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ON SHARP LOWER BOUND OF THE GAP FOR THE FIRST TWO EIGENVALUES IN THE SCHRÖDINGER OPERATOR

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Abstract. [8] is a deep study in the sharp lower bound estimate of the gap for the first two eigenvalues in Schrödinger operator on a smooth bounded convex domain in $\mathbb{R}^n$. In this paper we give another simple proof of the main result in [8]. Although the methods used in here due to [8] on the whole, to some extent we deal with the singularity of some function and also simplify greatly calculation in [8].

1. Introduction

The study in the estimates of the eigenvalues has a long history. Meanwhile, there are many works in this field. Among these works, Li-Yau's results (e.g., [2-6]) and Zhong's results (e.g., [7, 8]) are all very well known. In 1979, the maximum principle method was used by Li [2] in proving eigenvalue estimates for compact manifolds. To be more precise, the maximum principle is used to deduce the gradient estimate on the eigenfunction, and then the eigenvalue estimates is obtained via the above gradient estimate. This method was then refined and used by many authors ([3, 7], etc) for obtaining sharp eigenvalue estimates.

Now we state some results on the lower bound estimate of the gap for the first two eigenvalues in Schrödinger operator on a smooth bounded convex domain in $\mathbb{R}^n$.

In 1985, Yau and others [6] use the method of the gradient estimate to deduce estimates on $\lambda_2 - \lambda_1$. one of the main results in [6] is the following theorem.

Theorem 1.1. (Singer-Wong-Yau-Yau). *Let Ω be a smooth strictly convex bounded domain in $\mathbb{R}^n$ and $W : \bar{\Omega} \mapsto \mathbb{R}$ a nonnegative convex function. Suppose λ_1 and λ_2 are*

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the first and second nonzero eigenvalues of (2.1)*, then the following pinch inequality holds*

$$\frac{\pi^2}{4d^2} \leqslant \lambda_2 - \lambda_1 \leqslant \frac{4n\pi^2}{D^2} + \frac{4(M-m)}{n}, \tag{1.1}$$

where d is the diameter of Ω, $D=$ the diameter of the largest inscribed ball in Ω,

$$M = \sup_{\bar\Omega} W \qquad \textit{and} \qquad m = \inf_{\bar\Omega} W.$$

Later in 1986, the above estimate of lower bound was improved by Yu-Zhong in [8] via following the similar techniques as [6] and [7] to $\lambda_2 - \lambda_1 \geqslant \frac{\pi^2}{d^2}$. Yu-Zhong's result was thought to be the sharp estimate of lower bound of the gap for the first two eigenvalues under the above assumption. Next we state Yu-Zhong's result asserted in [8] as follows.

Theorem 1.2. $(\mathbf{Yu-Zhong})$. *Let Ω be a smooth strictly convex bounded domain in $\mathbb{R}^n$ and $W : \bar\Omega \mapsto \mathbb{R}$ a nonnegative convex function. Suppose λ_1 and λ_2 are the first and second nonzero eigenvalues of* (2.1)*. Then*

$$\lambda_2 - \lambda_1 \geqslant \frac{\pi^2}{d^2}, \tag{1.2}$$

where d is the diameter of Ω.

In present paper we give another simple proof of Theorem 1.2. Our argument is based on many early works, e.g., Li-Yau [5], Singer-Wong-Yau-Yau [6] and Yu-Zhong [8]. One feature of this argument is that it avoids various kinds of trouble from the singularity of $|\nabla u|^2/(1-u^2)$ in those papers. Although our argument many ways analogous to [8], we can readily handle the above singularity, and reduce the difficulty in calculation to a certain extent.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a smooth strictly convex bounded domain and $W : \bar\Omega \mapsto \mathbb{R}$ a nonnegative convex smooth function. Consider the following Dirichlet eigenvalue problem of Schrödinger equation

$$\begin{cases} -\Delta f + Wf = \lambda f \qquad \text{in} \quad \Omega, \\ f = 0 \qquad \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

According to a result of [12], the eigenvalues of the Dirichlet problem (2.1) can be arranged in nondecreasing order as follows

