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POINTWISE MULTIPLIERS ON INHOMOGENEOUS BESOV AND TRIEBEL-LIZORKIN SPACES IN THE SETTING OF SPACES OF HOMOGENEOUS TYPE

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Abstract. Using the discrete Calderón reproducing formula in [9] and the Plancherel-Pôlya characterization of the inhomogeneous Besov and Triebel-Lizorkin spaces developed in [5], we prove pointwise multipliers on inhomogeneous Besov and Triebel-Lizorkin spaces in the setting of spaces of homogeneous type.

1. INTRODUCTION

The multiplier theory of function spaces has been studied for a long time and a lot of results have been obtained. We know that the multiplier theory is one of the important parts in the studies of the Gleason problem, function space properties and general operator theory. The pointwise multipliers on $\mathbb{R}^d$ are studied as a part of the researches of function spaces in several monographs, cf. [8, 12, 14-20]. Pointwise multipliers have been found many important applications in partial differential equations.

However, it was not clear how to generalize the pointwise multipliers $\mathbb{R}^d$ to spaces of homogeneous type introduced by Coifman and Weiss because the Fourier transform is no longer available. The main purpose of this paper is to establish pointwise multipliers on inhomogeneous Besov and Triebel-Lizorkin spaces in the setting of spaces of homogeneous type. We first recall some necessary definitions. A *quasi-metric* ρ on a set X is a function $\rho : X \times X \to [0, \infty)$ satisfying:

(i) $\rho(x, y) = 0$ if and only if $x = y$;
(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
(iii) There exists a constant $A \in [1, \infty)$ such that for all $x, y, z \in X$,

$$\rho(x, y) \le A[\rho(x, z) + \rho(z, y)].$$

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Any quasi-metric defines a topology, for which the balls $B(x,r) = \{y \in X : \rho(y,x) < r\}$ for all $x \in X$ and all $r > 0$ form a basis.

The following spaces of homogeneous type are variants of those introduced by Coifman and Weiss in [2].

Definition 1.1. ([4, 10]). Let $d > 0$ and $0 < \theta \leq 1$. A *space of homogeneous type* $(X, \rho, \mu)_{d,\theta}$ is a set X together with a quasi-metric ρ and a nonnegative Borel measure μ on X with supp $\mu = X$, and there exists a constant $0 < C < \infty$ such that for all $0 < r <$ diam X and all $x, x', y \in X$,

$$\mu(B(x,r)) \sim r^d, \tag{1.1}$$

$$|\rho(x,y) - \rho(x',y)| \leq C\rho(x,x')^\theta[\rho(x,y) + \rho(x',y)]^{1-\theta}. \tag{1.2}$$

Mac´as and Segovia in [13] have proved that spaces $(X, \rho, \mu)_{d,\theta}$ for $d = 1$ are just spaces of homogeneous type in the sense of Coifman and Weiss, whose definition only requires that ρ is a quasi-metric without property (1.2) and μ satisfies the doubling condition which is weaker than (1.1).

In what follows, we assume the following conventions. We denote by $f \lesssim g$ and $f \gtrsim g$ to denote $f \leq Cg$ and $f \geq Cg$, respectively. If $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $a, b \in \mathbb{R}$, set $a \wedge b \doteq \min\{a, b\}, a \vee b \doteq \max\{a, b\}$. Throughout, we also denote by C a positive constant independent of main parameters involved, which may vary at different occurrences. Constants with subscripts do not change through the whole paper. For any $q \in [1, \infty]$, we denote by q' its conjugate index, that is, $\frac{1}{q} + \frac{1}{q'} = 1$. Let B be a set and we will denote by χ_B the characteristic function of B and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

To state the definition of the inhomogeneous Besov and Triebel-Lizorkin spaces, we need the following definitions.

Definition 1.2. ([4, 10]). A sequence $\{S_k\}_{k\in\mathbb{Z}_+}$ of operators is said to be an *approximation to the identity* if $S_k(x,y)$, the kernel of S_k, is a function from $X \times X$ into $\mathbb{C}$ such that for all $k \in \mathbb{Z}_+$ and all x, x', y and y' in X, and some $0 < \epsilon \leq \theta$,

$$|S_k(x,y)| \lesssim \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{d+\epsilon}}; \tag{1.3}$$

$$|S_k(x,y) - S_k(x',y)| \lesssim \left(\frac{\rho(x,x')}{2^{-k} + \rho(x,y)}\right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{d+\epsilon}} \tag{1.4}$$

for $\rho(x,x') \leq \frac{1}{2A}(2^{-k} + \rho(x,y))$;

$$|S_k(x,y) - S_k(x,y')| \lesssim \left(\frac{\rho(y,y')}{2^{-k} + \rho(x,y)}\right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{d+\epsilon}} \tag{1.5}$$

for $\rho(y,y') \le \frac{1}{2A}(2^{-k}+\rho(x,y))$;

$$|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|$$
$$\lesssim \left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$$

for $\rho(x,x') \le \frac{1}{2A}(2^{-k}+\rho(x,y))$ and $\rho(y,y') \le \frac{1}{2A}(2^{-k}+\rho(x,y))$;

$$\int S_k(x,y)d\mu(x)=1 \tag{1.6}$$

for all $k \in \mathbb{Z}_+$;

$$\int S_k(x,y)d\mu(y)=1 \tag{1.7}$$

for all $k \in \mathbb{Z}_+$.

The existence of such an approximation to the identity is due to Coifman, See [3] for more details. The following space of test functions plays a key role in this paper; see [5, 9, 11].

Definition 1.3. Fix two exponents $0<\beta\le\theta$ and $\gamma>0$. A function f defined on X is said to be a *test function* of type (β,γ) centered at $x_0\in X$ with width $r>0$ if f satisfies the following conditions:

$$|f(x)| \le C\frac{r^{\gamma}}{(r+\rho(x,x_0))^{d+\gamma}}; \tag{1.8}$$

$$|f(x)-f(x')| \le C\left(\frac{\rho(x,x')}{r+\rho(x,x_0)}\right)^{\beta}\frac{r^{\gamma}}{(r+\rho(x,x_0))^{d+\gamma}} \tag{1.9}$$

for $\rho(x,x') \le \frac{1}{2A}(r+\rho(x,x_0))$.

If f is a test function of type (β,γ) centered at x_0 with width $r>0$, we write $f\in\mathcal{M}(x_0,r,\beta,\gamma)$, and the norm of f in $\mathcal{M}(x_0,r,\beta,\gamma)$ is defined by

$$\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} = \inf\{C\ge 0 : (1.8) \text{ and } (1.9) \text{ hold}\}.$$

We denote by $\mathcal{M}(\beta,\gamma)$ the class of all $f\in\mathcal{M}(x_0,1,\beta,\gamma)$. It is easy to see that $\mathcal{M}(x_1,r,\beta,\gamma)=\mathcal{M}(\beta,\gamma)$ with the equivalent norms for all $x_1\in X$ and $r>0$. Furthermore, it is also easy to check that $\mathcal{M}(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{M}(\beta,\gamma)$.

In what follows, we let $\widetilde{\mathcal{M}(\beta,\gamma)}$ be the completion of the space $\mathcal{M}(\theta,\theta)$ in $\mathcal{M}(\beta,\gamma)$ when $0<\beta,\gamma\le\theta$. Obviously, $\widetilde{\mathcal{M}(\theta,\theta)}=\mathcal{M}(\theta,\theta)$. Moreover, $f\in\widetilde{\mathcal{M}(\beta,\gamma)}$ if and only if $f\in\mathcal{M}(\beta,\gamma)$ when $0<\beta,\gamma\le\theta$ and there exists a sequence $\{f_n\}_{n\in\mathbb{N}}\subset$

$\mathcal{M}(\theta,\theta)$ such that $\|f-f_n\|_{\mathcal{M}(\beta,\gamma)}\to 0$ as $n\to\infty$. If $f\in\widetilde{\mathcal{M}}(\beta,\gamma)$, we then define $\|f\|_{\widetilde{\mathcal{M}}(\beta,\gamma)}=\|f\|_{\mathcal{M}(\beta,\gamma)}$. Obviously, $\widetilde{\mathcal{M}}(\beta,\gamma)$ is a Banach space and we also have $\|f\|_{\widetilde{\mathcal{M}}(\beta,\gamma)}=\lim_{n\to\infty}\|f_n\|_{\mathcal{M}(\beta,\gamma)}$ for the above $\{f_n\}_{n\in\mathbb{N}}$.

We denote by $(\widetilde{\mathcal{M}}(\beta,\gamma))'$ the dual space of $\widetilde{\mathcal{M}}(\beta,\gamma)$ consisting of all linear functionals $\mathcal{L}$ from $\widetilde{\mathcal{M}}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists a constant C such that for all $f\in\widetilde{\mathcal{M}}(\beta,\gamma)$,

$$|\mathcal{L}(f)|\leq C\|f\|_{\widetilde{\mathcal{M}}(\beta,\gamma)}.$$

We denote by $<h,f>$ the natural pairing of elements $h\in(\widetilde{\mathcal{M}}(\beta,\gamma))'$ and $f\in\widetilde{\mathcal{M}}(\beta,\gamma)$. Since $\widetilde{\mathcal{M}}(x_1,r,\beta,\gamma)=\widetilde{\mathcal{M}}(\beta,\gamma)$ with the equivalent norms for all $x_1\in X$ and $r>0$, therefore, for all $h\in(\widetilde{\mathcal{M}}(\beta,\gamma))'$, $<h,f>$ is well defined for all $f\in\widetilde{\mathcal{M}}(x_0,r,\beta,\gamma)$ with $x_0\in X$ and $r>0$.

The following constructions, which provide an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type, were given by Christ in [1].

Lemma 1.4. *Let X be a space of homogeneous type. Then there exists a collection $\{Q_\alpha^k\subset X : k\in\mathbb{Z}_+,\alpha\in I_k\}$ of open subsets, where I_k is some (possible finite) index set, and constants $\delta\in(0,1)$ and $C_1,C_2>0$ such that*

(i) *$\mu(X\setminus\cup_\alpha Q_\alpha^k)=0$ for each fixed k and $Q_\alpha^k\cap Q_\beta^k=\emptyset$ if $\alpha\neq\beta$;*

(ii) *for any α,β,k,l with $l\geq k$, either $Q_\beta^l\subset Q_\alpha^k$ or $Q_\beta^l\cap Q_\alpha^k=\emptyset$;*

(iii) *for each (k,α) and each $l<k$ there is a unique β such that $Q_\alpha^k\subset Q_\beta^l$;*

(iv) *$diam(Q_\alpha^k)\leq C_1\delta^k$;*

(v) *each Q_α^k contains some ball $B(z_\alpha^k,C_2\delta^k)$, where $z_\alpha^k\in X$.*

In fact, we can think of Q_α^k as being a dyadic cube with a diameter roughly δ^k and centered at z_α^k. In what follows, we always suppose $\delta=1/2$. See [10] for how to remove this restriction. Also, in the following, for $k\in\mathbb{Z}_+,\tau\in I_k$, we will denote by $Q_\tau^{k,\nu},\nu=1,\cdots,N(k,\tau,M)$, the set of all cubes $Q_{\tau'}^{k+M}\subset Q_\tau^k$, where M is a fixed large positive integer.

