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DISTANCE THREE LABELINGS FOR DIRECT PRODUCTS OF THREE COMPLETE GRAPHS

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Abstract. The distance 3 labeling number $\lambda_G(j_0, j_1, j_2)$ for a graph $G = (V, E)$ is the smallest integer α such that there is a function $f : V \to [0, \alpha]$, satisfying $|f(u)-f(v)| \geq j_{\delta-1}$ for any pair of vertices u, v of distance $\delta \leq 3$. In this paper, we determine the distance 3 labeling number $\lambda_G(j, k, 1)$ for the direct product $G = K_n \times K_m \times K_2$ $(n \geq m \geq 3)$ of 3 complete graphs under various conditions on j and k. As a consequence, we have the radio number $\mathrm{rn}(G) = 2mn - 1$.

1. INTRODUCTION

The channel assignment problem introduced by Hale [9] is the motivation for the various labeling problems in graphs. For a graph $G = (V, E)$ and nonnegative integers $\alpha, j_0, j_1, \cdots, j_{d-1}$, an $L(j_0, j_1, \cdots, j_{d-1})$*-labeling* (or *distance* d*-labeling*) is an integer valued function $f : V \to [0, \alpha]$ such that for $u, v \in V$ with $\delta = \text{dist}(u, v)$, the labeling condition $|f(u) - f(v)| \geq j_{\delta-1}$ is satisfied. The *labeling number* $\lambda_G(j_0, j_1, \cdots, j_{d-1})$ for G is the smallest integer α such that there is an $L(j_0, j_1, \cdots, j_{d-1})$-labeling $f : V \to [0, \alpha]$. The distance 2 labeling has been studied the most. In particular, the labeling number $\lambda_G(2, 1)$ is referred to as the λ*-number* for G, and is denoted by $\lambda(G)$. After the work by Griggs and Yeh [8], special attention has been paid to the λ-number of graphs.

The *radio labeling* for G is the $L(d, d-1, \cdots, 1)$-labeling f, where $d = \text{diam}(G)$. In this case, the labeling condition between two vertices u and v is $|f(u) - f(v)| \geq d - \delta + 1$, where $\delta = \text{dist}(u, v)$. The *radio number* for G, denoted by $\mathrm{rn}(G)$, is the minimum span of radio labelings for G. If a graph has diameter 2, then $\lambda(G) = \mathrm{rn}(G)$.

Liu and Zhu [16] completely determined the radio numbers for paths and cycles. Liu [13] treated the radio number for trees. Liu and Xie [17] determined the radio

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number for the squares of paths and squares of some cycles. Recently Li, Mak and Zhou [15] found the optimal radio labelings of complete m-ary trees. There are some results on the distance 3 labelings for graphs. In particular, $\lambda_G(1,1,1)$ and $\lambda_G(2,1,1)$ have been computed when G is a path, cycle, grid, complete binary tree, or cube [1, 2, 3, 4, 7, 13, 16, 19]. Recently, Chia et al. [6] gave the upper bound of $\lambda_G(3,2,1)$ when G is a general graph, and when G is a tree. They also computed $\lambda_G(3,2,1)$ for some classes of G. In addition, $L(h,1,1)$-labelings of trees and outplanar graphs are studied recently in [5, 12].

The *direct product* of n graphs $G_i = (V_{G_i}, E_{G_i}), (i = 1, 2, \cdots, n)$ is the graph $G = G_1 \times G_2 \times \cdots \times G_n$ that the vertex set is $V_{G_1} \times V_{G_2} \times \cdots \times V_{G_n}$ and $[(u_1, u_2, \cdots, u_n), (v_1, v_2, \cdots, v_n)]$ belongs to the edge set if $(u_i, v_i) \in E_{G_i}$ for all i. It is easy to see that $G = K_n \times K_m \times K_l (n \geq m \geq l)$ is of diameter 2 when $l \geq 3$, and of diameter 3 when $l = 2$ and $m \geq 3$. Further, $G = K_n \times K_m \times K_l$ is disconnected when $l = m = 2$, in which case, G is isomorphic to two copies of $K_n \times K_2$. Haque and Jha [10] and Lam et al. [14] studied the $L(j,k)$-labeling number for the multiple direct product of complete graphs.

In this paper, we show that when $G = K_n \times K_m \times K_2,\ n \geq m \geq 3$, the distance 3 labeling number $\lambda_G(j,k,1)$ for G satisfies

$$\lambda_G(j,k,1) = \begin{cases} 2mn-1 & \text{if } \ k=1 \text{ and } 1 \leq j \leq n+2. \\ (mn-1)k+1 & \text{if } \ k \geq 2 \text{ and } k \leq j \leq 2k-1. \\ (mn-1)k+m & \text{if } \ k \geq 2 \text{ and } j = 2k. \\ (m-1)j+(mn-2m+1)k+1 & \text{if } \ k \geq 2 \text{ and } 2k+1 \leq j \leq nk. \end{cases}$$

As a consequence, we prove that the radio number $\mathrm{rn}(G)$ for G is $2mn-1$.

2. Preliminaries and Some Lemmas

Henceforth, we assume that $n \geq m \geq 3$, $j \geq k \geq 2$, $G = K_n \times K_m \times K_2 = (V, E)$, and $f : V \to [0, \alpha]$ is an $L(j,k,1)$-labeling. The vertices $x_1, x_2, \cdots, x_p$ are on the *same row*, *same column*, or *same floor* if they have the same first, second, or third components, respectively. Two vertices u and v on the same floor are said to be *set oblique* if they are on different rows and columns. Since the distance between two vertices on the same floor is 2, if $x_1, x_2, \cdots, x_p$ are on the same floor, $f(x_1) < f(x_2) < \cdots < f(x_p)$ and $1 \leq t < s \leq p$, and then, we have $f(x_s) - f(x_t) \geq k(t-s)$. Let $V_i = \{(u_0, u_1, u_2) | u_2 = i\}$ for $i = 0, 1$ be the two floors of V.