$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots .$$

Let f_1 and f_2 be the first and second eigenfunctions of (2.1), respectively. It is well known that (e.g. [12]) $f_1(x) > 0$, $x \in \Omega$, and $u = f_2/f_1$ is smooth to the boundary $\partial\Omega$ of Ω (e.g. [6], p.331). Direct computation implies that

$$\Delta u + \lambda u + 2\nabla u \cdot \nabla(\log f_1) = 0 \qquad \text{in} \quad \Omega, \tag{2.2}$$

where $\lambda = \lambda_2 - \lambda_1 > 0$. Clearly, $\log f_1$ is well-defined since $f_1 > 0$ on Ω. Without loss of generality, we may assume that

$$\max u = 1, \qquad \min u = -k \qquad \text{and} \qquad 0 < k \leqslant 1.$$

Set

$$\begin{cases} \tilde{u} = (u - \frac{1-k}{2})/\frac{1+k}{2} \\ a = \frac{1-k}{1+k}, \quad 0 \leqslant a < 1. \end{cases}$$

So (2.2) can be rewitten as follows

$$\begin{cases} \Delta\tilde{u} + \lambda(\tilde{u} + a) + 2\nabla\tilde{u} \cdot \nabla \log f_1 = 0 \qquad \text{in} \quad \Omega, \\ \max \tilde{u} = 1, \qquad \min \tilde{u} = -1. \end{cases} \tag{2.3}$$

Now if we set $\theta = \arcsin \tilde{u}$, Then $\tilde{u} = \sin\theta$ and $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$. Define a subset of $\bar{\Omega}$ as follows

$$\Sigma_* = \{x \in \bar{\Omega} \ : \ \theta(x) = \frac{\pi}{2} \quad \text{or} \quad \theta(x) = -\frac{\pi}{2}\}.$$

By (2.3), a straight forward calculation shows that θ satisfies

$$\cos\theta \cdot \Delta\theta - \sin\theta \cdot |\nabla\theta|^2 + \lambda(\sin\theta + a) + 2\cos\theta \cdot \nabla\theta \cdot \nabla \log f_1 = 0 \qquad \text{in} \quad \Omega. \tag{2.4}$$

In particular,

$$\Delta\theta = \frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 - \frac{\lambda(\sin\theta + a)}{\cos\theta} - 2\nabla\theta \cdot \nabla \log f_1 \qquad \text{in} \quad \Omega \setminus \Sigma_*. \tag{2.5}$$

From (2.4), we know that

$$|\nabla\theta|^2 = \lambda(1 - a) \qquad \text{as} \quad \theta = -\frac{\pi}{2}, \tag{2.6}$$

and

$$|\nabla\theta|^2 = \lambda(1 + a) \qquad \text{as} \quad \theta = \frac{\pi}{2}. \tag{2.7}$$

We also define a function $F : (-\frac{\pi}{2}, \frac{\pi}{2}) \mapsto \mathbb{R}$ as follows

$$F(\theta_0) = \max_{x\in\Omega,\, \theta(x)=\theta_0} |\nabla\theta(x)|^2 \,, \quad \forall\ \theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}). \tag{2.8}$$

Obviously, F is well-defined. Actually, $F(\theta_0)$ is not something but an extreme value of f with condition $\theta(x) = \theta_0$. It is very easy to verify that $F(\theta)$ is continuous in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Moreover, by (2.6) and (2.7), if we define

$$F(-\frac{\pi}{2}) = F(-\frac{\pi}{2} + 0) = \lambda(1-a),$$

and

$$F(\frac{\pi}{2}) = F(\frac{\pi}{2} - 0) = \lambda(1+a),$$

then $F(\theta)$ can be extended a continuous function on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

3. A Rough Estimate of $|\nabla\theta|^2$

Firstly, in a similar way owing to [6, 8] and [10], we get the following lemma.

Lemma 3.1. *Let $g(\theta)$ be a smooth function defined on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Assume that*

$$G(x) = |\nabla\theta|^2 + g\big(\theta(x)\big)$$

arrives on its maximum at $p \in \partial\Omega \setminus \Sigma_*$. Then $\nabla\theta(p) = 0$.