Now, we can introduce the inhomogeneous Besov spaces $B_p^{\alpha,q}(X)$ and Triebel-Lizorkin spaces $F_p^{\alpha,q}(X)$ via the approximation to the identity in Definition 1.2. Note that the Besov and Triebel-Lizorkin spaces have been already investigated for many decades in the study of partial differential equations, interpolation theory, approximation theory.

Definition 1.5. ([11]). Suppose $\{S_k\}_{k\in\mathbb{Z}_+}$ is an approximation to identity. Set $D_0=S_0$, and $D_k=S_k-S_{k-1}$ for $k\in\mathbb{N}$. Let M be a fixed large positive integer, $Q_\tau^{0,\nu}$ be as above. The *inhomogeneous Besov space* $B_p^{s,q}(X)$ for $-\epsilon<s<\epsilon$, $\max\left(\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+s}\right)<p\leq\infty$, $0<q\leq\infty$ is the collection of all $f\in(\widetilde{\mathcal{M}}(\beta,\gamma))'$ for

some β and γ satisfying:

$$\max(0, s, -s + d(\frac{1}{\min\{p,1\}} - 1)) < \beta < \epsilon, \frac{d}{\min\{p,1\}} - d < \gamma < \epsilon \tag{1.10}$$

such that

$$\|f\|_{B_p^{s,q}(X)} = \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p \right\}^{\frac{1}{p}} + \left\{ \sum_{k=1}^{\infty} \left[2^{ks} \|D_k(f)\|_{L^p(X)} \right]^q \right\}^{\frac{1}{q}} < \infty.$$

The *inhomogeneous Triebel-Lizorkin space* $F_p^{s,q}(X)$ for $-\epsilon < s < \epsilon$, $\max\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < p < \infty$ and $\max\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < q \leq \infty$ is the collection of $f \in (\widetilde{\mathcal{M}}(\beta, \gamma))'$, for some β and γ satisfying (1.10) such that

$$\|f\|_{F_p^{s,q}(X)} = \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p \right\}^{\frac{1}{p}} + \left\| \left\{ \sum_{k=1}^{\infty} \left[2^{k\alpha} |D_k(f)| \right]^q \right\}^{\frac{1}{q}} \right\|_{L^p(X)} < \infty,$$

where $m_{Q_\tau^{0,\nu}}(|D_0(f)|)$ be averages of $|D_0(f)|$ over $Q_\tau^{0,\nu}$.

The restrictions (1.10) guarantee that the definitions of the *inhomogeneous Besov space* $B_p^{s,q}(X)$ for $\max\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < p \leq \infty$, $0 < q \leq \infty$ and the *inhomogeneous Triebel-Lizorkin space* $F_p^{s,q}(X)$ for $\max\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < p < \infty$ and $\max\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < q \leq \infty$ are independent of the choices of β and γ satisfying these conditions and $B_p^{s,q}(X) \cup F_p^{s,q}(X) \subset (\widetilde{\mathcal{M}}(\beta, \gamma))'$, $\widetilde{\mathcal{M}}(\beta, \gamma) \subset \mathcal{M}(\beta, \gamma) \subset B_p^{s,q}(X) \cap F_p^{s,q}(X)$. See [Remark 2, 11] for more details.

The classical scale of inhomogeneous Besov spaces and Triebel-Lizorkin spaces contains many well-known function spaces. For example,

$$B_\infty^{\alpha,\infty}(X) = F_\infty^{\alpha,\infty}(X) = \mathcal{C}^\alpha(X), \alpha > 0,$$

where the Hölder-Zygmund space $\mathcal{C}^\alpha(X)$ is defined as the collection of f such that

$$\|f\|_{\mathcal{C}^\alpha(X)} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} < \infty.$$

If $-\theta < s < \theta$ and $1 < p < \infty$, then $H_p^s(X) = F_p^{s,2}(X)$ are the Bessel-potential spaces (Lebesgue spaces, Liouville spaces). If $m = 0, 1, 2, ...$ and $1 < p < \infty$, then $W_p^m(X) = H_p^m(X) = F_p^{m,2}(X)$ are the usual Sobolev spaces. If $\theta > s > 0, 1 < p < \infty$ and $1 \le q \le \infty$, then $B_p^{s,q}(X)$ coincides with the classical Besov-Lipschitz spaces $\Lambda_p^{s,q}(X)$. Furthermore, if $\frac{d}{d+\epsilon} < p \le 1$, then $F_p^{0,2}(X) = h^p(X)$ are the inhomogeneous Hardy spaces, which are closely related to the Hardy spaces $H^p(X)$ in [3] (More precisely: The homogeneous spaces $\dot{F}_p^{0,2}(X)$ coincide with the usual Hardy spaces $H_p(X)$). We shall use here the notation $F_p^{s,2}(X) = H_p^s(X), s \in (-\theta, \theta), \frac{d}{d+\epsilon} < p \le \infty$. Then these spaces will be denoted as the (inhomogeneous) Hardy-Sobolev spaces, which include the above Lebesgue-Sobolev spaces for $1 < p < \infty$.

The inhomogeneous Besov and Triebel-Lizorkin spaces have the following Plancherel-Pôlya characterizations in [5], which will be one of the the basic tools to prove the main results of this paper.

Lemma 1.6. *Let $s \in (-\theta, \theta)$. Let $\{D_k\}_{k\in\mathbb{Z}_+}$ be as in Definition 1.5. Then, if max$\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < p \le \infty$, $0 < q \le \infty$, for all $f \in \widetilde{\mathcal{M}}(\beta, \gamma)$ with β, γ satisfying (1.10), we have*

$$\begin{aligned}
\|f\|_{B_p^{s,q}(X)} &\sim \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p\right\}^{\frac{1}{p}} \\
&\quad + \left\{\sum_{k=1}^{\infty}\left[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)} \mu(Q_\tau^{k,\nu})\left(2^{ks}\inf_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|\right)^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}} \\
&\sim \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p\right\}^{\frac{1}{p}} \\
&\quad + \left\{\sum_{k=1}^{\infty}\left[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)} \mu(Q_\tau^{k,\nu})\left(2^{ks}\sup_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|\right)^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}.
\end{aligned}$$

If $\max\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < p < \infty$ and $\max\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < q \le \infty$, for all $f \in \widetilde{\mathcal{M}}(\beta, \gamma)$ with β, γ satisfying (1.10), we have

$$\|f\|_{F_p^{s,q}(X)} \sim \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p\right\}^{\frac{1}{p}}$$

$$+\left\|\left\{\sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)}\left[2^{ks}\inf_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|\chi_{Q_\tau^{k,\nu}}(\cdot)\right]^q\right\}^{\frac{1}{q}}\right\|_{L^p(X)}$$

$$\sim\left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p\right\}^{\frac{1}{p}}$$

$$+\left\|\left\{\sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)}\left[2^{ks}\sup_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|\chi_{Q_\tau^{k,\nu}}(\cdot)\right]^q\right\}^{\frac{1}{q}}\right\|_{L^p(X)}.$$

We now introduce the following Definition of the pointwise multiplier.

Definition 1.7. Suppose that g is a given function on X. Then g is called a pointwise multiplier for $B_p^{s,q}(X)$ if $f\to gf$ admits a bounded linear mapping from $B_p^{s,q}(X)$ into itself. Similarly, g is called a pointwise multiplier for $F_p^{s,q}(X)$ if $f\to gf$ admits a bounded linear mapping from $F_p^{s,q}(X)$ into itself.

The main results in this paper are the following theorems.

Theorem 1.8. *Let $0<\epsilon\le\theta, -\epsilon<s<\epsilon$, $\max\Big(\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+s}\Big)<p\le\infty$, $0<q\le\infty$ and $\alpha>\max\Big(s,\frac{d}{\min\{p,1\}}-d-s\Big)$, then $g\in\mathcal{C}^\alpha(X)$ with $0<\alpha<\epsilon$ is a pointwise multiplier for $B_p^{s,q}(X)$. Moreover, there exists a positive constant c such that*

$$\|gf\|_{B_p^{s,q}(X)}\le c\|g\|_{\mathcal{C}^\alpha(X)}\|f\|_{B_p^{s,q}(X)} \tag{1.11}$$

holds for all $g\in\mathcal{C}^\alpha(X)$ and $f\in B_p^{s,q}(X)$.

Theorem 1.9. *Let $0<\epsilon\le\theta, -\epsilon<s<\epsilon$, $\max\Big(\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+s}\Big)<p<\infty$ and $\max\Big(\frac{d}{d+\epsilon},\ \frac{d}{d+\epsilon+s}\Big)<q\le\infty$ and $\alpha>\max\Big(s,\frac{d}{\min\{p,q,1\}}-d-s\Big)$, then $g\in\mathcal{C}^\alpha(X)$ with $0<\alpha<\epsilon$ is a pointwise multiplier for $F_p^{s,q}(X)$. Moreover, there exists a positive constant c such that*

$$\|gf\|_{F_p^{s,q}(X)}\le c\|g\|_{\mathcal{C}^\alpha(X)}\|f\|_{F_p^{s,q}(X)} \tag{1.12}$$

holds for all $g\in\mathcal{C}^\alpha(X)$ and $f\in F_p^{s,q}(X)$.

We would like to point out that the study of pointwise multipliers is one of important problems in the theory of function spaces. It has attracted a lot of attentions in the decades since starting with [18]. Pointwise multipliers in general spaces

$$B_p^{s,q}(\mathbb{R}^d)\ \text{ and }\ F_p^{s,q}(\mathbb{R}^d),\ \text{ where }\ 0<p\le\infty, 0<q\le\infty, s\in\mathbb{R}$$

have been studied in great detail in [Ch.4, 16] and in the more recent paper [17]. As for our own contributions we refer to [2.8, 19].

Theorems 1.8 and 1.9 were proved in [19] for pointwise multipliers of inhomogeneous Besov and Triebel-Lizorkin spaces on $\mathbb{R}^d$ based on Fourier transform. In the present setting, however, we do not have the Fourier transform at our disposal. Hence the idea used in [19] does not work for this more general setting. A new idea to prove Theorem 1.8 and 1.9 is to use the discrete Calderón reproducing formula, which was developed in [9]. Therefore this scheme easily extends to geometrical settings where the Fourier transform does not exist. The Fourier transform is missing but a version of pointwise multiplier is still present.

We would also like to point out that the above restrictions for α are sharp in the following sense. Let $s \in (-\theta, \theta)$, $\max\Big(\frac{d}{d+\epsilon},\ \frac{d}{d+\epsilon+s}\Big) < p \le \infty$, $0 < q \le \infty$ and $\alpha > \max\Big(s, \frac{d}{\min\{p,1\}} - d - s\Big)$, then there exists a function $g \in \mathcal{C}^\alpha(\mathbb{R}^d)$ which is not a pointwise multiplier for $B_p^{s,q}(\mathbb{R}^d)$. The conditions $s \in (-\theta, \theta)$, $\max\Big(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\Big) < p < \infty$, $\max\Big(\frac{d}{d+\epsilon},\ \frac{d}{d+\epsilon+s}\Big) < q \le \infty$ and $\alpha > \max\Big(s, \frac{d}{\min\{p,q,1\}} - d - s\Big)$, were imposed to guarantee (1.12). Although the index is not sharp for $F_p^{s,q}(\mathbb{R}^d)$ in the above sense (see [14]), but probably $\max\Big(s, \frac{d}{\min\{p,q,1\}} - d - s\Big)$ is also natural if $q < \min\{p, 1\}$ for spaces of homogeneous type.

