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WILLMORE SURFACES AND F-WILLMORE SURFACES IN SPACE FORMS

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Abstract. Let M^2 be a compact F-Willmore surface in the n-dimensional space form $\mathbb{N}^n(c)$ of constant curvature c. Denote by ϕ_{ij}^{α} the trace free part of the second fundamental form $h = (h_{ij}^{\alpha})$, and by $\mathbf{H}$ the mean curvature vector of M^2. Let Φ be the square of the length of ϕ_{ij}^{α} and $H = |\mathbf{H}|$. If $F'(\Phi) \geq 0$, then $\int_M \Big\{F''(\Phi)\Big[\frac{1}{2}|\nabla\Phi|^2 - \sum_{\alpha,i,j}\phi_{ij}^{\alpha}\Phi_j H_i^{\alpha}\Big] + F(\Phi)H^2 + F'(\Phi)(2c - K(n)\Phi)\Phi\Big\}dv \leq 0$. The constant function $K(n) = 1$ when $n = 3$ and $K(n) = \frac{3}{2}$ when $n \geq 4$. Similarly, $\int_M \Big\{F''(\Phi)\Big[\frac{1}{2}|\nabla\Phi|^2 - \sum_{\alpha,i,j}\phi_{ij}^{\alpha}\Phi_j H_i^{\alpha}\Big] + F(\Phi)H^2 + F'(\Phi)(2c - K(n)\Phi)\Phi\Big\}dv \geq 0$, if $F'(\Phi) \leq 0$. We also prove the following: If M^2 is a compact Willmore surface in the n-dimensional space form $\mathbb{N}^n(c)$. Then $\int_M \Phi(C(n)(c + \frac{H^2}{2}) - \Phi) \leq 0$, where $C(n) = 2$ when $n = 3$ and $C(n) = \frac{4}{3}$ when $n \geq 4$. If $0 \leq \Phi \leq C(n)(c + \frac{H^2}{2})$, then either $\Phi = 0$ and M is totally umbilical sphere, or $\Phi = C(n)(c + \frac{H^2}{2})$. In the latter case, either M is the Clifford torus in S^3 of $\mathbb{N}^n(c)$, or M is the Veronese surface in S^4 of $\mathbb{N}^n(c)$.

1. Introduction

Let $\mathbb{N}^n(c)$ be an n-dimensional space form of constant curvature c, namely,

$$\mathbb{N}^n(c) = \begin{cases} \mathbb{S}^n(c) = \{x \in \mathbb{R}^{n+1} : \langle x, x\rangle = \frac{1}{c}\}, & \text{if } c > 0, \\ \mathbb{R}^n, & \text{if } c = 0, \\ \mathbb{H}^n(c) = \{x \in \mathbb{R}_1^{n+1} : \langle x, x\rangle_1 = \frac{1}{c}, x^{n+1} > 0\}, & \text{if } c < 0, \end{cases}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^{n+1}$ and

$$\langle x, y\rangle_1 = x^1y^1 + \cdots + x^ny^n - x^{n+1}y^{n+1}$$

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is the standard Lorentzian inner product on $\mathbb{R}_1^{n+1}$. When $c = 1, 0, -1$, $\mathbb{N}^n(c)$ is the standard unit sphere $S^n(1)$, the Euclidean space $\mathbb{R}^n$ and the hyperbolic space $H^n(-1)$, respectively. Let M^2 be a compact surface in the n-dimensional space form $\mathbb{N}^n(c)$ and h be the second fundamental form of M^2. If S denotes the square of the length of the second fundamental form, $\mathbf{H}$ denotes the mean curvature vector, and H denotes the mean curvature of M^2, then we have

$$S = |h|^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2, \quad \mathbf{H} = \sum_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{2}\sum_i h_{ii}^\alpha, \quad H = |\mathbf{H}|,$$

where e_α, $3 \le \alpha \le n$, are orthonormal vector fields of M^2 in $\mathbb{N}^n(c)$. Denote by ϕ_{ij}^α the tensor $h_{ij}^\alpha - H^\alpha \delta_{ij}$ of the trace free part of the second fundamental form h and Φ the square of the length of (ϕ_{ij}^α).