Proof. We pick an orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ around p such that e_1 is the unit normal of $\partial\Omega$ pointing outward to Ω. We also denote below by $\frac{\partial}{\partial x_1}$ the restriction on $\partial\Omega$ of the directional derivative corresponding to e_1. By the maximality of $G(x)$ at p, we also have

$$0 \leqslant \frac{\partial G(p)}{\partial x_1} = 2\sum_{i=1}^{n} \theta_i(p)\cdot\theta_{i1}(p) + g'\big(\theta(p)\big)\cdot\theta_1(p) \tag{3.1}$$

In addition, we know by (2.5) that

$$\nabla\theta\cdot\nabla\log f_1 = \frac{\nabla\theta\cdot\nabla f_1}{f_1} = \frac{1}{f_1}\sum_{i=1}^{n}\theta_i\cdot(f_1)_i$$

achieves finite value on $\partial\Omega \setminus \Sigma_*$. But $f_1 = 0$ on $\partial\Omega$, thus

$$\theta_1\cdot(f_1)_1 + \sum_{i=2}^{n}\theta_i\cdot(f_1)_i = \sum_{i=1}^{n}\theta_i\cdot(f_1)_i = 0 \qquad \text{on} \quad \partial\Omega\setminus\Sigma_*. \tag{3.2}$$

Since $f_1 \equiv 0$ on $\partial\Omega$ and e_i $(2 \leqslant i \leqslant n)$ are all the tangent vectors of $\partial\Omega$,

$$(f_1)_i\,|_{\partial\Omega} = 0 \qquad \text{for all} \quad 2\leqslant i\leqslant n.$$

Hence, (3.2) can be reduced to

$$\theta_1 \cdot (f_1)_1 = 0 \qquad \text{on} \quad \partial\Omega \setminus \Sigma_*.$$

However, Hopf's lemma asserts that $(f_1)_1(p) = \frac{\partial f_1(p)}{\partial x_1} \neq 0$. Therefore,

$$\theta_1(p) = 0. \tag{3.3}$$

Putting (3.3) into (3.1), we then have

$$0 \leqslant \frac{\partial G(p)}{\partial x_1} = 2\sum_{i=2}^{n} \theta_i(p) \cdot \theta_{i1}(p) \tag{3.4}$$

Note $\theta_1(p) = 0$ and recall the definition of second fundamental form of a hypersurface in $\mathbb{R}^n$, one can derive

$$\theta_{i1} = -\sum_{j=2}^{n} h_{ij}\theta_j, \tag{3.5}$$

where $(h_{ij})_{2\leqslant i,j\leqslant n}$ is the second fundamental form of $\partial\Omega$ relative to e_1.

It is known that Ω is strictly convex if and only if $(h_{ij})_{2\leqslant i,j\leqslant n}$ is positive definite. Putting (3.5) into (3.4), we thus have

$$0 \leqslant \frac{\partial G(p)}{\partial x_1} = -2\sum_{i,j=2}^{n} \theta_i(p) h_{ij}(p)\theta_j(p) \leqslant 0$$

Hence, $\theta_i(p) = 0$, $2 \leqslant i \leqslant n$. By (3.3) again, we have $\nabla\theta(p) = 0$. ■

As [7] points out that the estimate of the upper bound of $|\nabla\theta|^2$ plays an important role in the estimate of the lower bound for $\lambda = \lambda_2 - \lambda_1$. In the following we establish a rough estimate for $|\nabla\theta|^2$.

Lemma 3.2. (*see [7]*) *The following estimate is valid*

$$|\nabla\theta(x)|^2 \leqslant \lambda(1+a), \qquad \forall\ x \in \Omega. \tag{3.6}$$

Moreover,

$$F(\theta) \leqslant \lambda(1+a). \tag{3.7}$$

Proof. Suppose that $|\nabla\theta|^2$ attains its local maximum at $x_0 \in \Omega$. Clearly, (2.6) and (2.7) imply that (3.6) holds in the case: $x_0 \in \Sigma_*$. Without loss of generality, we may assume further that $x_0 \in \bar{\Omega} \setminus \Sigma_*$. Thus $\theta_0 = \theta(x_0) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We easily know from Lemma 3.1 that $\nabla\theta(x_0) = 0$ if $x_0 \in \partial\Omega \setminus \Sigma_*$, . Obviously, the conclusion is valid in this case. So, we suppose that $x_0 \in \Omega \setminus \Sigma_*$ in the rest of the proof. One compute easily that

$$\frac{1}{2}\,(|\nabla\theta|^2)_j = \frac{1}{2}\sum_i(\theta_i^2)_j = \sum_i \theta_i\theta_{ij},$$

and

$$\begin{aligned} \frac{1}{2}\,\Delta(|\nabla\theta|^2) &= \frac{1}{2}\,\sum_j(|\nabla\theta|^2)_{jj} = \sum_j \big(\sum_i \theta_i\theta_{ij}\big)_j \\ &= \sum_{i,j}\big(\theta_{ij}^2 + \theta_i\theta_{ijj}\big) = \left|\nabla^2\theta\right|^2 + \nabla\theta\cdot\nabla(\Delta\theta). \end{aligned} \tag{3.8}$$