Lemma 1. *Let $x_1, x_2, x_3 \in V_0$ and $y_1, y_2 \in V_1$. If x_1 and x_2 are set oblique, then there exist $x \in \{x_1, x_2, x_3\}$ and $y \in \{y_1, y_2\}$ such that x and y are adjacent.*

Proof. Suppose that no element of x_1 and x_2 is adjacent to any of y_1 and y_2. Since $x_1 = (u_0, u_1, 0)$ and $x_2 = (v_0, v_1, 0)$ are set oblique, $u_0 \neq v_0$ and $u_1 \neq v_1$.

Since the two non adjacent vertices share some components, we have only two vertices of V_1—$(u_0, v_1, 1)$ and $(v_0, u_1, 1)$—that are adjacent to neither x_1 nor x_2. As such, there exist y_1 and y_2 that are set oblique. Similarly, x_1 and x_2 are the only vertices of V_0 that are adjacent to neither y_1 nor y_2. Hence, x_3 is adjacent to y_1 or y_2. ∎

Lemma 2. *If $x_1, x_2, \cdots, x_{n+1} \in V_i$ for some $i = 0, 1$, then there exists a pair of vertices that are set oblique among $x_1, x_2, \cdots, x_{n+1}$.*

Proof. Suppose there exists no pair of vertices among vertices $x_1, x_2, \cdots, x_{n+1}$ that are set oblique. Since x_1 and x_2 are not set oblique, they are on the same row or the same column. Since $n \geq m$, we can assume that they are on the same row. Since there are $n + 1$ vertices on one floor, there exists x_i that is not on the same row as x_1. Since x_1 and x_2 are not on the same column, x_i is not on the same column as x_1 or x_2. As such either x_i and x_1 or x_i and x_2 are set oblique. ∎

Lemma 3. *For $p \geq 3$, let $x_1, x_2, \cdots, x_p \in V_i$ for some $i = 0, 1$. If there exists a pair of vertices among $x_1, x_2, \cdots, x_p$ that are set oblique, then there is an integer s $(1 \leq s \leq p - 2)$ such that $\{x_s, x_{s+1}, x_{s+2}\}$ contains a pair of vertices set oblique.*

Proof. Among the pair (r, t), where x_r and x_t are set oblique, choose (r_0, t_0) such that $|t - r|$ takes the minimum. We can assume that $r_0 < t_0$. It suffices to show that $t_0 - r_0 \leq 2$. On the contrary, assume that $t_0 - r_0 \geq 3$. If x_{r_0} and x_{r_0+1} are not set oblique, then they are on the same row or the same column. We can assume that they are on the same row. Since x_{r_0+2} and x_{t_0} are not set oblique, they are on the same row or the same column. If they are on the same row, then x_{r_0} and x_{r_0+2} are on the same column. Since x_{r_0} and x_{r_0+1} are on the same row, and x_{r_0+2} and x_{t_0} are on the same row, x_{r_0+1} and x_{r_0+2} are on distinct rows. Since x_{r_0} and x_{r_0+1} are on distinct columns, x_{r_0+1} and x_{r_0+2} are on distinct columns. As such, x_{r_0+1} and x_{r_0+2} are set oblique. This is a contradiction. If x_{r_0+2} and x_{t_0} are on the same column, since x_{r_0} and x_{r_0+2} are not set oblique, then they are on the same row. Then, $x_{r_0+1} = x_{r_0+2}$. This is also a contradiction. Hence, $t_0 - r_0 \leq 2$. ∎

For $S \subset V$ and labeling f, we define the *span* $\operatorname{span}(f : S)$ of f on S as the maximum of $|f(u) - f(v)|$ for $u, v \in S$. The *span* $\operatorname{span}(f)$ of f is $\operatorname{span}(f : V)$. Let $|S| = 2n + 1$, $S \cap V_0 = \{x_1, x_2, \cdots, x_p\}$ and $S \cap V_1 = \{y_1, y_2, \cdots, y_q\}$ with $f(x_1) < f(x_2) < \cdots < f(x_p)$ and $f(y_1) < f(y_2) < \cdots < f(y_q)$. Without loss of generality, we can assume that $p \geq q$, and hence, we have $p \geq n + 1$. From Lemmas 2 and 3, there exists s $(1 \leq s \leq p - 2)$ such that $\{x_s, x_{s+1}, x_{s+2}\}$ contains a pair of vertices that are set oblique. Then, $p + q = 2n + 1$ and we have the following propositions.

Proposition 1. *If $\{x_s, x_{s+1}, x_{s+2}\}$ contains a pair of vertices that are set oblique, $f(y_2) < f(x_{s+2})$, and $f(x_s) < f(y_{q-1})$, then $\operatorname{span}(f : S) \geq j + (n - 2)k$.*

Proof. Since $p-3 \geq n-2$ and $\lceil \frac{p+q-6}{2} \rceil \geq \lceil \frac{2n+1-6}{2} \rceil = n-2$, it is enough to show that $\mathrm{span}(f:S) \geq \min\{j+(p-3)k, j+\lceil \frac{p+q-6}{2} \rceil k\}$.

Since $f(y_2) < f(x_{s+2})$ and $f(x_s) < f(y_{q-1})$, there exist α, β such that α is the smallest number h such that $f(y_h) > f(x_s)$ and β is the largest number h such that $f(y_h) < f(x_{s+2})$. If $\alpha \geq \beta+2$, then $f(x_{s+2}) < f(y_{\beta+1}) \leq f(y_{\alpha-1}) < f(x_s)$. This is a contradiction. Hence, $\alpha \leq \beta+1$. Since $(s+q-\alpha+1)+(\beta+p-s-1) = p+q-\alpha+\beta \geq p+q-1 = 2n$, there are three possible cases:

Case 1: $s+q-\alpha+1 \geq \lceil \frac{p+q}{2} \rceil = n+1$.

