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LAGRANGIAN DUALITY FOR VECTOR OPTIMIZATION PROBLEMS WITH SET-VALUED MAPPINGS

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Abstract. In this paper, by using a alternative theorem, we establish Lagrangian conditions and duality results for set-valued vector optimization problems when the objective and constant are nearly cone-subconvexlike multifunctions in the sense of E-weak minimizer.

1. Introduction

Optimality conditions and duality theorems for optimization problems of single-valued functions satisfying convexity or weaker conditions have been studied by many authors, see [1-8]. In particular, in works of [3-6], Lagrangian conditions and duality theorems for convexlike functions and a class of quasiconvex functions were discussed.

In recent years, many authors have generalized the single-valued functions to set-valued mappings, for its extensive applications in many fields such as mathematical programming [9], economics [10] and differential inclusions [11]. In particular, Lagrangian conditions and duality theorems were discussed when the objective and constraint are convex, preinvex, subconvexlike and nearly convexlike set-valued mappings in [12-16] and [17], respectively.

Recently, Yang, Li and Wang [18] introduced a new class of generalized convexity for set-valued functions, called nearly cone-subconvexlike, which is a generalization of the set-valued functions mentioned above. They obtained a alternative theorem, a Lagrangian multiplier theorem and two scalarization theorems. Sach [19] showed some characterizations of nearly cone-subconvexlikeness and established some saddle

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theorems under nearly cone-subconvexlikeness conditions for set-valued vector optimization. Some related works, we refer to [20].

In this paper, under nearly cone-subconvexlikeness, Lagrangian conditions and duality results for set-valued vector optimization problems are obtained in the sense of E-weak minimizer by using the alternative theorem of Yang, Li and Wang [18].

2. Preliminaries

Throughout this paper, let X be a nonempty subset of a real linear topological vector space; Y and Z be real linear topological vector spaces with topological dual spaces Y^* and Z^*, respectively. Let $C \subset Y$ and $D \subset Z$ be pointed closed convex cones with $intC \neq \emptyset$ and $intD \neq \emptyset$. The nonnegative dual cone C^+ of C is defined by

$$C^+ = \{\phi \in Y^* : \phi(y) \geq 0, \ \forall y \in C\},$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form with respect to the dual between Y^* and Y.

Let $F : X \to 2^Y$ and $G : X \to 2^Z$ be two set-valued mappings with nonempty value. We consider the following vector optimization problem with set-valued mappings:

$$\text{(P)} \qquad \begin{array}{ll} \min & F(x) \\ s.t. & G(x) \cap (-D) \neq \emptyset. \end{array}$$

Let K denote the set of all feasible points for the problem (P), i.e.,

$$K = \{x \in X \mid G(x) \cap (-D) \neq \emptyset\}.$$

Let $E \subset Y$ be a nonempty subset, and let $\varepsilon \in C$.

Definition 2.1

(i) A point $x_0 \in K$ is said to be a weak efficient solution of problem (P), if there exists $y_0 \in F(x_0)$ such that $(F(K) - y_0) \cap (-intC) = \emptyset$. The pair (x_0, y_0) is said to be a weak minimizer of problem (P).

(ii) A point $x_0 \in K$ is said to be an ε-weak efficient solution of problem (P), if there exists $y_0 \in F(x_0)$ such that $(F(K) - y_0 + \varepsilon) \cap (-intC) = \emptyset$. The pair (x_0, y_0) is said to be ε-weak minimizer of problem (P).

(iii) A point $x_0 \in K$ is said to be an E-weak efficient solution of problem (P), if there exists $y_0 \in F(x_0)$ such that $(F(K) - y_0 + E) \cap (-intC) = \emptyset$. The pair (x_0, y_0) is said to be E-weak minimizer of problem (P).

It is clear that the set of weak efficient solutions is contained in the set of ε-weak efficient solutions. Some relationships between ε-weak efficient solutions and E-weak efficient solutions were investigated in [21] as follows:

(i) if $E = \{\varepsilon\}$, then an E-weak efficient solution of problem (P) becomes a ε-weak efficient solution of problem (P);

(ii) if x_0 is an E-weak efficient solution of problem (P) and there exists $\varepsilon' \in E$ such that $\varepsilon - \varepsilon' \in C$, then x_0 is an ε-weak efficient solution of problem (P);

(iiii) if x_0 is an ε-weak efficient solution of problem (P) and $E - \varepsilon \subset C$, then x_0 is an E-weak efficient solution of problem (P).