Putting (2.5) into (3.8), we have

$$\begin{aligned} \frac{1}{2}\,\Delta(|\nabla\theta|^2) &= \left|\nabla^2\theta\right|^2 + \nabla\theta\cdot\nabla\big[\frac{\sin\theta}{\cos\theta}\cdot|\nabla\theta|^2 - \frac{\lambda(\sin\theta+a)}{\cos\theta} \\ &\quad -2\nabla\theta\cdot\nabla\log f_1\big] \\ &= \left|\nabla^2\theta\right|^2 + \nabla\theta\cdot\nabla\big(\frac{\sin\theta}{\cos\theta}\big)\cdot|\nabla\theta|^2 + \nabla\theta\cdot\frac{\sin\theta}{\cos\theta}\cdot\nabla(|\nabla\theta|^2) \\ &\quad -\lambda\cdot\nabla\theta\cdot\big[\nabla\big(\frac{\sin\theta}{\cos\theta}\big) + a\cdot\nabla\big(\frac{1}{\cos\theta}\big)\big] \\ &\quad -2(\nabla\theta\cdot\nabla^2\theta)\cdot\nabla\log f_1 - 2\nabla\theta\cdot(\nabla\theta\cdot\nabla^2\log f_1). \end{aligned} \tag{3.9}$$

A direct calculation leads to that

$$\nabla\big(\frac{\sin\theta}{\cos\theta}\big) = \frac{\nabla(\sin\theta)\cdot\cos\theta - \sin\theta\cdot\nabla(\cos\theta)}{\cos^2\theta} = \frac{\nabla\theta}{\cos^2\theta}, \tag{3.10}$$

and

$$\nabla\big(\frac{1}{\cos\theta}\big) = \frac{-1}{\cos^2\theta}\cdot(-\sin\theta)\cdot\nabla\theta = \frac{\sin\theta\cdot\nabla\theta}{\cos^2\theta}. \tag{3.11}$$

Putting (3.10) and (3.11) into (3.9), we obtain

$$\begin{aligned} \frac{1}{2}\,\Delta(|\nabla\theta|^2) &= \left|\nabla^2\theta\right|^2 + \frac{|\nabla\theta|^4}{\cos^2\theta} + \nabla\theta\cdot\frac{\sin\theta}{\cos\theta}\cdot\nabla(|\nabla\theta|^2) \\ &\quad -\lambda\cdot|\nabla\theta|^2\cdot\frac{1+a\sin\theta}{\cos^2\theta} - \nabla(|\nabla\theta|^2)\cdot\nabla\log f_1 \\ &\quad -2\nabla\theta\cdot(\nabla\theta\cdot\nabla^2\log f_1). \end{aligned} \tag{3.12}$$

Since W and Ω are all convex, $\log f_1$ is concave by assumption, according to a result of Brascamp and Liep [11], $\log f_1$ is concave, i. e., $(\nabla^2\log f_1)$ is non-positive definite. Thus

$$\nabla\theta\cdot(\nabla\theta\cdot\nabla^2\log f_1) \leqslant 0. \tag{3.13}$$

In addition, according to maximum principle, we easily know that at x_0

$$\nabla(|\nabla\theta|^2) = 0 \qquad \text{and} \qquad \Delta(|\nabla\theta|^2) \leqslant 0, \tag{3.14}$$

Noticing that (3.13) and (3.14), we deduce from (3.12) that at x_0

$$0 \geqslant \frac{|\nabla\theta|^4}{\cos^2\theta} - \lambda \cdot |\nabla\theta|^2 \cdot \frac{1 + a\sin\theta}{\cos^2\theta}.$$

Dividing by $|\nabla\theta|^2$ and multiplying by $\cos^2\theta$ successively, it follows that at x_0

$$0 \geqslant |\nabla\theta|^2 - \lambda(1 + a\sin\theta).$$

Hence we have

$$|\nabla\theta(x_0)|^2 \leqslant \lambda(1 + a\sin\theta_0) \leqslant \lambda(1 + a).$$