Case 2: $\beta+p-s-1 \geq \lceil \frac{p+q}{2} \rceil = n+1$.

Case 3: $s+q-\alpha+1 = \beta+p-s-1 = n$.

Case 1. $s+q-\alpha+1 \geq \lceil \frac{p+q}{2} \rceil$.
From Lemma 1, some $x \in \{x_s, x_{s+1}, x_{s+2}\}$ is adjacent to some $y \in \{y_\alpha, y_{\alpha+1}\}$. If $f(y_{\alpha+1}) > f(x_{s+2})$, then $f(y_{\alpha+1}) - f(x_s) \geq |f(x)-f(y)| \geq j$. As such,

$$\begin{aligned}\mathrm{span}(f:S) &\geq f(y_q) - f(x_1)\\ &= (f(y_q) - f(y_{\alpha+1})) + (f(y_{\alpha+1}) - f(x_s)) + (f(x_s) - f(x_1))\\ &\geq (q-\alpha-1)k + j + (s-1)k = j + (s+q-\alpha-2)k\\ &\geq j + (n-2)k.\end{aligned}$$

If $f(x_{s+2}) > f(y_{\alpha+1})$, then $f(x_{s+2}) - f(x_s) \geq |f(x)-f(y)| \geq j$. Thus,

$$\begin{aligned}\mathrm{span}(f:S) &\geq f(x_p) - f(x_1)\\ &= (f(x_p) - f(x_{s+2})) + (f(x_{s+2}) - f(x_s)) + (f(x_s) - f(x_1))\\ &\geq (p-s-2)k + j + (s-1)k = j + (p-3)k \geq j + (n-2)k.\end{aligned}$$

Case 2. $\beta+p-s-1 \geq \lceil \frac{p+q}{2} \rceil = n+1$.
From Lemma 1, some $x \in \{x_s, x_{s+1}, x_{s+2}\}$ is adjacent to some $y \in \{y_{\beta-1}, y_\beta\}$. If $f(y_{\beta-1}) < f(x_s)$, then $f(x_{s+2}) - f(y_{\beta-1}) \geq |f(x)-f(y)| \geq j$. Hence,

$$\begin{aligned}\mathrm{span}(f:S) &\geq f(x_p) - f(y_1)\\ &= (f(x_p) - f(x_{s+2})) + (f(x_{s+2}) - f(y_{\beta-1})) + (f(y_{\beta-1}) - f(y_1))\\ &\geq (p-s-2)k + j + (\beta-2)k = j + (p-s+\beta-4)k\\ &\geq j + \lceil \frac{p+q-6}{2} \rceil k\\ &\geq j + (n-2)k.\end{aligned}$$

If $f(y_{\beta-1}) > f(x_s)$, then $f(x_{s+2}) - f(x_s) \geq |f(x)-f(y)| \geq j$. Therefore,

$$\begin{aligned}\mathrm{span}(f:S) &\geq f(x_p) - f(x_1)\\ &= (f(x_p) - f(x_{s+2})) + (f(x_{s+2}) - f(x_s)) + (f(x_s) - f(x_1))\\ &\geq (p-s-2)k + j + (s-1)k = j + (p-3)k\\ &\geq j + (n-2)k.\end{aligned}$$

Case 3. $s + q - \alpha + 1 = \beta + p - s - 1 = \frac{p+q-1}{2} = n$.
From Lemma 1, some $x \in \{x_s, x_{s+1}, x_{s+2}\}$ is adjacent to some $y \in \{y_\alpha, y_\beta\}$. Since $\alpha = \beta + 1$, $f(y_\beta) = f(y_{\alpha-1}) < f(x_s)$ and $f(y_\alpha) = f(y_{\beta+1}) > f(x_{s+2})$. If $y = y_\alpha$, $f(y_\alpha) - f(x_s) \geq |f(y) - f(x)| \geq j$. As such,

$$\begin{aligned}\text{span}(f : S) &\geq f(y_q) - f(x_1)\\ &= (f(y_q) - f(y_\alpha)) + (f(y_\alpha) - f(x_s)) + (f(x_s) - f(x_1))\\ &\geq (q-\alpha)k + j + (s-1)k = j + (q + s - \alpha - 1)k\\ &= j + (n-2)k.\end{aligned}$$

If $y = y_\beta$, then $f(x_{s+2}) - f(y_\beta) \geq |f(y) - f(x)| \geq j$. Thus,

$$\begin{aligned}\text{span}(f : S) &\geq f(x_p) - f(y_1)\\ &= (f(x_p) - f(x_{s+2})) + (f(x_{s+2}) - f(y_\beta)) + (f(y_\beta) - f(y_1))\\ &\geq (p - s - 2)k + j + (\beta - 1)k = j + (p + \beta - s - 3)k\\ &= j + (n-2)k.\end{aligned}$$

■

Proposition 2. *Let* $|S| = p + q = 2n + 1$ *and* $j \geq 2k$.

(1) *If* $q \leq 2$, *then* $\text{span}(f : S) \geq (2n-2)k$.
(2) *If* $q \geq 3$, *then* $\text{span}(f : S) \geq j + (n-2)k$.
(3) *If* $j = 2k$, *then* $\text{span}(f : S) \geq nk + 1$.

Proof.