The following two examples show that the ε-weak efficient solution and the E-weak efficient solution are totally different.

Example 2.1. Let $K = (0, 2) \times [0, 2]$, $Y = R^2$,

$$C = R^2_+ = \{(x_1, x_2) \in R^2 : x_1 \geq 0, x_2 \geq 0\},$$

$$F(x_1, x_2) = \{(x_1, x_2) \cup (\frac{3}{2}, \frac{\sqrt{3}}{2} + 1) : \ x_2 = \sqrt{1 - (x_1 - 1)^2} + 1\}.$$

Let $\varepsilon = (\frac{1}{2}, 0)$ and $E = -2 \times [\frac{1}{4}, 1]$. It is easy to compute that the ε-weak efficient solutions set S_ε and the E-weak efficient solutions set S_E, respectively, as follows:

$$S_\varepsilon = \{(x_1, x_2) : x_2 = \sqrt{1 - (x_1 - 1)^2} + 1, \ x_1 \in (0, \frac{1}{2}]\},$$

and

$$S_E = \{(x_1, x_2) : x_2 = \sqrt{1 - (x_1 - 1)^2} + 1, \ x_1 \in (0, \frac{1}{4}] \cup [\frac{7}{4}, 2)\}.$$

Then $S_\varepsilon \not\subset S_E$ and $S_E \not\subset S_\varepsilon$.

Example 2.2. Let $K = (-1, 1) \times [0, 1]$, $Y = R^2$,

$$C = \{(x, y) \in R^2 : x \geq 0, \ y \geq x\},$$

$$F(x, y) = \{(x, y) \cup (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) : \ y = \sqrt{1 - x^2}\}.$$

Let $\varepsilon = (\frac{1}{2}, \frac{1}{2})$ and $E = [1 - \frac{\sqrt{39}}{8}, \frac{1}{2}] \times \frac{5}{8}$. It is easy to compute that the ε-weak efficient solutions set S_ε and the E-weak efficient solutions set S_E, respectively, as follows:

$$S_\varepsilon = \{(x, y) : \ y = \sqrt{1 - x^2}, \ x \in (-1, -\frac{1}{2}] \cup [0, 1)\},$$

and

$$S_E = \{(x, y) : \ y = \sqrt{1 - x^2}, \ x \in (-1, -\frac{\sqrt{39}}{8}] \cup [\frac{\sqrt{47} - 9}{16}, 1)\}.$$

Thus, $S_E \not\subset S_\varepsilon$ and $S_\varepsilon \not\subset S_E$.

Definition 2.3. Let X be a convex set. A set-valued function $F : X \to 2^Y$ is said to be C-convex on X if, for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, one has

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C.$$

Definition 2.4. [15].

(i) A set-valued function $F : X \to 2^Y$ is said to be C-convexlike on X if, for all $x_1, x_2 \in X$ and $\lambda \in (0, 1)$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(X) + C.$$

(ii) A set-valued function $F : X \to 2^Y$ is said to be C-subconvexlike on X if, there exists $\theta \in intC$ such that for all $x_1, x_2 \in X$, $\lambda \in (0, 1)$, and $\varepsilon > 0$,

$$\varepsilon\theta + \lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(X) + C.$$

Remark 2.1. In Definition 2.3, X may be a nonconvex set.

Remark 2.2. From [15], we know that

(i) F is C-convexlike on X if and only if $F(X) + C$ is a convex set;

(ii) F is C-subconvexlike on X if and only if $F(X) + intC$ is a convex set.

Lemma 2.1. *If $F : X \to 2^Y$ is C-convexlike on X, then F is C-subconvexlike on X.*

Definition 2.4. [17] A set-valued function $F : X \to 2^Y$ is said to be nearly C-convexlike on X if $cl(F(X) + C)$ is a convex set.

Remark 2.3. If $F(X) + intC$ is a convex set, then $cl(F(X) + C)$ is a convex set, because $cl(F(X) + C) = cl(F(X) + intC)$.

In order to prove Theorem 2.1, we need the following lemma.