The proof is complete. ■

4. The Estimate of $F(\theta)$

In the sequel, without loss of generality, we may assume $0 < a < 1$. In fact, $F(\theta) \leqslant \lambda$ is the best estimate of F when $a = 0$. What we want now is get a more precise estimate on $F(\theta)$ than Lemma 3.2. For this purpose, let us introduce the function $\phi(\theta) : \Omega \mapsto \mathbb{R}$ such that

$$F(\theta) = \lambda[1 + a\phi(\theta)]. \tag{4.1}$$

By Lemma 3.2, it is also easy to see that $\phi(\theta) \leqslant 1$. We shall also need the following technique lemma to estimate accurately $\phi(\theta)$.

Lemma 4.1. *(see [7]) Assume that $h : [-\frac{\pi}{2}, -\frac{\pi}{2}] \mapsto \mathbb{R}$ is a nondecreasing function, i.e., $h'(\theta) \geqslant 0$, and satisfies*

(1) $h(\theta) \geqslant \phi(\theta)$,

(2) *there exists some $\theta_0 \in (-\frac{\pi}{2}, -\frac{\pi}{2})$, such that $h(\theta_0) = \phi(\theta_0) \geqslant -1$.*

Then the following estimate holds

$$\phi(\theta_0) \leqslant \sin\theta_0 - \sin\theta_0 \cdot \cos\theta_0 \cdot h'(\theta_0) + \frac{\cos^2\theta_0}{2} \cdot h''(\theta_0). \tag{4.2}$$

Proof. Set

$$\mathcal{E}(x) = \frac{1}{2}\Big\{ |\nabla\theta(x)|^2 - \lambda\big[1 + ah\big(\theta(x)\big)\big]\Big\}.$$

Obviously, $\mathcal{E}(x) \leqslant 0$ for all $x \in \bar{\Omega}$. By (2.8), we know that there exists some $x_0 \in \bar{\Omega} \setminus \Sigma_*$ such that $\theta(x_0) = \theta_0$ and $F(\theta_0) = |\nabla\theta(x_0)|^2$. Thus $\mathcal{E}(x)$ achieves its maximum 0 at x_0, i. e.,

$$|\nabla\theta(x_0)|^2 = \lambda[1 + a\phi(\theta_0)] = \lambda[1 + ah(\theta_0)]. \tag{4.3}$$

Obviously, it is easy to verify that the hypothesis of Lemma 3.1 is satisfied if $2\,\mathcal{E}(x)$ is being used in place of $G(x)$. By Lemma 3.1, if $x_0 \in \partial\Omega \setminus \Sigma_*$, then $\nabla\theta(x_0) = 0$. Since $0 < a < 1$ and $\phi(\theta_0) \geqslant -1$,

$$0 = \mathcal{E}(x_0) = \frac{1}{2}\Big\{ |\nabla\theta(x_0)|^2 - \lambda\big[1 + ah(\theta_0)\big]\Big\} \leqslant -\frac{\lambda(1-a)}{2} < 0.$$

But this is a contradiction. Hence, $x_0 \in \Omega \setminus \Sigma_*$. It is obvious that at x_0

$$\nabla\mathcal{E} = 0 \qquad \text{and} \qquad \Delta\mathcal{E} \leqslant 0, \tag{4.4}$$

by maximum principle again. Direct computation shows that

$$\mathcal{E}_j = \sum_i \theta_i \cdot \theta_{ij} - \frac{\lambda a}{2} h'(\theta) \cdot \theta_j,$$

namely,

$$\nabla\mathcal{E} = \frac{1}{2}\left[\nabla(|\nabla\theta|^2) - \lambda a h'(\theta) \cdot \nabla\theta\right] = \nabla\theta \cdot \nabla^2\theta - \frac{\lambda a}{2} h'(\theta) \cdot \nabla\theta.$$

Since $\nabla\mathcal{E} = 0$ at x_0,

$$\nabla(|\nabla\theta|^2) = 2\nabla\theta \cdot \nabla^2\theta = \lambda a h'(\theta_0) \cdot \nabla\theta \qquad \text{at} \quad x_0. \tag{4.5}$$