A brief description of the contents of this paper is as follows. In Section 2 we prove Theorem 1.8. The proof of Theorem 1.9 will be given in Section 3.

2. Proof of Theorem 1.8

In this section, we will prove Theorem 1.8. Since there is no the Fourier transforms on spaces of homogeneous type, the proof of Theorem 1.8 is quite different from the proof of Theorem 2.8.2 in [19]. The key new ingredient in the proof of Theorem 1.8 is to apply the following discrete Calderón reproducing formulae established in [9]. This formula can be stated as follows.

Lemma 2.1. *Suppose that $\{S_k\}_{k\in\mathbb{Z}_+}$ is an approximation to the identity as in Definition 1.2. Set $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. Then there exist functions $\widetilde{D}_{Q_\tau^{0,\nu}}, \tau \in I_0$ and $\nu \in \{1, \dots, N(0, \tau, M)\}$ and $\{\widetilde{D}_k(x, y)\}_{k\in\mathbb{N}}$ such that for any fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$, $k \in \mathbb{N}, \tau \in I_k$ and $\nu \in \{1, \dots, N(k, \tau, M)\}$ and all $f \in (\widetilde{\mathcal{M}}(\beta, \gamma))'$ with $0 < \beta, \gamma < \epsilon$,*

$$\begin{aligned} f(x) &= \sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) m_{Q_\tau^{0,\nu}}(D_0(f)) \widetilde{D}_{Q_\tau^{0,\nu}}(x) \\ &+ \sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau,M)} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k(x, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}) \end{aligned} \tag{2.1}$$

where $diam(Q_\tau^{k,\nu}) \sim 2^{k+M}$ *for* $k \in \mathbb{Z}_+, \tau \in I_k, \nu \in \{1, \dots, N(k,\tau,M)\}$ *and a fixed large* $M \in \mathbb{N}$, *the series converges in the norm of* $f \in B_p^{s,q}(X)$ *with* $\max\Big(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\Big) < p < \infty$, $0 < q \le \infty, -\epsilon < s < \epsilon$, *and* $F_p^{s,q}(X)$ *for* $f \in F_p^{s,q}(X)$ *with* $\max\Big(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\Big) < p < \infty$ *and* $\max\Big(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\Big) < q < \infty, -\epsilon < s < \epsilon$, *and* $\widetilde{\mathcal{M}}(\beta', \gamma')$ *for* $f \in \widetilde{\mathcal{M}}(\beta, \gamma)$ *with* $\beta' < \beta$ *and* $\gamma' < \gamma$, *and* $(\widetilde{\mathcal{M}}(\beta', \gamma'))'$ *for* $f \in (\widetilde{\mathcal{M}}(\beta, \gamma))'$ *with* $\epsilon > \beta' > \beta$ *and* $\epsilon > \gamma' > \gamma$. *Moreover,* $\widetilde{D}_k(x,y), k \in \mathbb{N}$, *satisfies for any given* $\epsilon \in (0,\theta)$, *all* $x, y \in X$ *the following conditions:*

$$|\widetilde{D}_k(x,y)| \le C\frac{2^{-k\epsilon'}}{(2^{-k}+\rho(x,y))^{d+\epsilon'}}; \tag{2.2}$$

$$|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \le C\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\epsilon'}\frac{2^{-k\epsilon'}}{(2^{-k}+\rho(x,y))^{d+\epsilon'}} \tag{2.3}$$

for $\rho(x,x') \le \frac{1}{2A}(2^{-k}+\rho(x,y))$;

$$\int_X \widetilde{D}_k(x,y)d\mu(y) = \int_X \widetilde{D}_k(x,y)d\mu(x) = 0.$$

$\widetilde{D}_{Q_\tau^{0,\nu}}(x)$ *for* $\tau \in I_0$ *and* $\nu \in \{1, \dots, N(0,\tau,M)\}$ *satisfies*

$$\int_X \widetilde{D}_{Q_\tau^{0,\nu}}(x)d\mu(x) = 1,$$

and

$$|\widetilde{D}_{Q_\tau^{0,\nu}}(x)| \le \frac{C}{(1+\rho(x,y))^{d+\epsilon}} \tag{2.4}$$

for all $x \in X$ *and* $y \in Q_\tau^{0,\nu}$ *and*

$$|\widetilde{D}_{Q_\tau^{0,\nu}}(x) - \widetilde{D}_{Q_\tau^{0,\nu}}(z)| \le C\left(\frac{\rho(x,z)}{1+\rho(x,y)}\right)^{\epsilon}\frac{1}{(1+\rho(x,y))^{d+\epsilon}} \tag{2.5}$$

for all $x, z \in X$ *and* $y \in Q_\tau^{0,\nu}$ *satisfying* $\rho(x,z) \le \frac{1}{2A}(1+\rho(x,y))$; *the constant* C *in* $(2.2)-(2.5)$ *is independent of* M.

To prove Theorem 1.8, we first show the following Lemma.

Lemma 2.2. *For any* $0 < \epsilon' < \epsilon \in (0,\theta]$, *let* $\{S_k(x,y)\}_{k\in\mathbb{Z}_+}$ *and* $\{G_k(x,y)\}_{k\in\mathbb{Z}_+}$ *be two approximations to the identity as in Lemma 2.1 above and* $D_k = S_k - S_{k-1}, E_k = G_k - G_{k-1}$ *for* $k \in \mathbb{N}$ *and* $D_0 = S_0, E_0 = G_0$. *Then for any given* $g \in \mathcal{C}^\alpha(X)$ *with* $0 < \alpha < \epsilon$,

$$(2.6)\qquad |E_k g \widetilde{D}_{Q_\tau^{0,\nu}}(x)| \lesssim \|g\|_{\mathcal{C}^\alpha(X)} 2^{-k(\epsilon'\wedge\alpha)} \frac{1}{1+\rho(x,y_\tau^{0,\nu}))^{d+\epsilon'}};$$

$$(2.7)\qquad |E_k g \widetilde{D}_{k'}(x,y)| \lesssim \|g\|_{\mathcal{C}^\alpha(X)} 2^{-|k-k'|(\epsilon'\wedge\alpha)} \frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')}+\rho(x,y))^{d+\epsilon'}},$$

where $\widetilde{D}_{Q_\tau^{0,\nu}}$ *and* $\widetilde{D}_{k'}$ *are given as in Lemma 2.1, and* $k \in \mathbb{Z}_+$ *and* $k' \in \mathbb{N}$.

Proof. We first show inequality (2.6) for the case $k = 0$. In this case, we have

$$|E_0 g \widetilde{D}_{Q_\tau^{0,\nu}}(x)| \le \|g\|_{\mathcal{C}^\alpha(X)} \int |D_0(x,z)\widetilde{D}_{Q_\tau^{0,\nu}}(z)|d\mu(z)$$

$$\lesssim \|g\|_{\mathcal{C}^\alpha(X)} \frac{1}{(1+\rho(x,y_\tau^{0,\nu})))^{d+\epsilon}}.$$

To prove the inequality (2.6) when $k \in \mathbb{N}$, we write

$$\begin{aligned}
&|E_k g \widetilde{D}_{Q_\tau^{0,\nu}}(x)| \\
&= |\int_X E_k(x,z) g(z) \widetilde{D}_{Q_\tau^{0,\nu}}(z) d\mu(z)| \\
&= |\int_X E_k(x,z)[g(z)\widetilde{D}_{Q_\tau^{0,\nu}}(z) - g(x)\widetilde{D}_{Q_\tau^{0,\nu}}(x)]d\mu(z)| \\
&\le \int_X |E_k(x,z)|[|g(z)||\widetilde{D}_{Q_\tau^{0,\nu}}(z) - \widetilde{D}_{Q_\tau^{0,\nu}}(x)| + |g(z)-g(x)||\widetilde{D}_{Q_\tau^{0,\nu}}(x)|]d\mu(z) \\
&\le \int_X |E_k(x,z)||g(z)|\widetilde{D}_{Q_\tau^{0,\nu}}(z) - \widetilde{D}_{Q_\tau^{0,\nu}}(x)|d\mu(z) \\
&\quad + \int_X |E_k(x,z)||g(z)-g(x)||\widetilde{D}_{Q_\tau^{0,\nu}}(x)|d\mu(z) \\
&\doteq B + D.
\end{aligned}$$

By the smoothness of $\widetilde{D}_{Q_\tau^{0,\nu}}(z)$ and using the fact $g \in L^\infty(X)$, it is easy to see that

$$B \lesssim \|g\|_{\mathcal{C}^\alpha(X)} 2^{-k\epsilon'} \frac{1}{(1+\rho(x,y_\tau^{0,\nu})))^{d+\epsilon}},$$

where $k \in \mathbb{N}$.

The estimate for D follows directly from

$$\begin{aligned}
D &\lesssim \|g\|_{\mathcal{C}^\alpha(X)} \int_X \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}} \rho(z,x)^\alpha \frac{1}{(1+\rho(x,y_\tau^{0,\nu}))^{d+\epsilon}} d\mu(z) \\
&\lesssim \frac{\|g\|_{\mathcal{C}^\alpha(X)} 2^{-k\alpha}}{(1+\rho(x,y_\tau^{0,\nu}))^{d+\epsilon}}.
\end{aligned}$$

This gives (2.6).

For (2.7), we only consider the case $k' \geq k \geq 0$ because the proof for the case $k \geq k' \geq 0$ is similar. We write

$$\begin{aligned} &|E_k g \widetilde{D}_{k'}(x,y)| \\ &= |\int_X E_k(x,z) g(z) \widetilde{D}_{k'}(z,y) d\mu(z)| \\ &= |\int_X [E_k(x,z) g(z) - E_k(x,y) g(y)] \widetilde{D}_{k'}(z,y) d\mu(z)| \\ &\leq \int_X |E_k(x,z) - E_k(x,y)| |g(z)| |\widetilde{D}_{k'}(z,y)| d\mu(z) \\ &\quad + \int_X |E_k(x,y)| |g(z) - g(y)| |\widetilde{D}_{k'}(z,y)| d\mu(z) \\ &\doteq E + F. \end{aligned}$$

For E, similar to the proof of B, we obtain

$$E \lesssim \|g\|_{\mathcal{C}^\alpha(X)} 2^{-(k'-k)\epsilon'} \frac{2^{-k\epsilon'}}{(2^{-k} + \rho(x,y))^{d+\epsilon'}}.$$

We then rewrite F to get

$$F \quad \lesssim \quad \|g\|_{\mathcal{C}^\alpha(X)} \int_X \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{d+\epsilon}} \rho(z,y)^\alpha \frac{2^{-k'\epsilon}}{(2^{-k'} + \rho(z,y))^{d+\epsilon}} d\mu(z).$$

Similar to the proof of D, we get

$$F \lesssim \|g\|_{\mathcal{C}^\alpha(X)} 2^{-(k'-k)\alpha} \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{d+\epsilon}},$$

where the fact $0 < \alpha < \epsilon$ is used. This finishes the proof of Lemma 2.2. ■

Now we show the following technical version of Theorem 1.8.