The Willmore functional is defined by

$$W(x) = \int_M \Phi dv.$$

This functional is invariant under conformal transformations of $\mathbb{N}^n(c)$. The critical surfaces of W are called Willmore surface. More precisely, M^2 is a Willmore surface if and only if

$$\triangle^\perp H^\alpha + \sum_{\beta,i,j} h_{ij}^\alpha h_{ij}^\beta H^\beta - 2H^2 H^\alpha = 0, \quad 3 \le \alpha \le n,$$

where $\triangle^\perp$ is the Laplacian in the normal bundle NM (see [1, 11, 15] and Theorem 3.5). In other words, M^2 is a Willmore surface if and only if

$$\triangle^\perp H^\alpha + \sum_{\beta,i,j} \phi_{ij}^\alpha \phi_{ij}^\beta H^\beta = 0, \qquad 3 \le \alpha \le n.$$

In the theory of Willmore surfaces in $S^n(c)$, the following integral inequality is well known.

Theorem 1.1. ([9]). *Let M^2 be a compact Willmore surface in an n-dimensional unit sphere $S^n(1)$. Then*

$$\int_M \Phi\Big(C(n) - \Phi\Big) dv \le 0,$$

where $C(n) = 2$ when $n = 3$ and $C(n) = \frac{4}{3}$ when $n \ge 4$. In particular, if

$$0 \le \Phi \le C(n),$$

then either $\Phi = 0$ and M is totally umbilical sphere, or $\Phi = C(n)$. In the latter case, either M is the Clifford torus or M is the Veronese surface.

Chang and Hsu improved the integral inequality and extended the above result, they proved the following theorem.

Theorem 1.2. ([5]). *Let M^2 be a compact Willmore surface in an n-dimensional unit sphere $S^n(1)$. Then*

$$\int_M \Phi\Big(C(n)(1+\frac{H^2}{2})-\Phi\Big)dv \leq 0,$$

where $C(n)=2$ when $n=3$ and $C(n)=\frac{4}{3}$ when $n\geq 4$. In particular, if

$$0 \leq \Phi \leq C(n)(1+\frac{H^2}{2}),$$

then either $\Phi=0$ and M is totally umbilical sphere, or $\Phi=C(n)(c+\frac{H^2}{2})$. In the latter case, either $n=3$ and M is the Clifford torus, or $n=4$ and M is the Veronese surface.

The first main result of this paper, we shall extend Chang and Hsu's result to space forms and prove the following theorem:

Theorem 1.3. *Let M^2 be a compact Willmore surface in the n-dimensional space form $\mathbb{N}^n(c)$ of constant curvature c. Then*

$$\int_M \Phi\Big(C(n)(c+\frac{H^2}{2})-\Phi\Big)dv \leq 0,$$

where $C(n)=2$ when $n=3$ and $C(n)=\frac{4}{3}$ when $n\geq 4$. In particular, if

$$0 \leq \Phi \leq C(n)(c+\frac{H^2}{2}),$$

then either $\Phi=0$ and M is totally umbilical sphere, or $\Phi=C(n)(c+\frac{H^2}{2})$. In the latter case, either M is the Clifford torus in $S^3(c)$ of $\mathbb{N}^n(c)$, or M is the Veronese surface in $S^4(c)$ of $\mathbb{N}^n(c)$.

Recently, many other types of Willmore functional are studied. In [2], Cai studied the following p-Willmore functional

$$W_p(x)=\int_M \Phi^p dv.$$

Liu and Jian (see [10]) introduced the F-Willmore functional of submanifold in space forms, which is defined as

$$W_F(x)=\int_M F(\Phi)dv,$$

where F is a given function satisfying

$$F \in C^3, \qquad F : [0, \infty) \to \mathbb{R}.$$

Obviously, the Willmore functional $W(x)$ and p-Willmore functional $W_p(x)$ are two special cases of F-Willmore functional $W_F(x)$.