Lemma 2.2. [22]. *Let C be a convex cone in Y with $intC \neq \emptyset$, and let S be a subset of Y. Then*

$$int[cl(S + C)] = S + intC.$$

Theorem 2.1. *If $F : X \to 2^Y$ is nearly C-convexlike on X, then $F(X) + intC$ is a convex set.*

Proof. Since F is nearly C-convexlike on X, then $cl(F(X) + C)$ is a convex set. Noting that the interior of a convex set is convex, it follows that $int[cl(F(X) + C)]$ is covex. By Lemma 2.2, we have that $F(X) + intC$ is a convex set. This completes the proof. ■

Corollary 2.1. *If $F : X \to 2^Y$ is nearly C-convexlike on X if and only if $F(X) + intC$ is convex.*

Definition 2.5 [18]. A set-valued function $F : X \to 2^Y$ is said to be nearly C-subconvexlike on X if and only if $clcone(F(X) + C)$ is a convex set.

Lemma 2.3. [18]. *If F is nearly C-convexlike on X, then F is nearly C-subconvexlike on X.*

From above definitions, lemmas and corollary, we have the following relationships:

C-convexity ⇒ C-convexlikeness ⇒ C-subconvexlikeness
⇔ nearly C-convexlike ⇒ nearly C-subconvexlikeness.

Example 2.3. This example illustrates that a nearly C-subconvexlike function is neither a nearly C-convexlike function nor a C-subconvexlike function. Let $X = [0, \infty) \times [0, \infty)$, $Y = R^2$,

$$C = R^2_+ = \{(x_1, x_2) \in R^2 : x_1 \geq 0, x_2 \geq 0\},$$

$$F(x_1, x_2) = \begin{cases} \{x_1\} \times [1, \infty), & \text{if} \quad x_1 \in [0, 1), \\ \{x_1\} \times [0, \infty), & \text{if} \quad x_1 \in [1, \infty). \end{cases}$$

It is easy to prove that $clcone(F(X)+C)$ is convex, i.e., F is nearly C-subconvexlike on X. But F is neither a nearly C-convexlike function nor a C-subconvexlike function, because $cl(F(X) + C)$ and $F(X) + intC$ are not convex.

Example 2.4. This example illustrates that a C-subconvexlike function is not a C-convexlike function. Let $X = \{(0, 1), (1, 0)\}$, $Y = R^2$,

$$C = R^2_+ = \{(x_1, x_2) \in R^2 : x_1 \geq 0, x_2 \geq 0\},$$

$$F(x_1, x_2) = \{(x_1, x_2)\} \cup (C \backslash \{(x, y) \in R^2 : x \geq 0, y \geq 0, x + y \leq 1\}).$$

It is easy to check that $F(X) + intC$ is convex, i.e., F is C-subconvexlike on X. But F is not a C-convexlike function, because $F(X) + C$ is not convex.

Lemma 2.4 If $u^* \in D^+ \setminus \{0\}$ and $u \in intD$, then $\langle u, u^* \rangle > 0$.

Lemma 2.5 [18]. *Let the set-valued function $F : X \longrightarrow 2^Y$ be nearly C-subconvexlike on X. Then, one and only one of the following statements is true:*

(i) *There exists $x \in X$ such that $F(x) \cap (-intC) \neq \emptyset$;*

(ii) *There exists $\varphi \in C^+ \backslash \{0\}$ such that $\varphi(y) \geq 0$ for all $y \in F(X)$.*

3. Main Results

In this section, let $L(Z,Y)$ denote the set of all linear continuous operators $\Lambda : Z \to Y$ with $\Lambda(D) \subset C$ and $E \subset intC$ be a subset.

Theorem 3.1. *Let $intC \neq \emptyset$ and $G(K) \cap (-intD) \neq \emptyset$. Assume that set-valued function $(F - y_0 + E, G)$ is nearly $(C \times D)$-subconvexlike on K. If (x_0, y_0) is E-weak minimizer of problem (P), then there exists $\Lambda \in L(Z,Y)$ such that (x_0, y_0) is E-weak minimizer of the following problem:*

$$(\overline{P}) \quad \min_{x \in K} \quad (F(x) + \Lambda(G(x))$$

and

$$-\Lambda(G(x_0) \cap (-D)) \subset (intC \cup \{0\}) \setminus (E + intC).$$

Proof. Let (x_0, y_0) be E-weak minimizer of problem (P). Then Definition 2.1 implies that $x_0 \in K$, $y_0 \in F(x_0)$ and

$$(F(K) - y_0 + E) \cap (-intC) = \emptyset.$$

Now we show that

$$(F(K) - y_0 + E, G(K)) \cap (-intC, -intD) = \emptyset.$$

Indeed, suppose by contradiction that there exists

$$(y_1, z_1) \in (F(x_1) - y_0 + E, G(x_1)) \cap (-intC, -intD)$$

for some $x_1 \in K$. Then there exists $\overline{y} \in F(x_1)$ such that $y_1 \in \overline{y} - y_0 + E$ and so

$$\overline{y} - y_0 \in y_1 - E \subset -intC - E \subset -intC - intC \subset -intC.$$