Proposition 2.3. *Suppose that* $-\epsilon < s < \epsilon$, *max*$\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < p \leq \infty$, $0 < q \leq \infty, \alpha > \max\left(s, \frac{d}{p\wedge 1} - d - s\right)$. *Then for any* $g \in \mathcal{C}^\alpha(X)$ *with* $0 < \alpha < \epsilon, f \in \widetilde{\mathcal{M}}(\beta, \gamma)$ *with* β *and* γ *satisfying* (1.10),

$$\|fg\|_{B_p^{s,q}(X)} \lesssim \|g\|_{\mathcal{C}^\alpha(X)} \|f\|_{B_p^{s,q}(X)}.$$

Proof. For any $g \in \mathcal{C}^\alpha(X)$, $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$ with β and γ satisfying (1.10), we have

$$\begin{aligned}\|fg\|_{B_p^{s,q}(X)} &= \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|E_0(fg)|)]^p\right\}^{\frac{1}{p}} \\ &\quad + \left\{\sum_{k=1}^{\infty}\left[2^{ks}\|E_k(fg)\|_{L^p(X)}\right]^q\right\}^{\frac{1}{q}} \\ &\doteq G+H.\end{aligned}$$

Using the discrete Calderón reproducing formula, we obtain

$$\begin{aligned}g(x)f(x) &= \sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) m_{Q_\tau^{0,\nu}}(D_0(f))g(x)\widetilde{D}_{Q_\tau^{0,\nu}}(x) \\ &\quad + \sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau,M)} \mu(Q_\tau^{k,\nu})g(x)\widetilde{D}_k(x,y_\tau^{k,\nu})D_k(f)(y_\tau^{k,\nu}).\end{aligned}$$

Applying Lemma 2.2 implies

$$\begin{aligned}G &\le \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})\left[\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'})|E_0 g\widetilde{D}_{Q_\tau^{0,\nu}}(y_{\tau'}^{0,\nu'})||m_{Q_{\tau'}^{0,\nu'}}(D_0(f))|\right]^p\right\}^{\frac{1}{p}} \\ &\quad + \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})\right. \\ &\qquad \left.\left[\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)} \mu(Q_{\tau'}^{k',\nu'})|E_0 g\widetilde{D}_{k'}(y_\tau^{0,\nu},y_{\tau'}^{k',\nu'})||D_{k'}(f)(y_{\tau'}^{k',\nu'})|\right]^p\right\}^{\frac{1}{p}} \\ &\lesssim \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\left[\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)} \frac{\|g\|_{\mathcal{C}^\alpha(X)}}{(1+\rho(y_\tau^{0,\nu},y_{\tau'}^{0,\nu'}))^{d+\epsilon'}}|m_{Q_{\tau'}^{0,\nu'}}(D_0(f))|\right]^p\right\}^{\frac{1}{p}} \\ &\quad + \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\left[\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)} \frac{2^{-k'(d+\epsilon'\wedge\alpha)}\|g\|_{\mathcal{C}^\alpha(X)}}{(1+\rho(y_\tau^{0,\nu},y_{\tau'}^{k',\nu'}))^{d+\epsilon'}}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|\right]^p\right\}^{\frac{1}{p}}.\end{aligned}$$

Applying the Hölder inequality for $p>1$ and

$$\left(\sum_k |a_k|\right)^p \le \sum_k |a_k|^p \tag{2.8}$$

for all $a_k \in \mathbb{C}$ and $p \le 1$, it follows that

$$\begin{aligned}
G &\lesssim \|g\|_{\mathcal{C}^\alpha(X)} \left\{ \sum_{\tau\in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \sum_{\tau'\in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} \frac{\mu(Q_\tau^{0,\nu})\mu(Q_{\tau'}^{0,\nu'})}{(1+\rho(y_\tau^{0,\nu}, y_{\tau'}^{0,\nu'}))^{(d+\epsilon')(p\wedge 1)}} |m_{Q_{\tau'}^{0,\nu'}}(D_0(f))|^p \right\}^{\frac{1}{p}} \\
&\quad + \|g\|_{\mathcal{C}^\alpha(X)} \left\{ \sum_{k'=1}^{\infty} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau',M)} 2^{-k'(d+s+\epsilon'\wedge\alpha)(p\wedge 1)} 2^{k'd} \right. \\
&\quad \left. \times \sum_{\tau\in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) \frac{1}{(1+\rho(y_\tau^{0,\nu}, y_{\tau'}^{k',\nu'}))^{(d+\epsilon')(p\wedge 1)}} \mu(Q_{\tau'}^{k',\nu'}) |2^{k's} D_{k'}(f)(y_{\tau'}^{k',\nu'})|^p \right\}^{\frac{1}{p}} \\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)} \left\{ \sum_{\tau'\in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'}) |m_{Q_{\tau'}^{0,\nu'}}(D_0(f))|^p \right\}^{\frac{1}{p}} \\
&\quad + \|g\|_{\mathcal{C}^\alpha(X)} \left\{ \sum_{k'=1}^{\infty} 2^{-k'(d+s+\epsilon'\wedge\alpha)(p\wedge 1)} 2^{k'd} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau',M)} \mu(Q_{\tau'}^{k',\nu'}) |2^{k's} D_{k'}(f)(y_{\tau'}^{k',\nu'})|^p \right\}^{\frac{1}{q}} \\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)} \left\{ \sum_{\tau'\in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'}) |m_{Q_{\tau'}^{0,\nu'}}(D_0(f))|^p \right\}^{\frac{1}{p}} \\
&\quad + \|g\|_{\mathcal{C}^\alpha(X)} \left\{ \sum_{k'=1}^{\infty} 2^{-k'(d+\epsilon'\wedge\alpha)(p\wedge 1)(\frac{q}{p}\wedge 1)} 2^{-k's(p\wedge 1)(\frac{q}{p}\wedge 1)} 2^{k'd(\frac{q}{p}\wedge 1)} \right. \\
&\quad \left. \times \left(\sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau',M)} \mu(Q_{\tau'}^{k',\nu'}) |2^{k's} D_{k'}(f)(y_{\tau'}^{k',\nu'})|^p \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)} \|f\|_{B_p^{s,q}(X)},
\end{aligned}$$

where we use the facts that $s > -(\epsilon' \wedge \alpha)$ when $p > 1$, and $\frac{d}{d+\epsilon'\wedge\alpha+s} < p \le \infty$ when $\max\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < p \le 1$.

Similarly, by Lemma 2.1 and Lemma 2.2, we write

$$\begin{aligned}
H &\le \left\{ \sum_{k=1}^{\infty} \left[2^{ks} \| \sum_{\tau'\in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'}) m_{Q_{\tau'}^{0,\nu'}}(D_0(f)) E_k g \widetilde{D}_{Q_{\tau'}^{0,\nu'}}(\cdot) \|_{L^p(X)} \right]^q \right\}^{\frac{1}{q}} \\
&\quad + \left\{ \sum_{k=1}^{\infty} \left[2^{ks} \| \sum_{k'=1}^{\infty} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau',M)} \mu(Q_{\tau'}^{k',\nu'}) E_k g \widetilde{D}_{k'}(\cdot, y_{\tau'}^{k',\nu'}) D_{k'}(f)(y_{\tau'}^{k',\nu'}) \|_{L^p(X)} \right]^q \right\}^{\frac{1}{q}} \\
&\doteq H_1 + H_2.
\end{aligned}$$

For the term H_1, by the estimate on $E_k g\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(z)$, when $s < \epsilon' \wedge \alpha$, we have,

$$\begin{aligned} H_1 &\lesssim \left\{ \sum_{k=1}^{\infty} \left[\int_X \sum_{\tau' \in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} [m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)]^p \frac{\|g\|_{\mathcal{C}^\alpha(X)}^p 2^{ksp} 2^{-k(\epsilon' \wedge \alpha)p}}{(1+\rho(x, y_{\tau'}^{0,\nu'}))^{(d+\epsilon')(p\wedge 1)}} d\mu(x) \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\ &\lesssim \|g\|_{\mathcal{C}^\alpha(X)} \left\{ \sum_{k=1}^{\infty} 2^{ksq} 2^{-k(\epsilon' \wedge \alpha)q} \left[\sum_{\tau' \in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'}) [m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)]^p \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\ &\lesssim \|g\|_{\mathcal{C}^\alpha(X)} \|f\|_{B_p^{s,q}(X)}. \end{aligned}$$

To obtain the estimate of H_2, by Lemma 2.2, it follows that

$$\begin{aligned} H_2 &\lesssim \left\{ \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \left[2^{k'd} 2^{-k'd(p\wedge 1)} 2^{-|k'-k|(\epsilon' \wedge \alpha)(p\wedge 1)} 2^{(k-k')s(p\wedge 1)} 2^{-(k\wedge k')d(1-(p\wedge 1))} \right]^{\frac{q}{p}\wedge 1} \right. \\ &\quad \left. \times \|g\|_{\mathcal{C}^\alpha(X)} \left[\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau',M)} \left(\mu(Q_{\tau'}^{k',\nu'})^{-\frac{s}{d}+\frac{1}{p}} \sup_{z \in Q_{\tau'}^{k',\nu'}} |D_{k'}(f)(z)| \right)^p \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\ &\lesssim \|g\|_{\mathcal{C}^\alpha(X)} \|f\|_{B_p^{s,q}(X)}, \end{aligned}$$

where we use the facts that $s < \epsilon' \wedge \alpha$ if $p > 1$ and $\frac{d}{d+(\epsilon' \wedge \alpha)+s} < p$ if $p \leq 1$.
This verifies Proposition 2.3. ■

Since for $f \in B_p^{s,q}(X)$, in general, f could be a distribution and hence the multiplication gf, even for $g \in \mathcal{C}^\alpha(X)$, does not make sense. For this propose, we need the following lemma. The proof of Theorem 1.8 then follows from Proposition 2.3 and this lemma.

Lemma 2.4. *For any $f \in B_p^{s,q}(X)$ with max$\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < p \leq \infty$, $0 < q \leq \infty$, $-\epsilon < s < \epsilon$, and $g \in \mathcal{C}^\alpha(X)$ with $\epsilon > \alpha > \max\left(s, \frac{d}{p\wedge 1} - d - s\right)$. There exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \in \widetilde{\mathcal{M}}(\epsilon,\epsilon)$, $\|f_n\|_{B_p^{s,q}(X)} \lesssim \|f\|_{B_p^{s,q}(X)}$ and $\lim_{n\to\infty} \langle gf_n, h\rangle$ converges for any $h \in \widetilde{\mathcal{M}}(\beta,\gamma)$ with β and γ satisfying* (1.10).

Assuming Lemma 2.4 for the moment, for $g \in \mathcal{C}^\alpha(X)$ and $f \in B_p^{s,q}(X)$, $\lim_{n\to\infty} \langle gf_n, h\rangle$ exists, where f_n is given by Lemma 2.4. Therefore, for $g \in \mathcal{C}^\alpha(X)$, $f \in B_p^{s,q}(X)$, we can define

$$\langle gf, h\rangle = \lim_{n\to\infty} \langle gf_n, h\rangle$$

for $h \in \mathcal{M}(\beta,\gamma)$ with (β,γ) satisfying (1.10). Here f_n is a sequence given by Lemma 2.4 and the limit is independent of the choice of f_n.