The critical point of $W_F(x)$ is called F-Willmore submanifold. For F-Willmore submanifold in space forms, Liu and Jian proved the following main result.

Theorem 1.4. ([10]). *Let M be an m-dimensional compact F-Willmore submanifold in space form $\mathbb{N}^n(c)$. Set*

$$b_k = inf_{u\in[0,C]} F^{(k)}(u), \qquad k = 1, 2,$$

where C is the positive constant such that $0 < \Phi < C$ on M. If $b_k > 0$, $k = 1, 2$, then

$$\int_M \Big[m^2H^2F(\Phi) - 2m\Phi H^2F'(\Phi) + 2mc\Phi F'(\Phi) - 2(2 - \frac{1}{n-m})\Phi^2F'(\Phi)\Big]dv \le 0.$$

Remark. If M is a compact F-Willmore surface in the unit sphere $S^n(1)$ and $F(\Phi) = \Phi$, the integral inequality implies that

$$\int_M \Phi\Big(C(n) - \Phi\Big)dv \le 0,$$

i.e., Theorem 1.1 is a corollary of Theorem 1.4. But note that Theorem 1.4 can not imply Theorem 1.2. So the integral inequality of Theorem 1.4 maybe can be improved.

In this paper, we shall consider a compact F-Willmore surface M in the space form $\mathbb{N}^n(c)$ and improve some result of Theorem 1.4. The second main result of this paper will improve the integral inequality of Theorem 1.4 in the case of F-Willmore surface and imply the Theorem 1.2 and Theorem 1.3. More precisely, the second main result is the following theorem.

Theorem 1.5. *Let M^2 be a compact F-Willmore surface in the n-dimensional space form $\mathbb{N}^n(c)$ of constant curvature c. If $F'(\Phi) \ge 0$, then*

$$\int_M \Big\{F''(\Phi)\Big[\frac{1}{2}|\nabla\Phi|^2 - \sum_{\alpha,i,j}\phi^\alpha_{ij}\Phi_jH^\alpha_i\Big] + F(\Phi)H^2 + F'(\Phi)(2c - K(n)\Phi)\Phi\Big\}dv \le 0.$$

If $F'(\Phi) \le 0$, then

$$\int_M \Big\{F''(\Phi)\Big[\frac{1}{2}|\nabla\Phi|^2 - \sum_{\alpha,i,j}\phi^\alpha_{ij}\Phi_jH^\alpha_i\Big] + F(\Phi)H^2 + F'(\Phi)(2c - K(n)\Phi)\Phi\Big\}dv \ge 0.$$

The constant function $K(n) = 1$ when $n = 3$ and $K(n) = \frac{3}{2}$ when $n \ge 4$.

Remark. If $\Phi \neq 0$, the equality holds if and only if either M^2 is minimal when $n \geq 4$ or $\phi_{111} = \phi_{122} = \frac{H_1}{2}$ and $\phi_{222} = \phi_{211} = \frac{H_2}{2}$ when $n = 3$.

Corollary 1.6. *Let M^2 be a compact F-Willmore surface in the n-dimensional space form $\mathbb{N}^n(c)$ of constant curvature c.*

(1) *Assume that $F'(\Phi) \geq 0$.; If $F''(\Phi) = 0$ or $F''(\Phi)$ and $\frac{1}{2}|\nabla\Phi|^2 - \sum_{\alpha,i,j} \phi_{ij}^\alpha \Phi_j H_i^\alpha$ have the same sign, then*

$$\int_M \Big\{F(\Phi)H^2 + F'(\Phi)(2c - K(n)\Phi)\Phi\Big\}dv \leq 0.$$

(2) *Assume that $F'(\Phi) \leq 0$. If $F''(\Phi) = 0$ or $F''(\Phi)$ and $\frac{1}{2}|\nabla\Phi|^2 - \sum_{\alpha,i,j} \phi_{ij}^\alpha \Phi_j H_i^\alpha$ have different signs, then*

$$\int_M \Big\{F(\Phi)H^2 + F'(\Phi)(2c - K(n)\Phi)\Phi\Big\}dv \geq 0.$$

Remark. If M is a compact F-Willmore surface in the space form $\mathbb{N}^n(c)$ and $F(\Phi) = \Phi$, the integral inequality can be written by

$$\int_M (2c + H^2 - K(n)\Phi)\Phi dv \leq 0.$$

This implies that

$$\int_M \Phi\Big(C(n)(c + \frac{H^2}{2}) - \Phi\Big)dv \leq 0,$$

i.e., Theorem 1.3 and Theorem 1.2 hold.

