\ &\le c\tfrac{m(\mathbb{D})}{\lambda^N}\Big(P\Big\{\tfrac{\|\mathbf{U}(\mathbf{t})\|_{\infty}}{\sigma_*(\|\mathbf{t}\|)}\ge x\Big\} \\ &\quad + \textstyle\sum_{n=1}^{\infty} 2^{N2^n} P\Big\{\tfrac{\|\mathbf{U}(\mathbf{t})\|_{\infty}}{\sigma_*(\|\mathbf{t}\|)} \ge x\sqrt{1+N\log 3}\cdot 2^{n/2}\Big\}\Big) \end{aligned} \tag{3.1}$$

for any $x\ge 1$, where $m(\mathbb{D})$ is the Lebesgue measure of $\mathbb{D}$.

Proof. For each $n=0,1,2,\cdots$, put $\varepsilon_n=\lambda\, 2^{-2^n}$, $\lambda>0$. Denote a diameter of any subset G of $\mathbb{D}$ by $d(G)$. Let $\{S_i^{(n)}\,;\ i=1,2,\cdots,N_{\varepsilon_n}(\mathbb{D})\}$ be a minimal ε_n-net of $\mathbb{D}$, where

$$N_{\varepsilon_n}(\mathbb{D}) = \min\{l;\, \mathbb{D}\subset \cup_{i=1}^{l} S_i^{(n)},\, d(S_i^{(n)})\le\varepsilon_n\}.$$

Then there is a positive constant c such that

$$N_{\varepsilon_n}(\mathbb{D}) \le c\,\frac{m(\mathbb{D})}{(\varepsilon_n)^N}. \tag{3.2}$$

For each $S_i^{(n)}$, choose one point $\mathbf{t}_i^{(n)}\in S_i^{(n)}\cap\mathbb{D}$ and put $\Delta_n=\cup_{i=1}^{N_{\varepsilon_n}(\mathbb{D})}\{\mathbf{t}_i^{(n)}\}$. Let $K_1>\sqrt{1+N\log 3}$ and $K=(2\sqrt{2}+2)K_1$. For $x\ge 1$, set

$$x_k = xK_1\,\varphi_*(\varepsilon_{k-1})2^{k/2}, \quad k\ge 1.$$

Letting $\delta_k=2^{(k-1)/2}$ for $k\ge 0$, it is clear that $2^{k/2}=(2\sqrt{2}+2)(\delta_k-\delta_{k-1})$. Thus

$$\sum_{k=1}^{\infty} x_k = xK\sum_{k=1}^{\infty}\varphi_*(\lambda\, 2^{-\delta_k{}^2})(\delta_k-\delta_{k-1}) \le xK\int_0^{\infty}\varphi_*(\lambda\, 2^{-y^2})dy.$$

Therefore

$$
\begin{aligned}
P\Big\{\sup_{\mathbf{t}\in\mathbb{D}}\|\mathbf{U}(\mathbf{t})\|_\infty &\ge x\Big(\Gamma + K\int_0^\infty \varphi_*(\lambda\, 2^{-y^2})dy\Big)\Big\} \\
&\le P\Big\{\sup_{\mathbf{t}\in\mathbb{D}}\ \|\mathbf{U}(\mathbf{t})\|_\infty \ge x\Gamma + \sum_{k=1}^\infty x_k\Big\} \\
&\le \lim_{n\to\infty} P\Big\{\sup_{\mathbf{t}\in\Delta_n}\ \|\mathbf{U}(\mathbf{t})\|_\infty \ge x\Gamma + \sum_{k=1}^n x_k\Big\}.
\end{aligned}
\tag{3.3}
$$

Put

$$
\begin{aligned}
B_0 &= \Big\{\sup_{\mathbf{t}\in\Delta_0}\ \|\mathbf{U}(\mathbf{t})\|_\infty \ge x\Gamma\Big\}, \\
B_n &= \Big\{\sup_{\mathbf{t}\in\Delta_n}\ \|\mathbf{U}(\mathbf{t})\|_\infty \ge x\Gamma + \sum_{k=1}^n x_k\Big\}, \quad n\ge 1.
\end{aligned}
$$

By induction, we have

$$
P(B_n) \le P(B_{n-1}) + P(B_n\cap B_{n-1}^c) \le P(B_0) + \sum_{n=1}^\infty P(B_n\cap B_{n-1}^c). \tag{3.4}
$$