Fatou's lemma and Proposition 2.3 imply

$$\|gf\|_{B_p^{s,q}(X)} \le \liminf_{n\to\infty} \|gf_n\|_{B_p^{s,q}(X)} \lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|f\|_{B_p^{s,q}(X)},$$

which completes the proof of Theorem 1.8.

Therefore, it remains to show lemma 2.4. To this end, we need the following Lemma.

Lemma 2.5. *Let* $\{S_k(x,y)\}_{k\in\mathbb{Z}_+}$ *be an approximation to the identity of order* ϵ *as in Lemma 2.1 above and* $D_k = S_k - S_{k-1}$ *for* $k \in \mathbb{N}$ *and* $D_0 = S_0$. *Then for any* $g \in \mathcal{C}^\alpha(X)$ *with* $0<\alpha<\epsilon$, $h \in \widetilde{\mathcal{M}}(\beta,\gamma)$ *with* β *and* γ *satisfying (1.10),*

$$|\langle \widetilde{D}_{Q_\tau^{0,\nu}} g, h\rangle| \lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)} \frac{1}{(1+\rho(y_\tau^{0,\nu},x_0))^{d+\gamma}}; \tag{2.9}$$

$$|\langle \widetilde{D}_k(\bullet,y) g, h\rangle| \lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)} 2^{-k(\beta\wedge\alpha)} \frac{1}{(1+\rho(y,x_0))^{d+\gamma'}}, \tag{2.10}$$

where $k\in\mathbb{N}, \gamma' = \gamma\wedge(\epsilon-\alpha)$.

Proof. We first prove inequality (2.9). In fact, we have

$$\begin{aligned}
&|\langle \widetilde{D}_{Q_\tau^{0,\nu}} g, h\rangle| \\
&\le \|g\|_{\mathcal{C}^\alpha(X)} \int_X |h(z)||\widetilde{D}_{Q_\tau^{0,\nu}}(z)|d\mu(z) \\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)} \frac{1}{(1+\rho(y_\tau^{0,\nu},x_0))^{d+\gamma}} \\
&\qquad \int_{\{z:\rho(y_\tau^{0,\nu},z)\le \frac{1}{2A}\rho(x_0,y_\tau^{0,\nu})\}} \frac{1}{(1+\rho(y_\tau^{0,\nu},z))^{d+\epsilon}} d\mu(z) \\
&\quad + \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)} \frac{1}{(1+\rho(y_\tau^{0,\nu},x_0))^{d+\gamma}} \\
&\qquad \int_{\{z:\rho(y_\tau^{0,\nu},z)> \frac{1}{2A}\rho(x_0,y_\tau^{0,\nu})\}} \frac{1}{(1+\rho(x_0,z))^{d+\gamma}} d\mu(z) \\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)} \frac{1}{(1+\rho(y_\tau^{0,\nu},x_0))^{d+\gamma}}.
\end{aligned}$$

That is, (2.9) holds.

To prove (2.10), we write

$$\begin{aligned}
&|\langle \widetilde{D}_k(\bullet,y) g, h\rangle| \\
&= \left|\int_X \widetilde{D}_k(x,y)g(x)h(x)d\mu(x)\right| \\
&= \left|\int_X \widetilde{D}_k(x,y)[g(x)h(x)-g(y)h(y)]d\mu(x)\right| \\
&\le \int_X |\widetilde{D}_k(x,y)||g(x)-g(y)||h(x)|d\mu(x) + \int_X |\widetilde{D}_k(x,y)||g(y)||h(x)-h(y)|]d\mu(x) \\
&\doteq I + II.
\end{aligned}$$

For I, we have

$$\begin{aligned}
I &\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}\int_X \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y)))^{d+\epsilon}}\rho(y,x)^\alpha \frac{1}{(1+\rho(x,x_0))^{d+\gamma}}d\mu(x)\\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k\alpha}\int_X \frac{2^{-k(\epsilon-\alpha)}}{(2^{-k}+\rho(x,y)))^{d+\epsilon-\alpha}}\frac{1}{(1+\rho(x,x_0))^{d+\gamma}}d\mu(x)\\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k\alpha}\frac{1}{(1+\rho(y,x_0))^{d+\gamma}}\int_{W_1} \frac{2^{-k(\epsilon-\alpha)}}{(2^{-k}+\rho(x,y)))^{d+\epsilon-\alpha}}d\mu(x)\\
&\quad +\|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k\alpha}\frac{1}{(1+\rho(y,x_0))^{d+\epsilon-\alpha}}\int_{W_2} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}}d\mu(x)\\
&\lesssim 2^{-k\alpha}\|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}\frac{1}{(1+\rho(y,x_0))^{d+\gamma'}},
\end{aligned}$$

where $W_1=\{z:\rho(x,y)<\frac{1}{2A}(1+\rho(x_0,y))\}$, $W_2=\{z:\rho(x,y)\geq\frac{1}{2A}(1+\rho(x_0,y))\}$ and $\gamma'=\gamma\wedge(\epsilon-\alpha)$.

To obtain the estimate of the II, we have

$$\begin{aligned}
II &\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}\int_X \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}\frac{\rho(x,y)^\beta}{(1+\rho(y,x_0))^{d+\gamma+\beta}}d\mu(x)\\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k\beta}\frac{1}{(1+\rho(y,x_0))^{d+\gamma+\beta}}\\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k\beta}\frac{1}{(1+\rho(y,x_0))^{d+\gamma}}.
\end{aligned}$$

Combining the estimate of I and II, (2.10) holds. This finishes the proof of Lemma 2.5. ■

Now we show Lemma 2.4.

Proof of Lemma 2.4. For any $f\in B_p^{s,q}(X)$, with $\max\left(\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+s}\right)<p\leq\infty$, $0<q\leq\infty$, $-\epsilon<s<\epsilon$, we denote

$$\begin{aligned}
f_n \doteq &\sum_{\tau=1}^{n}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q_\tau^{0,\nu})m_{Q_\tau^{0,\nu}}(D_0(f))\widetilde{D}_{Q_\tau^{0,\nu}}(x)\\
&+\sum_{k=1}^{n}\sum_{\tau=1}^{n}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q_\tau^{k,\nu})\widetilde{D}_k(x,y_\tau^{k,\nu})D_k(f)(y_\tau^{k,\nu}).
\end{aligned}$$

It is easy to see that $f_n\in\widetilde{\mathcal{M}}(\epsilon,\epsilon)$. To see $\|f_n\|_{B_p^{s,q}(X)}\leq C\|f\|_{B_p^{s,q}(X)}$, by the definition of $B_p^{s,q}(X)$, we write

$$\begin{aligned}\|f_n\|_{B_p^{s,q}(X)} &= \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|E_0(f_n)|)]^p\right\}^{\frac{1}{p}}\\ &\quad+\left\{\sum_{k=1}^{\infty}\left[2^{ks}\|E_k(f_n)\|_{L^p(X)}\right]^q\right\}^{\frac{1}{q}}\\ &\doteq J+K.\end{aligned}$$

Using the basic estimates of $E_0\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(x)$ and $E_0\widetilde{D}_{k'}(x,y)$ in Lemma 2.2 or [9], we obtain

$$\begin{aligned}J &\lesssim \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\left[\sum_{\tau'=1}^{n}\sum_{\nu'=1}^{N(0,\tau',M)}\frac{1}{(1+\rho(y_\tau^{0,\nu},y_{\tau'}^{0,\nu'}))^{d+\epsilon'}}|m_{Q_{\tau'}^{0,\nu'}}(D_0(f))|\right]^p\right\}^{\frac{1}{p}}\\ &\quad+\left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\left[\sum_{k'=1}^{n}\sum_{\tau'=1}^{n}\sum_{\nu'=1}^{N(k',\tau',M)}\frac{2^{-k'(d+\epsilon')}}{(1+\rho(y_\tau^{0\nu},y_{\tau'}^{k',\nu'}))^{d+\epsilon'}}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|\right]^p\right\}^{\frac{1}{p}}.\end{aligned}$$

Applying the Plancherel-Pôlya characterization of the inhomogeneous Besov spaces developed in [4] and the Hölder inequality for $p>1$ and (2.8) for $p\le 1$, it follows that

$$\begin{aligned}J &\lesssim \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\sum_{\tau'=1}^{n}\sum_{\nu'=1}^{N(0,\tau',M)}\frac{1}{(1+\rho(y_\tau^{0,\nu},y_{\tau'}^{0,\nu'}))^{(d+\epsilon')(p\wedge 1)}}|m_{Q_{\tau'}^{0,\nu'}}(D_0(f))|^p\right\}^{\frac{1}{p}}\\ &\quad+\left\{\sum_{k'=1}^{n}2^{-k'(d+\epsilon')(p\wedge1)(\frac{q}{p}\wedge1)}2^{-k's(p\wedge1)(\frac{q}{p}\wedge1)}2^{k'd(\frac{q}{p}\wedge1)}\right.\\ &\quad\times\left.\left(\sum_{\tau'=1}^{n}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})|2^{k's}D_{k'}(f)(y_{\tau'}^{k',\nu'})|^p\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\ &\lesssim \|f\|_{B_p^{s,q}(X)},\end{aligned}$$

where $\frac{d}{d+\epsilon'+s}<p\le\infty$.

Similarly, we write

$$K\lesssim\left\{\sum_{k=1}^{\infty}\left[2^{ks}\|\sum_{\tau'=1}^{n}\sum_{\nu'=1}^{N(0,\tau',M)}\mu(Q_{\tau'}^{0,\nu'})m_{Q_{\tau'}^{0,\nu'}}(D_0(f))E_k\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(\cdot)\|_{L^p(X)}\right]^q\right\}^{\frac{1}{q}}$$

$$+\left\{\sum_{k=1}^{\infty}\left[2^{ks}\|\sum_{k'=1}^{n}\sum_{\tau'=1}^{n}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})E_k\widetilde{D}_{k'}(\cdot,y_{\tau'}^{k',\nu'})D_{k'}(f)(y_{\tau'}^{k',\nu'})\|_{L^p(X)}\right]^q\right\}^{\frac{1}{q}}$$

$$\doteq K_1+K_2.$$

For the term K_1, from the basic estimate $E_k\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(z)$ in Lemma 2.2 or [9], we have when $s<\epsilon'$,

$$K_1\lesssim\left\{\sum_{k=1}^{\infty}\left[\int_X\sum_{\tau'=1}^{n}\sum_{\nu'=1}^{N(0,\tau',M)}[m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)]^p\frac{2^{ksp}2^{-k\epsilon' p}}{(1+\rho(x,y_{\tau'}^{0,\nu'}))^{(d+\epsilon')(p\wedge 1)}}d\mu(x)\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

$$\lesssim\left\{\sum_{k=1}^{\infty}2^{ksq}2^{-k\epsilon' q}\left[\sum_{\tau'=1}^{n}\sum_{\nu'=1}^{N(0,\tau',M)}\mu(Q_{\tau'}^{0,\nu'})[m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)]^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

$$\lesssim\|f\|_{B_p^{s,q}(X)}.$$

To obtain the estimate of K_2, by the basic estimate (2.7) of Lemma 2.2, it follows that

$$K_2\lesssim\left\{\sum_{k=1}^{\infty}\sum_{k'=1}^{n}\left[2^{k'd}2^{-k'd(p\wedge 1)}2^{-|k'-k|\epsilon'(p\wedge 1)}2^{(k-k')s(p\wedge 1)}2^{-(k\wedge k')d(1-(p\wedge 1))}\right]^{\frac{q}{p}\wedge 1}\right.$$

$$\left.\times\left[\sum_{\tau'=1}^{n}\sum_{\nu'=1}^{N(k',\tau',M)}\left(\mu(Q_{\tau'}^{k',\nu'})^{-\frac{s}{d}+\frac{1}{p}}\sup_{z\in Q_{\tau'}^{k',\nu'}}|D_{k'}(f)(z)|\right)^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

$$\lesssim\|f\|_{B_p^{s,q}(X)},$$

where $s<\epsilon'$, $\frac{d}{d+\epsilon'+s}<p\leq\infty$. That is, $\|f_n\|_{B_p^{s,q}(X)}\lesssim\|f\|_{B_p^{s,q}(X)}$ holds.