By directly calculating and applying (2.5), we also obtain

$$\begin{aligned} \frac{1}{2}\Delta\big[\lambda(1+ah)\big] &= \frac{1}{2}\sum_j \big[\lambda(1+ah)\big]_{jj} = \frac{\lambda a}{2}\sum_j \big(h' \cdot \theta_j\big)_j \\ &= \frac{\lambda a}{2}\sum_j (h'' \cdot \theta_j^2 + h' \cdot \theta_{jj}) = \frac{\lambda a}{2}(h'' \cdot |\nabla\theta|^2 + h' \cdot \Delta\theta) \\ &= \frac{\lambda a}{2}\Big\{h'' \cdot |\nabla\theta|^2 + h' \cdot \Big[\frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 - \frac{\lambda(\sin\theta + a)}{\cos\theta} \\ &\quad -2\nabla\theta \cdot \nabla\log f_1\Big]\Big\}. \end{aligned} \tag{4.6}$$

Combining (3.12) with (4.6), we hence obtain

$$\begin{aligned} \Delta\mathcal{E} \;=\; & \big|\nabla^2\theta\big|^2 + \frac{|\nabla\theta|^4}{\cos^2\theta} + \nabla\theta \cdot \frac{\sin\theta}{\cos\theta} \cdot \nabla(|\nabla\theta|^2) \\ & -\lambda \cdot |\nabla\theta|^2 \cdot \frac{1 + a\sin\theta}{\cos^2\theta} - 2(\nabla\theta \cdot \nabla^2\theta) \cdot \nabla\log f_1 \\ & -2\nabla\theta \cdot (\nabla\theta \cdot \nabla^2\log f_1) - \frac{\lambda a}{2}\Big\{h'' \cdot |\nabla\theta|^2 \\ & +h' \cdot \Big[\frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 - \frac{\lambda(\sin\theta + a)}{\cos\theta} - 2\nabla\theta \cdot \nabla\log f_1\Big]\Big\}. \end{aligned} \tag{4.7}$$

Inserting (4.5) into (4.7), it is easy to deduce that at x_0

$$\begin{aligned}\Delta\mathcal{E} =& \left|\nabla^2\theta\right|^2 + \frac{|\nabla\theta|^4}{\cos^2\theta} + \lambda a h' \cdot \frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 \\ & -\lambda \cdot |\nabla\theta|^2 \cdot \frac{1 + a\sin\theta}{\cos^2\theta} - \lambda a h' \cdot \nabla\theta \cdot \nabla \log f_1 \\ & -2\nabla\theta \cdot (\nabla\theta \cdot \nabla^2 \log f_1) - \frac{\lambda a}{2}\Big\{h'' \cdot |\nabla\theta|^2 \\ & + h' \cdot \big[\frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 - \frac{\lambda(\sin\theta + a)}{\cos\theta} - 2\nabla\theta \cdot \nabla \log f_1\big]\Big\}.\end{aligned}$$

Rearranging the terms, the above equality reduces to

$$\begin{aligned}\Delta\mathcal{E} =& \left|\nabla^2\theta\right|^2 + \frac{|\nabla\theta|^4}{\cos^2\theta} + \lambda a h' \cdot \frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 \\ & -\lambda \cdot |\nabla\theta|^2 \cdot \frac{1 + a\sin\theta}{\cos^2\theta} - 2\nabla\theta \cdot (\nabla\theta \cdot \nabla^2 \log f_1) \\ & -\frac{\lambda a}{2}\Big\{h'' \cdot |\nabla\theta|^2 + h' \cdot \big[\frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 - \frac{\lambda(\sin\theta + a)}{\cos\theta}\big]\Big\}.\end{aligned} \tag{4.8}$$