2. Notations and Auxiliary Lemmas

Throughout this paper, let M^2 be a compact surface isometrically immersed in the n-dimensional complete and simply connected space form $\mathbb{N}^n(c)$ with constant curvature c. We shall use the following ranges of indices

$$1 \leq A, B, C, \cdots \leq n, \qquad 1 \leq i, j, k, \cdots \leq 2, \qquad 3 \leq \alpha, \beta, \gamma, \cdots \leq n.$$

Choose a local orthonormal frame field $\{e_A\}$ in $\mathbb{N}^n(c)$ such that, restricted to M, $\{e_i\}$ are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection $1-$forms of $\mathbb{N}^{n+p}(c)$ respectively. Then the structure equations are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0,$$
$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} \tilde{R}_{ABCD}\omega_C \wedge \omega_D,$$

where $\tilde{R}_{ABCD}$ are the components of the Riemannian curvature tensor of $\mathbb{N}^n(c)$. We restrict to a neighborhood of $x : M^2 \hookrightarrow \mathbb{N}^n(c)$. Let θ_A, θ_{AB} be the restriction of $\{\omega_A\}$ and $\{\omega_{AB}\}$ to M. Then we have $\theta_\alpha = 0$. Taking its exterior derivative, we get

$$\sum_i \theta_{\alpha i} \wedge \theta_i = 0.$$

By Cartan's lemma we have

$$\theta_{i\alpha} = \sum_j h^\alpha_{ij}\theta_j, \qquad h^\alpha_{ij} = h^\alpha_{ji},$$

from which we can defined the second fundamental form h and the mean curvature vector $\mathbf{H}$ of $x : M^2 \hookrightarrow \mathbb{N}^n(c)$ as following

$$h = \sum_{\alpha,i,j} h^\alpha_{ij}\theta_i \otimes \theta_j \otimes e_\alpha, \qquad \mathbf{H} = \frac{1}{2}\sum_{\alpha,i} h^\alpha_{ii}e_\alpha = \sum_\alpha H^\alpha e_\alpha.$$

Let R_{ijkl} denote the Riemannian curvature tensor of M, the Gauss equations are

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2.1}$$

$$R_{ik} = c\delta_{ik} + 2\sum_\alpha H^\alpha h^\alpha_{ik} - \sum_{\alpha,j} h^\alpha_{ij}h^\alpha_{jk}, \tag{2.2}$$

$$2K = 2c + 4H^2 - S, \tag{2.3}$$

$$R_{\alpha\beta kl} = \sum_i (h^\alpha_{ki}h^\beta_{il} - h^\alpha_{li}h^\beta_{ik}), \tag{2.4}$$

were K is the Gaussian curvature of M^2. Since M^2 is a two-dimensional surface, we have

$$R_{ijkl} = K(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{2.5}$$

$$R_{ik} = K\delta_{ik}. \tag{2.6}$$

The covariant derivative of h^α_{ij} with components h^α_{ijk} is defined by

$$\sum_k h^\alpha_{ijk}\theta_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\theta_{ki} + \sum_k h^\alpha_{ik}\theta_{kj} + \sum_\beta h^\beta_{ij}\theta_{\beta\alpha}.$$

The second covariant derivative of h^α_{ij} with components h^α_{ijkl} is defined by

$$\sum_l h^\alpha_{ijkl}\theta_l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ljk}\theta_{li} + \sum_l h^\alpha_{ilk}\theta_{lj} + \sum_l h^\alpha_{ijl}\theta_{lk} + \sum_\beta h^\beta_{ijk}\theta_{\beta\alpha}.$$