Next we prove that $\lim_{n\to\infty}\langle gf_n,h\rangle$ converges for any $h\in\widetilde{\mathcal{M}}(\beta,\gamma)$ with β and γ satisfying (1.10). We consider the following four cases respectively:

(I) $1<p<\infty$ and $1<q<\infty$;
(II) $1<p<\infty$ and $0<q\leq 1$;
(III) $1<p<\infty$ and $q=\infty$ or $p=\infty$ and $0<q\leq\infty$;
(IV) $\max\left(\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+s}\right)<p\leq 1$.

Suppose $g\in\mathcal{C}^\alpha(X), h\in\widetilde{\mathcal{M}}(\beta,\gamma)$ for β and γ satisfying (1.10), $n,m\in\mathbb{N}, m<n$. We now consider the case (I). By duality and Proposition 2.3, we have

$$|\langle f_n-f_m,gh\rangle|\leq\|f_n-f_m\|_{B_p^{s,q}}\|gh\|_{B_{p'}^{-s,q'}}\lesssim\|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{B_{p'}^{-s,q'}(X)}\|f_n-f_m\|_{B_p^{s,q}}.$$

Note that $\|h\|_{B^{-s,q'}_{p'}(X)} \lesssim \|h\|_{\widetilde{\mathcal{M}}(\beta,\gamma)}$ and $\|f_n - f_m\|_{B^{s,q}_p}$ tends to zero as n, m tend to infinity. This implies that $|\langle f_n - f_m, gh\rangle| \to 0$ as $n, m \to \infty$ when $s \in (-\epsilon, \epsilon)$, $1 < p < \infty$ and $1 < q < \infty$ and hence the case (I) is concluded.

For case (II), by (2.9) and (2.10), we have

$$\sum_{\tau=m+1}^{n} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q^{0,\nu}_\tau)| < \widetilde{D}_{Q^{0,\nu}_\tau}, (gh) > |^{p'}$$
$$\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)} \left\{ \sum_{i\geq \log 2^m} \int_{2^i<\rho(x_0,y)\leq 2^{i+1}} \frac{1}{(1+\rho(y,x_0))^{(d+\gamma')p'}} d\mu(y) \right\}^{\frac{1}{p'}}$$
$$\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)} m^{-[\gamma'+\frac{d}{p}]},$$

and

$$\sup_{k\in\mathbb{N}, m+1\leq k\leq n} 2^{-ks}\|\widetilde{D}^*_k(gh)\|_{L^{p'}(X)}$$
$$\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)} \sup_{m+1\leq k} 2^{-k[s+\beta\wedge\alpha]} \left\{ \int_X \frac{1}{(1+\rho(y,x_0))^{(d+\gamma')p'}} d\mu(y) \right\}^{\frac{1}{p'}}$$
$$\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)} 2^{-m[s+\beta\wedge\alpha]},$$

and

$$\sup_{k\in\mathbb{N}, 1\leq k\leq n} \left(\sum_{\tau=m+1}^{n} \sum_{\nu=1}^{N(k,\tau,M)} \mu(Q^{k,\nu}_\tau)|2^{-ks}\widetilde{D}^*_k(gh)(y^{k,\nu}_\tau)|^{p'} \right)^{\frac{1}{p'}}$$
$$\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)} \sup_{k\in\mathbb{N}, 1\leq k\leq n} 2^{-k[s+\beta\wedge\alpha]} m^{-[\gamma'+\frac{d}{p}]}$$
$$\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)} m^{-[\gamma'+\frac{d}{p}]}.$$

Applying the Hölder inequality for $p > 1$ and (2.8) for $q \leq 1$, and (2.9) and (2.10), it follows that

$$|\langle f_n - f_m, gh\rangle|$$
$$\leq \left\{ \sum_{\tau=m+1}^{n} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q^{0,\nu}_\tau)[m_{Q^{0,\nu}_\tau}(|D_0(f)|)]^p \right\}^{\frac{1}{p}}$$
$$\left\{ \sum_{\tau=m+1}^{n} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q^{0,\nu}_\tau)|\widetilde{D}_{Q^{0,\nu}_\tau}(gh)|^{p'} \right\}^{\frac{1}{p'}}$$

$$
\begin{aligned}
&+\left\{\sum_{k=m+1}^{n}\left[2^{ks}\|D_k(f)\|_{L^p(X)}\right]^q\right\}^{\frac{1}{q}}\sup_{k\in\mathbb{N},m+1\le k\le n}2^{-ks}\|\widetilde{D}_k^*(gh)\|_{L^{p'}(X)}\\
&+\left\{\sum_{k=1}^{n}\left(\sum_{\tau=m+1}^{n}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q_\tau^{k,\nu})|2^{ks}D_k(f)(y_\tau^{k,\nu})|^p\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&\times\sup_{k\in\mathbb{N},1\le k\le n}\left(\sum_{\tau=m+1}^{n}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q_\tau^{k,\nu})|2^{-ks}\widetilde{D}_k^*(gh)(y_\tau^{k,\nu})|^{p'}\right)^{\frac{1}{p'}}\\
&\lesssim\ \|g\|_{\mathcal{C}^\alpha(X)}\|f\|_{B_p^{s,q}(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}\left\{m^{-(\frac{d}{p}+\gamma')}+2^{-m(\beta\wedge\alpha+s)}\right\}.
\end{aligned}
$$

Using the fact that $s+\beta\wedge\alpha>0$, we see that for $s\in(-\epsilon,\epsilon)$, $1<p<\infty$ and $0<q\le 1$, $|\langle f_n-f_m,gh\rangle|\to 0$ as $n,m\to\infty$.

For case (III), if $p=\infty, q=\infty$, we obtain

$$
\begin{aligned}
&|\langle f_n-f_m,gh\rangle|\\
\le&\sup_{x\in X}|D_0(f)(x)|\left|\sum_{\tau=m+1}^{n}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q_\tau^{0,\nu})\widetilde{D}_{Q_\tau^{0,\nu}}(gh)\right|\\
&+\sup_{k\in\mathbb{N}}\sup_{x\in X}|2^{ks}D_k(f)(x)|\sum_{k=m+1}^{n}\int_X|2^{-ks}\widetilde{D}_k^*(gh)(x)|d\mu(x)\\
&+\sup_{k\in\mathbb{N}}\sup_{x\in X}|2^{ks}D_k(f)(x)|\left|\sum_{k=1}^{n}\sum_{\tau=m+1}^{n}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q_\tau^{k,\nu})2^{-ks}\widetilde{D}_k^*(gh)(y_\tau^{k,\nu})\right|\\
\le&\|f\|_{B_\infty^{s,\infty}(X)}\left\{\left|\sum_{\tau=m+1}^{n}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q_\tau^{0,\nu})\widetilde{D}_{Q_\tau^{0,\nu}}(gh)\right|\right.\\
&\qquad+\sum_{k=m+1}^{n}\int_X|2^{-ks}\widetilde{D}_k^*(gh)(x)|d\mu(x)\\
&\qquad\left.+\left|\sum_{k=1}^{n}\sum_{\tau=m+1}^{n}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q_\tau^{k,\nu})2^{-ks}\widetilde{D}_k^*(gh)(y_\tau^{k,\nu})\right|\right\}.
\end{aligned}
$$

Applying Proposition 2.3, then $gh\in B_1^{-s,1}(X)$. Note that the terms in brace are remainder of $(gh)_n-(gh)_m$ in the norm of $B_1^{-s,1}(X)$, which go to zero as n,m tend to infinity. Thus $|\langle f_n-f_m,gh\rangle|\to 0$ as $n,m\to\infty$ when $s\in(-\epsilon,\epsilon)$, $p=\infty$ and $q=\infty$.

The estimate for case $\infty > p > 1, q = \infty$ or $p = \infty, q \neq \infty$ is similar to case (III) above.

For case (IV), applying (2.8) for $\max\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < p \leq 1$, we have

$$\begin{aligned}
&|\langle f_n - f_m, gh\rangle| \\
&\leq \left\{ \sum_{\tau=m+1}^{n} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p \right\}^{\frac{1}{p}} \sup_{m<\rho(x_0,y_\tau^{0,\nu})\leq n} |\widetilde{D}_{Q_\tau^{0,\nu}}(gh)| \\
&+ \sum_{k=m+1}^{n} 2^{ks}\|D_k(f)\|_{L^p(X)} \sup_{y\in X} 2^{-ks}\mu(Q_\tau^{k,\nu})^{1-\frac{1}{p}}|\widetilde{D}_k^*(gh)(y)| \\
&+ \sum_{k=1}^{n} 2^{ks}\|D_k(f)\|_{L^p(X)} \sup_{m<\rho(x_0,y)\leq n} 2^{-ks}\mu(Q_\tau^{k,\nu})^{1-\frac{1}{p}}|\widetilde{D}_k^*(gh)(y)|.
\end{aligned}$$

By (2.10), we have

$$\begin{aligned}
&\sup_{m<\rho(x_0,y_\tau^{k,\nu})\leq j} 2^{-ks}\mu(Q_\tau^{k,\nu})^{1-\frac{1}{p}}|\widetilde{D}_k^*(gh)(_\tau^{k,\nu})| \\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k[s+\beta\wedge\alpha-n(\frac{1}{p}-1)]} \sup_{m<\rho(x_0,y_\tau^{k,\nu})\leq j} \frac{1}{(1+\rho(y_\tau^{k,\nu},x_0))^{d+\gamma'}} \\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k[s+\beta\wedge\alpha-n(\frac{1}{p}-1)]}m^{-[\gamma'+d]},
\end{aligned}$$

and

$$\sup_{y_\tau^{k,\nu}\in X} 2^{-ks}\mu(Q_\tau^{k,\nu})^{1-\frac{1}{p}}|\widetilde{D}_k^*(gh)(_\tau^{k,\nu})| \lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k[s+\beta\wedge\alpha-n(\frac{1}{p}-1)]}.$$