By virtue of (4.3) and (4.4), we derive from (4.8) that at x_0

$$\begin{aligned}0 \geqslant& \left|\nabla^2\theta\right|^2 + \frac{\lambda^2(1+ah)^2}{\cos^2\theta} + \lambda^2 a h'(1+ah)\frac{\sin\theta}{\cos\theta} \\ & -\lambda^2(1+ah)\frac{1 + a\sin\theta}{\cos^2\theta} - 2\nabla\theta \cdot (\nabla\theta \cdot \nabla^2 \log f_1) \\ & -\frac{\lambda a}{2}\Big\{h'' \cdot |\nabla\theta|^2 + h' \cdot \big[\frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 - \frac{\lambda(\sin\theta + a)}{\cos\theta}\big]\Big\} \\ =& \left|\nabla^2\theta\right|^2 + \frac{\lambda^2(1+ah)^2}{\cos^2\theta} - \lambda^2(1+ah)\frac{1 + a\sin\theta}{\cos^2\theta} \\ & -2\nabla\theta \cdot (\nabla\theta \cdot \nabla^2 \log f_1) + \frac{\lambda^2 a}{2}\Big\{ - h''(1+ah) \\ & + h' \cdot \big[\frac{\sin\theta}{\cos\theta}(1+ah) + \frac{(\sin\theta + a)}{\cos\theta}\big]\Big\}.\end{aligned} \tag{4.9}$$

Obviously, we easily know by (3.13) that the first term and the fourth term on the right-hand side can be dropped because they are nonnegative. After dividing by $\lambda^2 a$, multiplying by $\cos^2\theta$ and rearranging the terms successively, we are led to

$$
\begin{aligned}
0 \geqslant\ & \frac{(1+ah)^2}{a} - \frac{(1+ah)(1+a\sin\theta)}{a} - \frac{h''(1+ah)\cos^2\theta}{2} \\
& + \frac{h'(1+ah)\cos\theta\sin\theta}{2} + \frac{h'\cos\theta(\sin\theta+a)}{2} \\
=\ & (1+ah)(h-\sin\theta) - \frac{h''(1+ah)\cos^2\theta}{2} \\
& + \frac{h'\cos\theta}{2}[(1+ah)\sin\theta + (\sin\theta+a)].
\end{aligned} \tag{4.10}
$$

Since $h(\theta_0) = \phi(\theta_0) \geqslant -1$ and $\phi(\theta_0) = \phi\big(\theta(x_0)\big) \leqslant 1$, then $|h(\theta_0)| \leqslant 1$.

From $|h| = |h(\theta)| \leqslant 1$ at x_0, and $0 < a < 1$, it follows that at x_0

$$a \geqslant ah\sin\theta \qquad \text{and} \qquad 1 + ah > 0.$$

Thus, at x_0

$$\sin\theta + a \geqslant \sin\theta + ah\sin\theta = (1+ah)\sin\theta \tag{4.11}$$

Hence, under the assumption that $h'(\theta) \geqslant 0$, using (4.11), we proceed by tackling with (4.10) at x_0 as follows

$$0 \geqslant (1+ah)(h-\sin\theta) + h'(1+ah)\cos\theta\sin\theta - \frac{h''(1+ah)\cos^2\theta}{2}.$$

Dividing by $1 + ah$, we have at x_0

$$0 \geqslant (h-\sin\theta) + h'\cos\theta\sin\theta - \frac{h''\cos^2\theta}{2}. \tag{4.12}$$

Obviously, (4.2) follows from (4.12) immediately. ∎

The remaining part of the present paper works exactly as in [7] and [8]. For the completeness we briefly sketch a proof of Theorem 1.2 below which only use the methods owing to [7] and [8]. We refer the reader to consult [7], [8] and [10] for more details.

Lemma 4.2. (*see [7]*) *Define function* ψ *as follows*

$$
\begin{cases}
\psi(\theta) = \frac{\frac{4}{\pi}(\theta+\cos\theta\sin\theta)-2\sin\theta}{\cos^2\theta}, & \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\
\psi(-\frac{\pi}{2}) = -1, \qquad \psi(\frac{\pi}{2}) = 1. &
\end{cases} \tag{4.13}
$$

Then $\psi \in C^0[-\frac{\pi}{2}, \frac{\pi}{2}] \cap C^2(-\frac{\pi}{2}, \frac{\pi}{2})$, *satisfies* $\psi'(\theta) \geqslant 0$ *and*

$$\psi(\theta) - \sin\theta + \sin\theta\cdot\cos\theta\cdot\psi'(\theta) - \frac{\cos^2\theta}{2}\cdot\psi''(\theta) = 0.$$

Moreover, the following properties

$$|\psi(\theta)| \leqslant 1 \qquad \textit{and} \qquad \psi(-\theta) = -\psi(\theta). \tag{4.14}$$

hold for any $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Using Lemma 4.1, Lemma 4.2 and the method of proof by contradiction, we easily deduce the following lemma. For convenience of the reader, we give a proof below which is due to Zhong-Yang [7].