(1) In this case, $p \geq 2n - 1$, and we have

$$\begin{aligned}\text{span}(f : S) &\geq f(x_p) - f(x_1)\\ &\geq (p-1)k \geq (2n-1-1)k = (2n-2)k.\end{aligned}$$

(2) From Lemmas 2 and 3, there exists s $(1 \leq s \leq p-2)$ such that $\{x_s, x_{s+1}, x_{s+2}\}$ contains a pair of vertices that are set oblique. Let $s_1 < s_2 < \cdots < s_h$ be all s such that $\{x_s, x_{s+1}, x_{s+2}\}$ contains a pair of vertices that are set oblique. From Lemmas 2 and 3, $s_{i+1} - s_i \leq n - 1$ for all $i = 1, 2, \cdots, h-1$.

Case 1. $f(y_{q-1}) < f(x_{s_1})$. If $\alpha = s_1$, from Lemma 2, $\alpha \leq n - 1$. If y_{q-1} is adjacent to some x_α, $x_{\alpha+1}$, and $x_{\alpha+2}$, since $f(x_{\alpha+2}) - f(y_{q-1}) \geq j$, we have

$$\begin{aligned}\text{span}(f : S) &\geq f(x_p) - f(y_1)\\ &= (f(x_p) - f(x_{\alpha+2})) + (f(x_{\alpha+2}) - f(y_{q-1})) + (f(y_{q-1}) - f(y_1))\\ &\geq (p - \alpha - 2)k + j + (q-2)k = j + (p + q - \alpha - 4)k\\ &\geq j + (2n + 1 - (n-1) - 4)k = j + (n-2)k.\end{aligned}$$

If y_{q-1} is adjacent to no x_α, $x_{\alpha+1}$, and $x_{\alpha+2}$, then from Lemma 1, y_q is adjacent to some x_α, $x_{\alpha+1}$, and $x_{\alpha+2}$. Moreover, from Lemma 1, y_{q-2} is adjacent to some x_α, $x_{\alpha+1}$, and $x_{\alpha+2}$. If $f(y_q) < f(x_\alpha)$, since $f(x_{\alpha+2})-f(y_q) \geq f(x)-f(y_q) \geq j$, we have

$$\begin{aligned}\text{span}(f:S) &\geq f(x_p) - f(y_1)\\ &= (f(x_p) - f(x_{\alpha+2})) + (f(x_{\alpha+2}) - f(y_q)) + (f(y_q) - f(y_1))\\ &\geq j + (p+q-\alpha-3)k \geq j + (n-2)k.\end{aligned}$$

If $f(y_q) > f(x_\alpha)$ and $\alpha \leq n-2$, then since y_{q-2} is adjacent to some x_α, $x_{\alpha+1}$, and $x_{\alpha+2}$, and since $f(x_{\alpha+2}) - f(y_{q-2}) \geq f(x) - f(y_{q-2}) \geq j$, we have

$$\begin{aligned}\text{span}(f:S) &\geq f(x_p) - f(y_1)\\ &= (f(x_p) - f(x_{\alpha+2})) + (f(x_{\alpha+2}) - f(y_{q-2})) + (f(y_{q-2}) - f(y_1))\\ &\geq j + (p+q-\alpha-5)k \geq j + (n-2)k.\end{aligned}$$

If $f(y_q) > f(x_\alpha)$ and $\alpha = n-1$, since x is adjacent to y_q, $|f(x) - f(y_q)| \geq j$. If $f(x) > f(y_q)$, then $f(x_{\alpha+2}) - f(x_\alpha) \geq f(x) - f(y) \geq j$. If $f(x) < f(y_q)$, then $f(y_q) - f(x_\alpha) \geq f(y_q) - f(x) \geq j$. As such,

$$\begin{aligned}\text{span}(f:S) &\geq \max\{f(x_{\alpha+2}) - f(x_1), f(y_q) - f(x_1)\}\\ &\geq \max\{f(x_{\alpha+2}) - f(x_\alpha), f(y_q) - f(x_\alpha)\} + (f(x_\alpha) - f(x_1))\\ &\geq j + (\alpha-1)k \geq j + (n-2)k.\end{aligned}$$

Case 2. $f(y_{q-1}) > f(x_{s_1})$. Let i be the largest t such that $f(x_{s_t}) < f(y_{q-1})$. If $f(x_{s_i+2}) > f(y_2)$, then from Proposition 1, $\text{span}(f:S) \geq j + (n-2)k$. Therefore, we can assume that $f(x_{s_i+2}) < f(y_2)$.

If $i = h$, then $f(x_{s_h+2}) < f(y_2)$. As in Case 1, we can prove that $\text{span}(f:S) \geq j + (n-2)k$.

As such, it suffices to prove the case $i \leq h-1$. If $\alpha = s_i$ and $\beta = s_{i+1}$, then $f(x_{\alpha+2}) < f(y_2) \leq f(y_{q-1}) < f(x_\beta)$. From Lemma 2, $\beta - \alpha \leq n-1$. From Lemma 1, there exist $x \in \{x_\alpha, x_{\alpha+1}, x_{\alpha+2}\}$, $x' \in \{x_\beta, x_{\beta+1}, x_{\beta+2}\}$, $y \in \{y_1, y_2\}$, and $y' \in \{y_{q-1}, y_q\}$ such that x and x' are adjacent to y and y', respectively.