If B_n occurs, then one can find $\mathbf{t}\in\Delta_n$ such that $\|\mathbf{U}(\mathbf{t})\|_\infty \ge x\Gamma + \sum_{k=1}^n x_k$. Since Δ_{n-1} covers $\mathbb{D}$, there is $S_{i_0}^{(n-1)}$ which contains $\mathbf{t}$. For this $S_{i_0}^{(n-1)}$, we choose a point $\mathbf{s}\in\Delta_{n-1}$. If, in addition, B_{n-1}^c occurs, then $\|\mathbf{U}(\mathbf{s})\|_\infty < x\Gamma + \sum_{k=1}^{n-1} x_n$ and hence we have $\|\mathbf{U}(\mathbf{t})-\mathbf{U}(\mathbf{s})\|_\infty \ge x_n$. These facts yield, for $n\ge 1$,

$$
P(B_n\cap B_{n-1}^c) \le \sum_{\substack{\mathbf{t}\in\Delta_n,\,\mathbf{s}\in\Delta_{n-1}\\ 0<\|\mathbf{t}-\mathbf{s}\|\le\varepsilon_{n-1}}} P\big\{\|\mathbf{U}(\mathbf{t})-\mathbf{U}(\mathbf{s})\|_\infty \ge x_n\big\}.
$$

Therefore, it follows from (3.2) and the stationarity of $\mathbf{U}(\mathbf{t})$ that

$$
\begin{aligned}
\sum_{n=1}^\infty P(B_n\cap B_{n-1}^c) &\le \sum_{n=1}^\infty \sum_{\substack{\mathbf{t}\in\Delta_n,\,\mathbf{s}\in\Delta_{n-1}\\ 0<\|\mathbf{t}-\mathbf{s}\|\le\varepsilon_{n-1}}} P\Big\{\|\mathbf{U}(\mathbf{t})-\mathbf{U}(\mathbf{s})\|_\infty \ge x_n\Big\} \\
&\le c\sum_{n=1}^\infty \frac{m(\mathbb{D})}{(\varepsilon_n)^N} P\Big\{\frac{\|\mathbf{U}(\mathbf{t})-\mathbf{U}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{t}-\mathbf{s}\|)} \ge \frac{x_n}{\varphi_*(\,\varepsilon_{n-1})}\Big\} \\
&\le c\,\frac{m(\mathbb{D})}{\lambda^N}\sum_{n=1}^\infty 2^{N2^n} P\Big\{\frac{\|\mathbf{U}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{t}\|)} \ge xK_1\, 2^{n/2}\Big\} \\
&\le c\,\frac{m(\mathbb{D})}{\lambda^N}\sum_{n=1}^\infty 2^{N2^n} P\Big\{\frac{\|\mathbf{U}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{t}\|)} \ge x\sqrt{1+N\log 3}\cdot 2^{n/2}\Big\}.
\end{aligned}
\tag{3.5}
$$

On the other hand,

$$\text{(3.6)}\quad P(B_0) \le c\,\frac{m(\mathbb{D})}{(\varepsilon_0)^N} P\Big\{\frac{\|\mathbf{U}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{t}\|)} \ge \frac{x\Gamma}{\sigma_*(\|\mathbf{t}\|)}\Big\} \le c\,\frac{m(\mathbb{D})}{\lambda^N} P\Big\{\frac{\|\mathbf{U}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{t}\|)} \ge x\Big\}.$$

Therefore, the inequality (3.1) follows from (3.3)-(3.6).

Proof of Proposition 2.3. Let $\mathbb{D}_T^2 = \{(\mathbf{s},\mathbf{t})\,;\, \|\mathbf{s}\| \le \|\mathbf{b}_T\|,\ \ \|\mathbf{t}\| \le \|\mathbf{b}_T\|\}$. In order to apply Lemma 3.1, we set

$$Z_i(\mathbf{s},\mathbf{t}) = \frac{X_i(\mathbf{s}+\mathbf{t}) - X_i(\mathbf{s})}{\sigma_*(\|\mathbf{b}_T\|)}, \quad i \ge 1$$

and

$$\varphi_i(z) = \frac{2\sigma_i(\sqrt{2}z)}{\sigma_*(\|\mathbf{b}_T\|)}\ , \quad z > 0.$$

Clearly,

$$E\{Z_i(\mathbf{s},\mathbf{t})\} = 0 \quad \text{and} \quad \Gamma = 1.$$

Letting $\mathbf{p} = (\mathbf{s}',\mathbf{t}')$ and $\mathbf{q} = (\mathbf{s}'',\mathbf{t}'')$, it follows that