From (2.9) and (2.10), and the Hölder inequality for $\infty \geq q > 1$, it follows that

$$\begin{aligned}
&|\langle f_n - f_m, gh\rangle| \\
&\leq \left\{ \sum_{\tau=m+1}^{n} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p \right\}^{\frac{1}{p}} \sup_{m<\rho(x_0,y_\tau^{0,\nu})\leq n} |\widetilde{D}_{Q_\tau^{0,\nu}}(gh)| \\
&+ \left\{ \sum_{k=m+1}^{n} \left[2^{ks}\|D_k(f)\|_{L^p(X)}\right]^q \right\}^{\frac{1}{q}} \left\{ \sum_{k=m+1}^{n} \left[\sup_{y\in X} 2^{-k\left[s-d\left(\frac{1}{p}-1\right)\right]}|\widetilde{D}_k^*(gh)(y)|\right]^{q'} \right\}^{\frac{1}{q'}} \\
&+ \left\{ \sum_{k=1}^{n} \left[2^{ks}\|D_k(f)\|_{L^p(X)}\right]^q \right\}^{\frac{1}{q}} \left\{ \sum_{k=1}^{n} \left[\sup_{m<\rho(x_0,y)\leq n} 2^{-k\left[s-d\left(\frac{1}{p}-1\right)\right]}|\widetilde{D}_k^*(gh)(y)|\right]^{q'} \right\}^{\frac{1}{q'}} \\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}\|f\|_{B_p^{s,q}(X)} \left[m^{-(d+\gamma')} + 2^{-m\left[s-d\left(\frac{1}{p}-1\right)\right]}2^{-m(\alpha\wedge\beta)}\right].
\end{aligned}$$

From (2.9) and (2.10), and (2.8) for $0 < q \le 1$, we also have

$$
\begin{aligned}
&|\langle f_n - f_m, gh\rangle| \\
&\le \left\{ \sum_{\tau=m+1}^{n} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p \right\}^{\frac{1}{p}} \sup_{y_\tau^{0,\nu}\in\{y_\tau^{0,\nu}|m<\rho(x_0,y_\tau^{0,\nu})\le n\}} |\widetilde{D}_{Q_\tau^{0,\nu}}(gh)| \\
&+ \left\{ \sum_{k=m+1}^{n} \left[2^{ks}\|D_k(f)\|_{L^p(X)}\right]^q \right\}^{\frac{1}{q}} \sup_{k\in\mathbb{N}, m<k\le n} \sup_{y\in X} 2^{-ks}\mu(Q_\tau^{k,\nu})^{\frac{1}{p'}} |\widetilde{D}_k^*(gh)(y)| \\
&+ \left\{ \sum_{k=1}^{n} \left[2^{ks}\|D_k(f)\|_{L^p(X)}\right]^q \right\}^{\frac{1}{q}} \sup_{k\in\mathbb{N}, 1\le k\le n} \sup_{y\in\{y|m<\rho(x_0,y)\le n\}} 2^{-ks}\mu(Q_\tau^{k,\nu})^{\frac{1}{p'}} |\widetilde{D}_k^*(gh)(y)| \\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}\|f\|_{B_p^{s,q}(X)} \left[m^{-(d+\gamma')} + 2^{-m\left[s-d\left(\frac{1}{p}-1\right)\right]}2^{-m(\alpha\wedge\beta)}\right]
\end{aligned}
$$

where we use the arbitrariness of $y_\tau^{k,\nu}$, and $\beta\wedge\alpha > \frac{d}{p} - d - s$ when $p \le 1$. This proves $|\langle f_n - f_m, gh\rangle| \to 0$ as $n, m \to \infty$ when $s \in (-\epsilon, \epsilon)$, $\max\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < p \le 1$ and $0 < q \le \infty$, and hence the proof of Lemma 2.4 is concluded. ■

3. Proof of Theorem 1.9

In this section, we prove Theorem 1.9. Roughly speaking, the idea of the proof is similar to Theorem 1.8. To be more precise, we first show the following result which is similar to Proposition 2.3.

Proposition 3.1. *For any $g \in \mathcal{C}^\alpha(X)$, $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$ with β and γ satisfying* (1.10),

$$\|fg\|_{F_p^{s,q}(X)} \lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|f\|_{F_p^{s,q}(X)},$$

where $0 < \epsilon \le \theta, -\epsilon < s < \epsilon$, max$\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < p < \infty$ and max$\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right) < q \le \infty$ and $\epsilon > \alpha > \max\left(s, \frac{d}{\min\{p,q,1\}} - d - s\right)$.

Proof. Using the Calderón reproducing formula, for any $g \in \mathcal{C}^\alpha(X)$, $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$, we write

$$
\begin{aligned}
&\|gf\|_{F_p^{s,q}(X)} \\
&\le \left\{ \sum_{\tau\in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \left[\sum_{\tau'\in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} m_{Q_\tau^{0,\nu}}(|E_0 g\widetilde{D}_{Q_{\tau'}^{0,\nu'}}|)|m_{Q_{\tau'}^{0,\nu'}}(D_0(f))| \right]^p \right\}^{\frac{1}{p}} \\
&\quad + \left\{ \sum_{\tau\in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \left[\sum_{k'=1}^{\infty} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau',M)} 2^{-k'd} m_{Q_\tau^{0,\nu}}(|E_0 g\widetilde{D}_{k'}(\cdot, y_{\tau'}^{k',\nu'})|)|D_{k'}(f)(y_{\tau'}^{k',\nu'})| \right]^p \right\}^{\frac{1}{p}}
\end{aligned}
$$

$$+\left\|\left\{\sum_{k=1}^{\infty}2^{ksq}\left[\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}|E_kg\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(\cdot)|m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)\right]^q\right\}^{\frac{1}{q}}\right\|_{L^p(X)}$$

$$+\left\|\left\{\sum_{k=1}^{\infty}2^{ksq}\left[\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}2^{-k'd}|E_kg\widetilde{D}_{k'}(\cdot,y_{\tau'}^{k',\nu'})||D_{k'}(f)(y_{\tau'}^{k',\nu'})|\right]^q\right\}^{\frac{1}{q}}\right\|_{L^p(X)}$$

$$\doteq L_1+L_2+L_3+L_4.$$

The estimate of L_1 is the same as the corresponding part of Besov spaces in Proposition 2.3. We only prove the estimates of L_2, L_3 and L_4. From [Lemma A.2, 7], the inequality (2.7), the Hölder inequality for $q>1$ and (2.8) for $q\le 1$, the Fefferman-Stein vector-valued maximal function inequality in [6], it follows that

$$\begin{aligned}L_2 &\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\left\|\sum_{k'=1}^{\infty}2^{-k'(d+s+\epsilon'\wedge\alpha-\frac{d}{r})}\left[\boldsymbol{M}\left(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}2^{k'sr}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|^r\chi_{Q_{\tau'}^{k',\nu'}}\right)\right]^{\frac{1}{r}}\right\|_{L^p(X)}\\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\left\|\left\{\sum_{k'=1}^{\infty}2^{-k'(d+s+\epsilon'\wedge\alpha-\frac{d}{r})(q\wedge 1)}\right.\right.\\
&\quad\times\left.\left.\left[\boldsymbol{M}\left(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}2^{k'sr}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|^r\chi_{Q_{\tau'}^{k',\nu'}}\right)\right]^{\frac{q}{r}}\right\}^{\frac{1}{q}}\right\|_{L^p(X)}\\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|f\|_{F_p^{s,q}(X)},\end{aligned}$$

where we have chosen r satisfying $\max\left(\frac{d}{d+\epsilon},\frac{d}{d+s+\epsilon'\wedge\alpha}\right)<r<\min\{1,p,q\}$.

By the inequality (2.6), similarly,

$$\begin{aligned}L_3 &\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\left\|\left\{\sum_{k=1}^{\infty}2^{-k(\epsilon'\wedge\alpha-s)q}\left[\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}\frac{1}{(1+\rho(\cdot,y_{\tau'}^{0,\nu'})^{d+\epsilon'}}m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)\right]^q\right\}^{\frac{1}{q}}\right\|_{L^p(X)}\\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\left\{\int_X\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}\frac{1}{(1+\rho(x,y_{\tau'}^{0,\nu'})^{(d+\epsilon')(p\wedge 1)}}[m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)]^p d\mu(x)\right\}^{\frac{1}{p}}\\
&\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\|f\|_{F_p^{s,q}(X)},\end{aligned}$$

where s can be any number in $(-\epsilon,\epsilon'\wedge\alpha)$.

To get the estimate of L_4, using Lemma A.2 in [7], the equality (2.7), the Hölder inequality if $q>1$ and (2.8) if $q\le 1$, the Fefferman-Stein vector-valued maximal function inequality in [6], we thus have

$$L_4\lesssim \|g\|_{\mathcal{C}^\alpha(X)}\left\|\left\{\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty}\left[2^{(k-k')s}2^{-|k'-k|(\epsilon'\wedge\alpha)}2^{-k'd}2^{(k\wedge k')d}2^{[k'-(k\wedge k')]d/r}\right]^{q\wedge 1}\right.\right.$$

$$\times\left[\boldsymbol{M}\left(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})^{-\frac{s}{d}}|D_{k'}(f)(y_{\tau'}^{k',\nu'})\chi_{Q_{\tau'}^{k',\nu'}}|\right)^r\right]^{\frac{q}{r}}\Bigg\}^{\frac{1}{q}}\Bigg\|_{L^p(X)}$$

$$\lesssim\|g\|_{\mathcal{C}^\alpha(X)}\left\|\left\{\sum_{k'=1}^{\infty}\left[\boldsymbol{M}\left(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})^{-\frac{\alpha}{d}}|D_{k'}(f)(y_{\tau'}^{k',\nu'}|^r\chi_{Q_{\tau'}^{k',\nu'}}\right)\right]^{\frac{q}{r}}\right\}^{\frac{1}{q}}\right\|_{L^p(X)}$$

$$\lesssim\|g\|_{\mathcal{C}^\alpha(X)}\|f\|_{F_p^{\alpha,q}(X)},$$

where $\max\left(\frac{d}{d+\epsilon},\frac{d}{d+s+\epsilon'\wedge\alpha}\right)<r<\min\{1,p,q\}$, s can be any number in $(-\epsilon,\epsilon\wedge\alpha)$. This verifies Proposition 3.1. ■

The following result is similar to Lemma 2.4.

Lemma 3.2. *For any $f\in F_p^{s,q}(X)$ with max$\left(\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+s}\right)<p<\infty$ and max$\left(\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+s}\right)<q\le\infty$, $-\epsilon<s<\epsilon$, and $g\in\mathcal{C}^\alpha(X)$ with $\epsilon>\alpha>\max\left(s,\frac{d}{\min\{p,q,1\}}-d-s\right)$. There exist a constant C and a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $\|f_n\|_{F_p^{s,q}(X)}\le C\|f\|_{F_p^{s,q}(X)}$ and $\lim_{n\to\infty}\langle gf_n,h\rangle$ converges for any $h\in\widetilde{\mathcal{M}}(\beta,\gamma)$ with β and γ satisfying* (1.10).