Case 2-1. $f(y) < f(x_{\alpha+2})$. Since $f(y_2) > f(x_{\alpha+2})$, we have $y = y_1$. If $p - \alpha \geq n$, then

$$\begin{aligned}\text{span}(f:S) &\geq f(x_p) - f(y_1)\\ &= (f(x_p) - f(x_{\alpha+2})) + (f(x_{\alpha+2}) - f(y_1)) \geq j + (n-2)k.\end{aligned}$$

If $p-\alpha \le n-1$, then $q+\alpha = (p+q)-(p-\alpha) \ge 2n+1-(n-1) = n+2$. From Lemma 1, some $x \in \{x_\alpha, x_{\alpha+1}, x_{\alpha+2}\}$ is adjacent to some $\tilde{y} \in \{y_2, y_3\}$, and we have $f(y_3) - f(x_\alpha) \ge |f(x) - f(\tilde{y})| \ge j$. Therefore,

$$\begin{aligned}\text{span}(f:S) &\ge f(y_q) - f(x_1)\\ &= (f(y_q)-f(y_3)) + (f(y_3)-f(x_\alpha)) + (f(x_\alpha)-f(x_1))\\ &\ge (q-3)k + j + (\alpha-1)k = j + (q+\alpha-4)k \ge j+(n-2)k.\end{aligned}$$

Case 2-2. $f(y') > f(x_\beta)$. We can show the case $\text{span}(f:S) \ge j+(n-2)k$ using a method similar to that used in Case 2-1.

Case 2-3. $f(x_{\alpha+2}) < f(y)$ and $f(y') < f(x_\beta)$. Since $f(y') - f(y) \ge f(y_{q-1}) - f(y_2) \ge (q-3)k$, we have

$$\begin{aligned}\text{span}(f:S) &\ge f(x_p) - f(x_1) = (f(x_p)-f(x_{\beta+2})) + (f(x_{\beta+2}) - f(y_{q-1}))\\ &\quad + (f(y_{q-1}) - f(y_2)) + (f(y_2)-f(x_\alpha)) + (f(x_\alpha)-f(x_1))\\ &\ge (p-\beta-2)k + j + (q-3)k + j + (\alpha-1)k\\ &= 2j + (p+q+\alpha-\beta-6)k \ge 2j + (2n+1-(n-1)-6)k\\ &= 2j+(n-4)k \ge j+(n-2)k.\end{aligned}$$

(3) If $p \ge n+2$, then

$$\text{span}(f:S) \ge f(x_p) - f(x_1) \ge (p-1)k \ge (n+1)k \ge nk+1.$$

If $p = n+1$, then $q = n$. From Lemmas 2 and 3, there exists t such that $1 \le t \le n-1$ and $\{x_t, x_{t+1}, x_{t+2}\}$ contains a pair of vertices that are set oblique. Then, from Lemma 1, there exist h, l such that $t \le h \le t+2$, $t \le l \le t+1$, and x_h is adjacent to y_l. If $f(x_h) < f(y_l)$ and $f(x_{t+2}) > f(y_l)$, then

$$\begin{aligned}\text{span}(f:S) =& f(x_{n+1}) - f(x_1) = (f(x_{n+1}) - f(x_{t+2}))\\ &+ (f(x_{t+2}) - f(y_l)) + (f(y_l) - f(x_h)) + (f(x_h)-f(x_1))\\ \ge& (n-t-1)k + 1 + 2k + (h-1)k \ge (n-t+h)k+1 \ge nk+1.\end{aligned}$$

If $f(x_h) < f(y_l)$ and $f(x_{t+2}) < f(y_l)$, then

$$\begin{aligned}\text{span}(f:S) &= (f(y_n) - f(y_l))\\ &\quad + (f(y_l) - f(x_{t+2})) + (f(x_{t+2}) - f(x_1))\\ &\ge (n-l)k + 1 + (t+2-1)k = (n-l+t+1)k+1 \ge nk+1.\end{aligned}$$

If $f(x_h) > f(y_l)$ and $f(x_t) > f(y_l)$, then

$$\begin{aligned}\text{span}(f:S) &= f(x_{n+1}) - f(y_1) = (f(x_{n+1}) - f(x_t)) + (f(x_t) - f(y_l))\\ &\quad + (f(y_l) - f(y_1))\\ &\ge (n+1-t)k + 1 + (l-1)k = (n-t+l)k+1 \ge nk+1.\end{aligned}$$

If $f(x_h) > f(y_l)$ and $f(x_t) < f(y_l)$, then

$$\begin{aligned}\text{span}(f:S) &= (f(x_{n+1}) - f(x_h)) + (f(x_h) - f(y_l)) \\ &\quad + (f(y_l) - f(x_t)) + (f(x_t) - f(x_1)) \\ &\geq (n+1-h)k + 2k + 1 + (t-1)k \\ &= (n-h+t+2)k + 1 \geq nk+1.\end{aligned}$$

■

3. Main Theorems

Theorem 1. *If $1 \leq j \leq n+2$, then the distance 3 number $\lambda_G(j,1,1)$ of $G = K_n \times K_m \times K_2$ $(n \geq m \geq 3)$ is $2mn-1$.*

Proof. Since the diameter of G is 3 and $|V| = 2mn$, $\lambda_G(j,1,1) \geq 2mn-1$. Let $\tilde{f} : V \to [0, N]$, and

$$\tilde{f}(u_0,u_1,u_2) = \begin{cases} u_0 + nu_1, & \text{if } u_2 = 0 \\ 2mn - u_0 - nu_1 - 1, & \text{if } u_2 = 1 \end{cases}$$

Let $u = (u_0,u_1,u_2)$ and $v = (v_0,v_1,v_2)$ be distinct vertices of G. If u and v are adjacent, since $u_0 \neq v_0$ and $u_1 \neq v_1$, $u_0 + v_0 \leq 2n-3$ and $u_1 + v_1 \leq 2m-3$. We can assume that $u_2 = 0$ and $v_2 = 1$. Then,

$$\begin{aligned}\tilde{f}(v) - \tilde{f}(u) &= 2mn - v_0 - nv_1 - 1 - u_0 - nu_1 \\ &= 2mn - (u_0+v_0) - (u_1+v_1)n - 1 \\ &\geq 2mn - (2n-3) - (2m-3)n - 1 = n+2 \geq j.\end{aligned}$$