$$\begin{aligned}
&E\{Z_i(\mathbf{p}) - Z_i(\mathbf{q})\}^2 \\
&= \frac{1}{\sigma_*^2(\|\mathbf{b}_T\|)} E\big\{(X_i(\mathbf{s}'+\mathbf{t}') - X_i(\mathbf{s}''+\mathbf{t}'')) - (X_i(\mathbf{s}') - X_i(\mathbf{s}''))\big\}^2 \\
&\le \frac{2}{\sigma_*^2(\|\mathbf{b}_T\|)} \big\{\sigma_i^2(\|(\mathbf{s}'+\mathbf{t}') - (\mathbf{s}''+\mathbf{t}'')\|) + \sigma_i^2(\|\mathbf{s}'-\mathbf{s}''\|)\big\} \\
&\le \frac{4}{\sigma_*^2(\|\mathbf{b}_T\|)} \sigma_i^2\big(\sqrt{2}\sqrt{\|\mathbf{s}'-\mathbf{s}''\|^2 + \|\mathbf{t}'-\mathbf{t}''\|^2}\,\big) \\
&= \varphi_i^2\big(\|\mathbf{p}-\mathbf{q}\|\big).
\end{aligned}$$

On the other hand, for any $\varepsilon > 0$, there exists a small constant $C_\varepsilon > 0$ such that

$$\text{(3.7)}\qquad K\int_0^\infty \varphi_*(\sqrt{2}\,C_\varepsilon\,\|\mathbf{b}_T\|\,2^{-y^2})\,dy < \varepsilon,$$

where $K > (2\sqrt{2}+2)\sqrt{1+2N\log 3}$ as in Lemma 3.1 and we take $\lambda = \sqrt{2}\,C_\varepsilon\,\|\mathbf{b}_T\|$. Indeed, using the regularity of $\sigma_*(\cdot)$, we have

$$\begin{aligned}
K\int_0^\infty \varphi_*(\sqrt{2}\,C_\varepsilon\,\|\mathbf{b}_T\|\,2^{-y^2})\,dy &= 2K\int_0^\infty \frac{\sigma_*(2\,C_\varepsilon\,\|\mathbf{b}_T\|\,2^{-y^2})}{\sigma_*(\|\mathbf{b}_T\|)}\,dy \\
&\le 2K\,(2\,C_\varepsilon)^{\alpha/2}\int_0^\infty 2^{-\alpha y^2/2}\,dy < \varepsilon,
\end{aligned}$$

provided C_ε is small enough. Put $v = x(1+\varepsilon),\ x \ge 1$, and set $\big\{\mathbf{U}(\mathbf{s},\mathbf{t}) := \big(Z_1(\mathbf{s},\mathbf{t}), Z_2(\mathbf{s},\mathbf{t}),\cdots\big)\,;\ (\mathbf{s},\mathbf{t}) \in \mathbb{D}_T^2\big\}$. Then $\{\mathbf{U}(\mathbf{s},\mathbf{t})\}$ is a centered strictly stationary l^∞-valued random field with $2N$ parameters. It follows from (3.7) and Lemma 3.1 that, for $v > 1$,

$$
\begin{aligned}
&P\Big\{\sup_{\|\mathbf{s}\|\le\|\mathbf{b}_T\|}\sup_{\|\mathbf{t}\|\le\|\mathbf{b}_T\|}\frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)}\ge v\Big\}\\
&\le P\Big\{\sup_{(\mathbf{s},\mathbf{t})\in\mathbb{D}_T^2}\|\mathbf{U}(\mathbf{s},\mathbf{t})\|_\infty\ge x\Big(\Gamma+K\int_0^\infty\varphi_*(\sqrt{2}\,C_\varepsilon\|\mathbf{b}_T\|\,2^{-y^2})dy\Big)\Big\}\\
&\le c\,\frac{\|\mathbf{b}_T\|^{2N}}{(\sqrt{2}\,C_\varepsilon\|\mathbf{b}_T\|)^{2N}}\Big(P\Big\{\frac{\|\mathbf{U}(\mathbf{s},\mathbf{t})\|_\infty}{\sigma_*(\|(\mathbf{s},\mathbf{t})\|)}\ge\frac{v}{1+\varepsilon}\Big\}\\
&\quad+\sum_{n=1}^\infty 2^{2N2^n}P\Big\{\frac{\|\mathbf{U}(\mathbf{s},\mathbf{t})\|_\infty}{\sigma_*(\|(\mathbf{s},\mathbf{t})\|)}\ge\frac{v}{1+\varepsilon}\sqrt{1+2N\log 3}\cdot 2^{n/2}\Big\}\Big)\\
&\le c_\varepsilon\Big(P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_T)\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)}\ge\frac{v}{1+\varepsilon}\Big\}+\sum_{n=1}^\infty 2^{2N2^n}P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_T)\|_\infty}{\sigma_*(\|\mathbf{b}_T\|)}\\
&\ge\frac{v}{1+\varepsilon}\sqrt{1+2N\log 3}\cdot 2^{n/2}\Big\}\Big).
\end{aligned}
$$

This completes the proof of Proposition 2.3.

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