Proof. For any $f\in F_p^{s,q}(X)$ with $\max\left(\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+s}\right)<p<\infty$ and $\max\left(\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+s}\right)<q\le\infty$, $-\epsilon<s<\epsilon$, and $g\in\mathcal{C}^\alpha(X)$ with $\alpha>\max\left(s,\frac{d}{\min(p,q,1)}-d-s\right)$, define f_n as in Lemma 2.4. Applying the same proof as in Lemma 2.4 gives $\|f_n\|_{F_p^{s,q}(X)}\le C\|f\|_{F_p^{s,q}(X)}$. We write

$$\begin{aligned}|\langle f_n-f_m,gh\rangle|\le&\left|\sum_{\tau=m+1}^{n}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q_\tau^{0,\nu})m_{Q_\tau^{0,\nu}}(D_0(f))\langle\widetilde{D}_{Q_\tau^{0,\nu}},gh\rangle\right|\\&+\left|\sum_{k=m+1}^{n}\sum_{\tau=1}^{n}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q_\tau^{k,\nu})\langle\widetilde{D}_k(\cdot,y_\tau^{k,\nu}),gh\rangle D_k(f)(y_\tau^{k,\nu})\right|\\&+\left|\sum_{k=1}^{n}\sum_{\tau=m+1}^{n}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q_\tau^{k,\nu})\langle\widetilde{D}_k(\cdot,y_\tau^{k,\nu}),gh\rangle D_k(f)(y_\tau^{k,\nu})\right|\\&\doteq R+T+Y.\end{aligned}$$

The estimate of R is similar to the corresponding part of Lemma 2.4. We now consider the estimate of T. By Hölder inequality if $p,q>1$, and (2.8) if $p,q\le1$, it follows that

$$T \le \begin{cases} \|f\|_{F_p^{s,q}(X)} \left\| \left\{ \sum_{k=m+1}^{n} |2^{-ks} \widetilde{D}_k^*(gh)|^{q'} \right\}^{\frac{1}{q'}} \right\|_{L_{p'}(X)} & : \ p>1, q>1, \\ \|f\|_{B_p^{s-\varepsilon,\infty}(X)} \sum_{k=m+1}^{n} \sup_{x\in X} |2^{k[(-s+\varepsilon)+d(\frac{1}{p}-1)]} \widetilde{D}_k^*(gh)(x)| & : \ p \le 1, q>1, \\ \|f\|_{F_p^{s,q}(X)} \left\| \sup_{m+1\le k\le n} |2^{-ks} \widetilde{D}_k^*(gh)| \right\|_{L_{p'}(X)} & : \ p>1, q\le 1, \\ \|f\|_{B_p^{s,1}(X)} \sup_{x\in X} \sup_{m+1\le k\le n} |2^{-k[s+d(1-\frac{1}{p})]} \widetilde{D}_k^*(gh)(x)| & : \ p\le 1, q\le 1, \end{cases}$$

$$\lesssim \begin{cases} \|f\|_{F_p^{s,q}(X)} \left\| \left\{ \sum_{k=m+1}^{n} |2^{-ks} \widetilde{D}_k^*(gh)|^{q'} \right\}^{\frac{1}{q'}} \right\|_{L_{p'}(X)}, & : \ p>1, q>1, \\ \|g\|_{\mathcal{C}^\alpha(X)} \|f\|_{F_p^{s,q}(X)} \|h\|_{\mathcal{M}(\beta,\gamma)} 2^{-m[s-\varepsilon+d(1-\frac{1}{p})+(\alpha\wedge\beta)]}, & : \ p\le 1, q>1, \\ \|g\|_{\mathcal{C}^\alpha(X)} \|f\|_{F_p^{s,q}(X)} \|h\|_{\mathcal{M}(\beta,\gamma)} 2^{-m[s+\alpha\wedge\beta]}, & : \ p>1, q\le 1, \\ \|g\|_{\mathcal{C}^\alpha(X)} \|f\|_{F_p^{s,q}(X)} \|h\|_{\mathcal{M}(\beta,\gamma)} 2^{-m[s+d(1-\frac{1}{p})+(\alpha\wedge\beta)]}, & : \ p\le 1, q\le 1 \end{cases},$$

where ε is a positive number with $s-\varepsilon+d(1-\frac{1}{p})+\alpha\wedge\beta>0$ when $p\le 1$ and $s+\alpha\wedge\beta>0$ if $p>1$, and the second inequality can be obtained by using the fact that

$$F_p^{s,q}(X) \subset B_p^{s,\max(p,q)}(X) \subset B_p^{s-\varepsilon,\infty}(X),$$

when $-\epsilon < s-\varepsilon$, see [11] and [19].

By Proposition 3.1, $gh \in F_{p'}^{-s,q'}(X)$ if $s\in(-\epsilon,\epsilon), p,q>1$. Thus by the Calderón reproducing that the series converges in the norm of $F_{p'}^{-s,q'}(X)$ for $gh\in F_{p'}^{-s,q'}(X)$ with $s\in(-\epsilon,\epsilon), p,q>1$ in Lemma 2.1, which proves $T\to 0$ as $n,m\to\infty$ when $\max\left(\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+s}\right)<p<\infty$, $\max\left(\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+s}\right)<q\le\infty$, $-\epsilon<s<\epsilon$, and $g\in\mathcal{C}^\alpha(X)$ with $\alpha>\max\left(s,\frac{d}{\min\{p,q,1\}}-d-s\right)$.

For Y, applying the Hölder inequality for $p,q>1$ and (2.8) for $p,q\le 1$ and (2.9) and (2.10), we also have

$$
Y \leq \begin{cases} \|f\|_{F_p^{s,q}(X)} \left\{ \int_{\{y|m<\rho(x_0,y)\leq n\}} \left\{ \sum_{k=1}^{n} |2^{-ks} \widetilde{D}_k^*(gh)(y)|^{q'} \right\}^{\frac{p'}{q'}} d\mu(y) \right\}^{\frac{1}{p'}} & : p>1, q>1, \\ \|f\|_{B_p^{s-\varepsilon,\infty}(X)} \sup_{k\in\mathbb{N}} \sup_{y\in\{y|m<\rho(x_0,y)\leq n\}} |2^{-k[s-\varepsilon+d(1-\frac{1}{p})]} \widetilde{D}_k^*(gh)(y)| & : p\leq 1, q>1, \\ \|f\|_{F_p^{s,q}(X)} \left\{ \int_{\{y|m<\rho(x_0,y)\leq n\}} \left\{ \sup_{1\leq k\leq n} |2^{-ks} \widetilde{D}_k^*(gh)(y)| \right\}^{p'} d\mu(y) \right\}^{\frac{1}{p'}} & : p>1, q\leq 1, \\ \|f\|_{B_p^{s,1}(X)} \sup_{y\in\{y|m<\rho(x_0,y)\leq n\}} \sup_{1\leq k\leq n} |2^{-k[s+d(1-\frac{1}{p})]} \widetilde{D}_k^*(gh)(y)| & : p\leq 1, q\leq 1, \end{cases}
$$

$$
\lesssim \begin{cases} \|g\|_{\mathcal{C}^\alpha(X)} \|f\|_{F_p^{s,q}(X)} \|h\|_{\mathcal{M}(\beta,\gamma)} m^{-(\gamma'+\frac{d}{p})} \left\{ \sum_{k=1}^{n} |2^{-k(s+\alpha\wedge\beta)q'} \right\}^{\frac{1}{q'}} & : p>1, q>1, \\ \|g\|_{\mathcal{C}^\alpha(X)} \|f\|_{F_p^{s,q}(X)} \|h\|_{\mathcal{M}(\beta,\gamma)} m^{-(d+\gamma')} \left\{ \sup_{k\in\mathbb{N}} 2^{-k[s-\varepsilon+d(1-\frac{1}{p})+\alpha\wedge\beta]q'} \right\}^{\frac{1}{q'}} & : p\leq 1, q>1, \\ \|g\|_{\mathcal{C}^\alpha(X)} \|f\|_{F_p^{s,q}(X)} \|h\|_{\mathcal{M}(\beta,\gamma)} m^{-(\gamma'+\frac{d}{p})} \sup_{k\in\mathbb{N}} 2^{-k[s+\alpha\wedge\beta]} & : p>1, q\leq 1, \\ \|g\|_{\mathcal{C}^\alpha(X)} \|f\|_{F_p^{s,q}(X)} \|h\|_{\mathcal{M}(\beta,\gamma)} m^{-(d+\gamma')} \sup_{k\in\mathbb{N}} 2^{-k[s+d(1-\frac{1}{p})+\alpha\wedge\beta]} & : p\leq 1, q\leq 1 \end{cases}
$$

$$
\lesssim \|g\|_{\mathcal{C}^\alpha(X)} \|f\|_{F_p^{s,q}(X)} \|h\|_{\mathcal{M}(\beta,\gamma)} \left[m^{-(\gamma'+\frac{d}{p})} + m^{-(d+\gamma')} \right],
$$

where ε is a positive number with $s-\varepsilon+d(1-\frac{1}{p})+\alpha\wedge\beta>0$ when $p\leq 1$ and $s+\alpha\wedge\beta>0$ if $p>1$, and we also use the fact that $F_p^{s,q}(X)\subset B_p^{s,\max(p,q)}(X)\subset B_p^{s-\varepsilon,\infty}(X)$, when $-\epsilon<s-\varepsilon$. This finishes the proof of $Y\to 0$ as $n,m\to\infty$, and hence the proof of Lemma 3.2 is concluded. ■

The above estimate shows $\lim_{n\to\infty}\langle gf_n, h\rangle$ exists and the limit is independent of the choice of f_n. Therefore, for $g\in\mathcal{C}^\alpha(X), f\in F_p^{s,q}(X)$ with $\max\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right)<p<\infty$ and $\max\left(\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+s}\right)<q\leq\infty$ $s\in(-\epsilon,\epsilon), \alpha>\max\left(s, \frac{d}{\min\{p,q,1\}}-d-s\right)$, we define

$$
\langle gf, h\rangle = \lim_{n\to\infty}\langle gf_n, h\rangle
$$

for any $h\in\widetilde{\mathcal{M}}(\beta,\gamma)$ with β and γ satisfying (1.10), f_n is fundamental sequence defined in Lemma 2.4.

We now prove Theorem 1.9.

Proof of Theorem 1.9. By Proposition 3.1 and Lemma 3.2, for any $g\in\mathcal{C}^\alpha(X), f\in F_p^{s,q}(X)$, Fatou's lemma implies that

$$\begin{aligned}
&\|gf\|_{F_p^{s,q}(X)}\\
&\leq \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|\lim_{n\to\infty}E_0(f_ng)|)]^p\right\}^{\frac{1}{p}}\\
&\quad+\left\|\left\{\sum_{k=1}^{\infty}\left[2^{k\alpha}\lim_{n\to\infty}|E_k(f_ng)|\right]^q\right\}^{\frac{1}{q}}\right\|_{L^p(X)}\\
&\leq \liminf_{n\to\infty}\|gf_n\|_{F_p^{s,q}(X)}\lesssim\|g\|_{C^\alpha(X)}\|f\|_{F_p^{s,q}(X)}.
\end{aligned}$$

We complete the proof of Theorem 1.9. ∎

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