If dist $(u,v) = 2$, then $u_2 = v_2$. If $u_2 = v_2 = 0$, since $u_0 \neq v_0$ or $u_1 \neq v_1$, $\tilde{f}(v) - \tilde{f}(u) = (v_0 - u_0) + (v_1 - u_1)n \neq 0$. Similarly, $\tilde{f}(v) \neq \tilde{f}(u)$ when $u_2 = v_2 = 1$. If $\text{dist}(u,v) = 3$, then $u_2 \neq v_2$. We can assume that $u_2 = 0$ and $v_2 = 1$. Since $\tilde{f}(u) = u_0 + u_1 n \leq mn - 1$ and $\tilde{f}(v) = 2mn - u_1 - v_1 n \geq mn$, $\tilde{f}(v) \neq \tilde{f}(u)$. Hence, $\tilde{f}$ is an $L(j,1,1)$-labeling for G. Thus, $\lambda_G(j,1,1) = 2mn-1$. ■

Table 1 represents the $L(j,1,1)$-labeling for $K_5 \times K_4 \times K_2$ when $1 \leq j \leq 7$. It shows that $\lambda_G(j,1,1) = 2mn - 1 = 39$.

Table 1. $L(j,1,1)$-labeling when $k = 1$ and $1 \leq j \leq 7$

$u_2 = 0$				$u_2 = 1$			
0	5	10	15	39	34	29	24
1	6	11	16	38	33	28	23
2	7	12	17	37	32	27	22
3	8	13	18	36	31	26	21
4	9	14	19	35	30	25	20

In Table 1, the number located in the (u_0+1)-th row and (u_1+1)-th column of the box over which u_2 is indicated, is the labeling of the vertex (u_0, u_1, u_2) of $K_5 \times K_4 \times K_2$. For example, the number 31 is labeled to the vertex $(3, 1, 2)$ of $K_5 \times K_4 \times K_2$. Other figures remain the same.

Theorem 2. *If* $2 \leq k \leq j \leq 2k-1$, *then the distance 3 labeling* $\lambda_G(j,k,1)$ *for* $G = K_n \times K_m \times K_2$ $(n \geq m \geq 3)$ *is* $(mn-1)k+1$.

Proof. Let f be an $L(j,k,1)$-labeling for G, $V_0 = \{x_1, x_2, \cdots, x_{mn}\}$, and $V_1 = \{y_1, y_2, \cdots, y_{mn}\}$. We can assume that $f(x_i) < f(x_{i+1})$ and $f(y_i) < f(y_{i+1})$ for all $i = 1, 2, \cdots, mn-1$, and $f(x_{mn}) < f(y_{mn})$. Then,

$$\begin{aligned}\lambda_G(j,k,1) &\geq f(y_{mn}) - f(x_1) = (f(y_{mn}) - f(x_{mn})) + (f(x_{mn}) - f(x_{mn-1})) \\ &\quad + (f(x_{mn-1}) - f(x_{mn-2})) + \cdots + (f(x_2) - f(x_1)) \geq 1 + (mn-1)k.\end{aligned}$$

Let $\tilde{f} : V \to [0, (mn-1)k+1]$, and

$$\tilde{f}(u_0, u_1, u_2) = \begin{cases} u_0 k + u_1 nk + u_2, & u_1 \text{ is even} \\ (n - u_0 - 1)k + u_1 nk + u_2, & u_1 \text{ is odd.} \end{cases}$$

Let $u = (u_0, u_1, u_2)$ and $v = (v_0, v_1, v_2)$ be distinct vertices of G. We can assume that $u_1 \leq v_1$. If u and v are adjacent, since $u_0 \neq v_0$ and $u_1 \neq v_1$, $u_0 + v_0 \leq 2n-3$ and $u_1 + v_1 \leq 2m-3$. If $v_1 - u_1 \geq 2$, since $\tilde{f}(u) \leq u_1 nk + (n-1)k + 1$ and $\tilde{f}(v) \geq v_1 nk \geq (u_1+2)nk$, $\tilde{f}(v) - \tilde{f}(u) \geq (n+1)k - 1 \geq 2k-1 \geq j$. If $v_1 - u_1 = 1$ and u_1 is even,

$$\begin{aligned}\tilde{f}(v) - \tilde{f}(u) &= (n - u_0 - v_0 - 1)k + (v_1 - u_1)nk \\ &\geq (n - (2n-3) - 1)k + nk - 1 = 2k - 1 \geq j.\end{aligned}$$

Similarly, we can get that $\tilde{f}(v) - \tilde{f}(u) \geq j$ when $v_1 - u_1 = 1$ and u_1 is odd.

If $\mathrm{dist}(u,v) = 2$, then $u \neq v$ and $u_2 = v_2$. We can assume that $u_1 \leq v_1$. If $u_1 < v_1$, then

$$\begin{aligned}\tilde{f}(v) - \tilde{f}(u) &= v_1 nk + u_2 - (u_1 nk + (n-1)k + u_2) \\ &\geq (v_1 - u_1)nk - (n-1)k \geq k.\end{aligned}$$

If $u_1 = v_1$, then

$$|\tilde{f}(v) - \tilde{f}(u)| = |v_0 - u_0|k \geq k.$$

If $\mathrm{dist}(u,v) = 3$, since $u_2 \neq v_2$, $\tilde{f}(v) \equiv v_2 \not\equiv u_2 \equiv \tilde{f}(u) \pmod{k}$. As such, $\tilde{f}(v) \neq \tilde{f}(u)$. Thus, $\tilde{f}$ is an $L(j,k,1)$-labeling for G. Hence, $\lambda_G(j,k,1) \leq (mn-1)k+1$. ∎

Table 2 represents the $L(j,k,1)$-labeling for $K_5 \times K_4 \times K_2$ when $2 \le k \le j \le 2k-1$. We can see that $\lambda_G(j,k,1) = (mn-1)k+1 = 19k+1$.