It follows that

$$G(x_1) \cap (-intD) \neq \emptyset$$

and

$$(F(x_1) - y_0 + E) \cap (-intC) \neq \emptyset,$$

which contradicts to $(F(K) - y_0 + E) \cap (-intC) = \emptyset$. Therefore,

$$(F(K) - y_0 + E, G(K)) \cap (-intC, -intD) = \emptyset.$$

From the nearly $(C \times D)$-subconvexlikeness of $(F - y_0 + E, G)$ on K and Lemma 2.5, we have that there exists $(\varphi, \phi) \in (C^+, D^+) \setminus \{(0,0)\}$ such that

$$(3.1) \quad \langle \varphi, y - y_0 + s \rangle + \langle \phi, z \rangle \geq 0, \ \forall\, x \in K, \ \forall\, y \in F(x), \ \forall\, s \in E, \ \forall\, z \in G(x).$$

We claim that $\varphi \neq 0$. In fact, if $\varphi = 0$, then $\phi \neq 0$ and

$$\langle \phi, z \rangle \geq 0, \quad \forall x \in K, \ \forall \ z \in G(x). \tag{3.2}$$

Since $G(K) \cap (-intD) \neq \emptyset$, there exists $\overline{x} \in K$ and $\overline{z} \in G(\overline{x}) \cap (-intD)$. Hence, $\langle \phi, \overline{z} \rangle < 0$, which contradicts (3.2). Therefore, $\varphi \neq 0$. Fix $c \in intC$ with $\langle \varphi, c \rangle = 1$ and define a map $\Lambda : Z \to Y$ as

$$\Lambda(z) = \langle \phi, z \rangle c, \ \forall \ z \in Z.$$

It is easy to check that $\Lambda \in L(Z, Y)$. Setting $x = x_0$, $y = y_0$ and $z = z_0 \in G(x_0) \cap (-D)$ in (3.1), then

$$\langle \varphi, s \rangle + \langle \phi, z_0 \rangle \geq 0, \ \forall \ s \in E. \tag{3.3}$$

It follows from $\phi \in D^+$ and $z_0 \in -D$ that

$$0 \geq \langle \phi, z_0 \rangle \geq -\langle \varphi, s \rangle, \ \forall \ s \in E. \tag{3.4}$$

From the left inequality of (3.4), we have

$$-\Lambda(z_0) = -\langle \phi, z_0 \rangle c \in intC \cup \{0\}.$$

From the right inequality of (3.4), we obtain

$$-\Lambda(z_0) \notin s + intC, \ \forall \ s \in E.$$

In fact, if $-\Lambda(z_0) \in s + intC$ for some $s \in E$, then

$$\varphi(\langle \phi, z_0 \rangle c + s) = \varphi(\Lambda(z_0) + s) < 0,$$

because $\varphi \in C^+ \setminus \{0\}$. It follows from $\langle \varphi, c \rangle = 1$ that

$$\langle \phi, z_0 \rangle + \langle \varphi, s \rangle < 0,$$

which contradicts (3.3). Therefore,

$$-\Lambda(z_0) \notin s + intC, \ \forall \ s \in E,$$

or equivalently,

$$-\Lambda(z_0) \notin E + intC.$$

Notice that z_0 is arbitrary in the set $G(x_0) \cap (-D)$, we have

$$-\Lambda(G(x_0) \cap (-D)) \subset (intC \cup \{0\}) \setminus (E + intC).$$

Suppose that (x_0, y_0) is not a E-weak minimizer of problem $(\overline{P})$. Then there exist $\overline{x} \in K$, $\overline{y} \in F(\overline{x})$, $s \in E$ and $\overline{z} \in G(\overline{x})$ such that

$$y_0 - (\overline{y} + \Lambda(\overline{z})) - s \in intC.$$

Since $\varphi \in C^+ \setminus \{0\}$, by Lemma 2.4, we have

$$\langle \varphi, y_0 - (\overline{y} + \Lambda(\overline{z})) - s \rangle > 0.$$