Lemma 4.3. (see [7]). *Assume that $\phi(\theta)$ and $\psi(\theta)$ are defined by* (4.1) *and* (4.13), *respectively. Then*

$$\phi(\theta) \leqslant \psi(\theta). \tag{4.15}$$

Proof. Assume that (4.15) is not true. Since $\phi(\pm\frac{\pi}{2}) = \pm 1 = \psi(\pm\frac{\pi}{2})$, then there exists some $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that

$$\sigma = \phi(\theta_0) - \psi(\theta_0) = \max_{-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}} \{\phi(\theta) - \psi(\theta)\} > 0. \tag{4.16}$$

Set $\widetilde{h}(\theta) = \psi(\theta) + \sigma$. Obviously, $\widetilde{h}'(\theta) = \psi'(\theta) \geqslant 0$,

$$\widetilde{h}(\theta) = \psi(\theta) + \sigma \geqslant \phi(\theta)$$

and

$$\widetilde{h}(\theta_0) = \phi(\theta_0) = \psi(\theta_0) + \sigma \geqslant -1 + \sigma > -1.$$

In place of $h(\theta)$ in Lemma 4.1 by $\widetilde{h}(\theta)$, we therefore get by Lemma 4.1 and Lemma 4.2 that

$$\begin{aligned} \phi(\theta_0) &\leqslant \sin\theta_0 - \sin\theta_0 \cdot \cos\theta_0 \cdot \widetilde{h}'(\theta_0) + \frac{\cos^2\theta_0}{2} \cdot \widetilde{h}''(\theta_0) \\ &= \sin\theta_0 - \sin\theta_0 \cdot \cos\theta_0 \cdot \psi'(\theta_0) + \frac{\cos^2\theta_0}{2} \cdot \psi''(\theta_0) = \psi(\theta_0). \end{aligned}$$

But this contradicts (4.16). ■

Corollary 4.1. (see [7]). *The following estimate holds.*

$$F(\theta) \leqslant \lambda[1 + a\psi(\theta)], \tag{4.17}$$

where $F(\theta)$ and $\psi(\theta)$ are defined by (2.8) *and* (4.13), *respectively.*

Our argument above establishes the inequality (4.17), which is the refined estimate of the upper bound for $F(\theta)$ as required.

5. Proof of Theorem 1.2

We stress here that although a reasoning similar to the one in [7] and [8] will give the claim, our proof is slightly different from [7] , [8] and [10]. We now use the estimate of $F(\theta)$ to prove Theorem 1.2 in the following.

Proof. (4.17) implies that

$$\lambda^{\frac{1}{2}} \geqslant \frac{|F(\theta)|^{\frac{1}{2}}}{\sqrt{1+a\psi(\theta)}} \geqslant \frac{|\nabla\theta|}{\sqrt{1+a\psi(\theta)}}, \tag{5.1}$$

where $\psi(\theta)$ is defined by (4.13).

Take $x_1, x_2 \in \Omega$ such that $\theta(x_1) = -\frac{\pi}{2}$, $\theta(x_2) = \frac{\pi}{2}$. We denote by d' the length of the straight line γ joining x_1 and x_2. Obviously, $d' \leqslant d$.

Using (4.14) and the the following inequality of analysis

$$\frac{1}{\sqrt{1-x}} + \frac{1}{\sqrt{1+x}} \geqslant 2, \quad \forall\, x \in (-1,1),$$

we derive from integrating (5.1) along the straight line γ that

$$\begin{aligned} \lambda^{\frac{1}{2}} d \geqslant \lambda^{\frac{1}{2}} d' &= \int_\gamma \lambda^{\frac{1}{2}} \mathrm{d}s \geqslant \int_\gamma \frac{1}{\sqrt{1+a\psi(\theta)}}\, |\nabla\theta|\, \mathrm{d}s \\ &\geqslant \int_\gamma \frac{1}{\sqrt{1+a\psi(\theta)}} \mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{1+a\psi(\theta)}} \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \Big[\frac{1}{\sqrt{1-a\psi(\theta)}} + \frac{1}{\sqrt{1+a\psi(\theta)}}\Big] \mathrm{d}\theta \\ &\geqslant \int_0^{\frac{\pi}{2}} 2\ \mathrm{d}\theta = \pi. \end{aligned}$$

This concludes the proof of the theorem. ■

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