Table 2. $L(j,k,1)$-labeling when $2 \le k \le j \le 2k-1$

$u_2 = 0$

0	$9k$	$10k$	$19k$
k	$8k$	$11k$	$18k$
$2k$	$7k$	$12k$	$17k$
$3k$	$6k$	$13k$	$16k$
$4k$	$5k$	$14k$	$15k$

$u_2 = 1$

1	$9k$+1	$10k$+1	$19k$+1
k+1	$8k$+1	$11k$+1	$18k$+1
$2k$+1	$7k$+1	$12k$+1	$17k$+1
$3k$+1	$6k$+1	$13k$+1	$16k$+1
$4k$+1	$5k$+1	$14k$+1	$15k$+1

Corollary 1. *The radio number* $\mathrm{rn}(G) = \lambda_G(3,2,1)$ *for* $G = K_n \times K_m \times K_2$ $(n \ge m \ge 3)$ *is* $2mn-1$.

Table 3 represents the radio number for $K_5 \times K_4 \times K_2$. We can see that $\mathrm{rn(G)} = 39$.

Table 3. Radio labeling

$u_2 = 0$

0	18	20	38
2	16	22	36
4	14	24	34
6	12	26	32
8	10	28	30

$u_2 = 1$

1	19	21	39
3	17	23	37
5	15	25	35
7	13	27	33
9	11	29	31

Theorem 3. *If* $k \ge 2$ *and* $j = 2k$*, then the distance 3 labeling* $\lambda_G(j,k,1)$ *for* $G = K_n \times K_m \times K_2$ $(n \ge m \ge 3)$ *is* $(mn-1)k+m$.

Proof. Let f be an $L(j,k,1)$-labeling for G. If $S = \{x_1, x_2, \cdots, x_{2mn}\}$ and $f(x_1) < f(x_2) < \cdots < f(x_{2mn})$, from Proposition 2 (3),

$$\begin{aligned}\mathrm{span}(f) =& f(x_{2mn}) - f(x_1)\\ =&(f(x_{2mn}) - f(x_{2mn-1})) + (f(x_{2mn-1}) - f(x_{2mn-3}))\\ &+ (f(x_{2mn-3}) - f(x_{2mn-5})) + \cdots + (f(x_{2mn-2n+3}) - f(x_{2mn-2n+1}))\\ &+ (f(x_{2mn-2n+1}) - f(x_{2mn-4n+1}))\\ &+ (f(x_{2mn-4n+1}) - f(x_{2mn-6n+1})) + \cdots + (f(x_{2n+1}) - f(x_1))\\ \ge& 1 + (n-1)k + (nk+1)(m-1) = (mn-1)k+m.\end{aligned}$$

Let $\tilde{f} : V \to [0, (mn-1)k+m]$, and

$$\tilde{f}(u_0,u_1,u_2) = \begin{cases} u_0k + (nk+1)u_1 + u_2, & u_1 \text{ is even} \\ (n-u_0-1)k + (nk+1)u_1 + u_2, & u_1 \text{ is odd.} \end{cases}$$

Then, using a method similar to that used in Theorems 1 and 2, we get that $\tilde{f}$ is an $L(j, k, 1)$-labeling for G. Hence, $\lambda_G(j, k, 1) = (mn-1)k + m$. ■

Table 4 represents an $L(j, k, 1)$-labeling for $K_5 \times K_4 \times K_2$ when $k \geq 2$ and $j = 2k$. We can see that $\lambda_G(j, k, 1) = (mn-1)k + m = 19k + 4$.

Table 4. $L(j, k, 1)$-labeling when $k \geq 2$ and $j = 2k$

$u_2 = 0$				$u_2 = 1$			
0	$9k$+1	$10k$+2	$19k$+3	1	$9k$+2	$10k$+3	$19k$+4
k	$8k$+1	$11k$+2	$18k$+3	k+1	$8k$+2	$11k$+3	$18k$+4
$2k$	$7k$+1	$12k$+2	$17k$+3	$2k$+1	$7k$+2	$12k$+3	$17k$+4
$3k$	$6k$+1	$13k$+2	$16k$+3	$3k$+1	$6k$+2	$13k$+3	$16k$+4
$4k$	$5k$+1	$14k$+2	$15k$+3	$4k$+1	$5k$+2	$14k$+3	$15k$+4

Theorem 4. *If* $k \geq 2$ *and* $2k + 1 \leq j \leq nk$*, then for* $G = K_n \times K_m \times K_2 (n \geq m \geq 3)$ *we have* $\lambda_G(j, k, 1) = (m-1)j + (mn - 2m + 1)k + 1$.

Proof. Let f be an $L(j, k, 1)$-labeling for G and $a_1, a_2, \cdots, a_{2mn}$ be the rearrangement of all elements of V such that $f(a_1) < f(a_2) < \cdots < f(a_{2mn})$. Since $j \leq nk$, $j + (n-2)k \leq (2n-2)k$. Hence, from Proposition 1, $f(a_{2n(i+1)+1}) - f(a_{2ni+1}) \geq j + (n-2)k$ for all $i = 0, 1, \cdots, m-1$. Thus,

$$\begin{aligned}
\mathrm{span}(f) &= f(a_{2mn}) - f(a_1) = (f(a_{2mn}) - f(a_{2mn-1})) \\
&+ (f(a_{2mn-1}) - f(a_{2mn-3})) + (f(a_{2mn-3}) - f(a_{2mn-5})) + \cdots \\
&+ (f(a_{2mn-2n+3}) - f(a_{2mn-2n+1})) + (f(a_{2mn-2n+1}) - f(a_{2mn-4n+1})) \\
&+ (f(a_{2mn-4n+1}) - f(a_{2mn-6n+1})) + \cdots + (f(a_{2n+1}) - f(a_1)) \\
&\geq 1 + (n-1)k + (m-1)(j + (n-2)k) \\
&= (m-1)j + (n - 1 + mn - 2m - n + 2)k + 1 \\
&= (m-1)j + (mn - 2m + 1)k + 1.
\end{aligned}$$