It follows from $\Lambda(z) = \langle \phi, z \rangle c$ and $\langle \varphi, c \rangle = 1$ that

$$\langle \varphi, \overline{y} - y_0 + s \rangle + \langle \phi, \overline{z} \rangle < 0.$$

This contradicts (3.1). Hence (x_0, y_0) is E-weak minimizer of problem $(\overline{P})$. This completes the proof. ■

Now, we consider the dual problem. Define a set-valued mapping $\Phi : L(Z, Y) \to 2^Y$ by

$$\Phi(\Lambda) = \{y \mid \exists x \in K \text{ such that } (x, y) \text{ is E-weak minimizer of problem } \overline{P}\}.$$

Consider the following maximum problem:

$$\text{(DP)} \qquad \max \quad \Phi(\Lambda) \qquad s.t. \quad \Lambda \in L(Z, Y).$$

A point $\Lambda \in L(Z, Y)$ is said to be a feasible point of problem (DP) if $\Phi(\Lambda) \neq \emptyset$. We say that (Λ_0, y_0) is E-weak maximizer of problem (DP) if Λ_0 is a feasible point of problem (DP), $y_0 \in \Phi(\Lambda_0)$, and there exists no feasible point $\Lambda \in L(Z, Y)$ such that

$$(y_0 - \Phi(\Lambda) + E) \cap (-intC) \neq \emptyset.$$

Theorem 3.2. (E-weak duality). *If Λ_0 is a feasible point of problem (DP) and x_0 is a feasible point of problem (P), then*

$$(F(x_0) - \Phi(\Lambda_0) + E) \cap (-intC) = \emptyset.$$

Proof. Since Λ_0 is a feasible point of problem (DP), for any $y \in \Phi(\Lambda_0)$, there exists $x \in K$ such that (x, y) is E-weak minimizer of problem $(\overline{P})$ corresponding to Λ_0. It follows that

$$[(F + \Lambda_0(G))(K) - y + E] \cap (-intC) = \emptyset. \tag{3.5}$$

Now we claim that

$$(F(x_0) - y + E) \cap (-intC) = \emptyset.$$

Indeed, if there is $y_0 \in F(x_0)$ and $s \in E$ such that $y_0 - y + s \in -intC$. Since x_0 is a feasible point of problem $(\overline{P})$, there exists $z_0 \in G(x_0) \cap (-D)$. From the fact that $\Lambda_0 \in L(Z, Y)$, it follows that $\Lambda_0 z_0 \in -C$. Thus,

$$y_0 + \Lambda_0 z_0 - y + s \in \Lambda_0 z_0 - intC \subset -intC,$$

or equivalently,

$$[(F + \Lambda_0(G))(K) - y + E] \cap (-intC) \neq \emptyset,$$

which contradicts (3.5). Notice that $y \in \Phi(\Lambda_0)$ is arbitrary. Therefore, we have

$$(F(x_0) - \Phi(\Lambda_0) + E) \cap (-intC) = \emptyset.$$

This completes the proof. ■

Theorem 3.3. (E-strong duality). *Let* $(F - y_0 + E, G)$ *be nearly* $(C \times D)$*-subconvexlike on* K. *If* (x_0, y_0) *is* E*-weak minimizer of problem* (P) *and* $G(K) \cap (-intD) \neq \emptyset$*, then there exists* $\Lambda_0 \in L(Z, Y)$ *such that* (Λ_0, y_0) *is* E*-weak maximizer of problem* (DP).

Proof. Suppose (x_0, y_0) is E-weak minimizer of problem (P) and $G(K) \cap (-intD) \neq \emptyset$, then by Theorem 3.1, there exists $\Lambda_0 \in L(Z, Y)$ such that (x_0, y_0) is E-weak minimizer of problem $(\overline{P})$ corresponding to Λ_0. It follows that Λ_0 is a feasible point of problem (DP), and $y_0 \in \Phi(\Lambda_0)$. By Theorem 3.2, we obtain

$$(y_0 - \Phi(\Lambda_0) + E) \cap (-intC) = \emptyset.$$

Thus, (Λ_0, y_0) is E-weak maximizer of problem (DP). This completes the proof. ■

Remark 3.1.

(i) If $E = \{\varepsilon\}$, F and G are subconvexlike, then Theorems 3.2 and 3.3 reduce to Theorems 5.1 and 5.2 of [16];

(ii) If $E = \{0\}$, F and G are nearly convexlike, then Theorems 3.2 and 3.3 reduce to Theorems 4.5 of [17].

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