Let $\tilde{f} : V \to [0, N]$, and

$$\tilde{f}(u_0, u_1, u_2) = \begin{cases} u_0 k + (j + (n-2)k)u_1 + u_2, & u_1 \text{ is even} \\ (n - u_0 - 1)k + (j + (n-2)k)u_1 + u_2, & u_1 \text{ is odd,} \end{cases}$$

where $N = (m-1)j + (mn - 2m + 1)k + 1$. Then, as in the previous theorems, we can show that $\tilde{f}$ is an $L(j, k, 1)$-labeling for G. As such, $\lambda_G(j, k, 1) \leq (m-1)j + (mn - 2m + 1)k + 1$. ■

Table 5 represents an $L(j, k, 1)$-labeling for $K_5 \times K_4 \times K_2$ when $k \leq 2$ and $2k+1 \leq j \leq nk$. We can show that $\lambda_G(j, k, 1) = (m-1)j + (mn-2m+1)k + 1 = 3j + 13k + 1$.

Table 5. $L(j,k,1)$-labeling when $k \geq 2$ and $2k+1 \leq j \leq nk$

$u_2 = 0$

0	$j+7k+1$	$2j+6k$	$3j+13k+1$
k	$j+6k+1$	$2j+7k$	$3j+12k+1$
$2k$	$j+5k+1$	$2j+8k$	$3j+11k+1$
$3k$	$j+4k+1$	$2j+9k$	$3j+10k+1$
$4k$	$j+3k+1$	$2j+10k$	$3j+9k+1$

$u_2 = 1$

1	$j+7k$	$2j+6k+1$	$3j+13k$
$k+1$	$j+6k$	$2j+7k+1$	$3j+12k$
$2k+1$	$j+5k$	$2j+8k+1$	$3j+11k$
$3k+1$	$j+4k$	$2j+9k+1$	$3j+10k$
$4k+1$	$j+3k$	$2j+10k+1$	$3j+9k$

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References

1. A. A. Bertossi and M. C. Pinotti, Mappings for conflict-free access of paths in bidimensional arrays, circular lists and complete trees, *J. Parallel and Distributed Computing*, **62** (2002), 1314-1333.

2. A. A. Bertossi and M. C. Pinotti, Channel assignment with separation for interference avoidance in wireless networks, *IEEE Trans. Parallel Distrib. Syst.*, **14** (2003), 222-235.

3. A. A. Bertossi and M. C. Pinotti, Approximate $L(\delta_1, \delta_2, \cdots, \delta_t)$-coloring of trees and interval graphs, *Networks*, **49** (2007), 204-216.

4. A. A. Bertossi, M. C. Pinotti and R. Tan, Efficient use of radio spectrum in weireless networks with channel separation between close stations, *DIAL-M 2000 4th International Workshop on Discrete Algorithms and Methods for Mobile Computing and Computations*, 2000.

5. T. Calamoneri, E. G. Fusco, R. B. Tan and P. Vocca, $L(h,1,1)$-labeling of outerplanar graphs, *Math. Methods Oper. Res.*, **69** (2009), 307-321.

6. M. L. Chia, D. Kuo, H. Liao, C. H. Yang and R. K. Yeh, $L(3,2,1)$ labeling of graphs, *Taiwanese J. Math.*, **15** (2011), 2439-2457.

7. J. Fiala, P. Golovach, J. Kratochvil, B. Lidický and D. Paulusma, Distance three labelings of trees, *Discrete Applied Math.*, **160(6)** (2012), 764-779.

8. J. R. Griggs and R. K. Yeh, Labeling graphs with a condition at distance two, *SIAM J. Disc. Math.*, **5** (1992), 586-595.

9. W. K. Hale, Frequency assignment: theory and application, *Proc. IEEE*, **68** (1980), 1497-1514.

10. E. Haque and P. K. Jha, $L(j,k)$-labelings of Kronecker products of complete graphs, *IEEE Trans. on Cir. & Sys. II*, **55** (2008), 70-73.

11. D. S. Kim, D. Z. Du and P. M. Pardalos, A coloring problem on the n-cable, *Discrete Applied Math.*, **103** (2000), 307-311.

12. D. King, C. J. Ras and S. Zhou, The $L(h, 1, 1)$-labeling problem for trees, *European J. Combinatorics*, **31** (2010), 1295-1306.

13. D. D. Liu, Radio number for trees, *Discrete Math.*, **308** (2008), 1153-1164.

14. P. Lam, W. Lin and J. Wu, $L(j, k)-$labelings for the products of complete graphs, *J. Comb. Optimm.*, **14** (2007), 219-227.

15. X. Li, V. Mak and S. Zhou, Optimal radio labelings of complete m-ary trees, *Discrete Applied Math.*, **158** (2010), 507-515.

16. D. D. Liu and X. Zhu, Multilevel distance labelings for paths and cycles, *SIAM J. Disc. Math.*, **19** (2005), 610-621.

17. D. D. Liu and M. Xie, Radio number for square paths, *Ars. Combinatoria*, **90** (2009), 307-319.

18. A. Sen, T. Roxbrorough and S. Medidi, Upper and lower bounds of a class of channel assignment problems in circular networks, *Proc. of IEEE INFOCOM '98*, **3** (1998), 1284-1291.

19. S. Zhou, A distance-labelling problem for hypercubes, *Discrete Applied Math.*, **156** (2008), 2846-2854.

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