Then the Codazzi equations and the Ricci identities are given by

(2.7) $$h^\alpha_{ijk} - h^\alpha_{ikj} = 0,$$

(2.8) $$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj} R_{mikl} + \sum_m h^\alpha_{im} R_{mjkl} + \sum_\beta h^\beta_{ij} R_{\beta\alpha kl}.$$

The Laplacian of h^α_{ij} is defined by $\triangle^\perp h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. Using (2.1), (2.7) and (2.8), we obtain

$$\begin{aligned}\sum_{\alpha,i,j} h^\alpha_{ij} \triangle^\perp h^\alpha_{ij} &= 2\sum_{\alpha,i,j} h^\alpha_{ij} H^\alpha_{ij} + 2cS - 2cH^2 + 2\sum_{\alpha,\beta,i,j,m} H^\beta h^\alpha_{ij} h^\alpha_{im} h^\beta_{mj} \\ &\quad - \sum_{\alpha,\beta,i,j,k,m} h^\alpha_{ij} h^\alpha_{km} h^\beta_{km} h^\beta_{ij} + \sum_{\alpha,\beta,i,j,k,m} h^\alpha_{ij} h^\alpha_{km} h^\beta_{mj} h^\beta_{ik} \\ &\quad - \sum_{\alpha,\beta,i,j,k,m} h^\alpha_{ij} h^\alpha_{im} h^\beta_{mk} h^\beta_{kj} + \sum_{\alpha,\beta,i,j,k} h^\alpha_{ij} h^\beta_{ki} R_{\beta\alpha jk}.\end{aligned}$$

Let ϕ^α_{ij} denote the tensor $h^\alpha_{ij} - H^\alpha \delta_{ij}$ and $\Phi = \sum_{\alpha,i,j} (\phi^\alpha_{ij})^2$ the square of the length of the trace free tensor ϕ^α_{ij}. It is easy to check that $\Phi = S - 2H^2$. The Codazzi equations and the Ricci identities can be written as

(2.9) $$\phi^\alpha_{ijk} - \phi^\alpha_{ikj} = H^\alpha_j \delta_{ik} - H^\alpha_k \delta_{ij},$$

(2.10) $$\phi^\alpha_{ijkl} - \phi^\alpha_{ijlk} = \sum_m \phi^\alpha_{mj} R_{mikl} + \sum_m \phi^\alpha_{im} R_{mjkl} + \sum_\beta \phi^\beta_{ij} R_{\beta\alpha kl}.$$

Lemma 2.1. *Let M^2 be a surface in the n-dimensional space form $\mathbb{N}^n(c)$ of constant curvature c. Then*

$$\frac{1}{2}\triangle\Phi = \sum_{\alpha,i,j,k} (\phi^\alpha_{ijk})^2 + 2\sum_{\alpha,i,j} \phi^\alpha_{ij} H^\alpha_{ij} + \Phi(2c + 2H^2 - \Phi) - \sum_{\alpha,\beta} R^2_{\alpha\beta 12}.$$

Proof. Using (2.9) and (2.10), we obtain

$$\begin{aligned}\triangle^\perp \phi^\alpha_{ij} &= \sum_k \phi^\alpha_{ijkk} \\ &= \sum_k \phi^\alpha_{kkij} + 2H^\alpha_{ij} - \triangle^\perp H^\alpha \delta_{ij} + \sum_{k,m} \phi^\alpha_{km} R_{mijk} + \sum_{k,m} \phi^\alpha_{mi} R_{mkjk} \\ &\quad + \sum_\beta H^\beta R_{\beta\alpha ji} + \sum_{\beta,k} \phi^\beta_{ki} R_{\beta\alpha jk}.\end{aligned}$$

Since $\sum_k \phi^\alpha_{kk} = 0$ and $R_{ijkl} = K(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$, we get

$$\triangle^\perp \phi^\alpha_{ij} = 2H^\alpha_{ij} - \triangle^\perp H^\alpha \delta_{ij} + 2K\phi^\alpha_{ij} + \sum_\beta H^\beta R_{\beta\alpha ji} + \sum_{\beta,k} \phi^\beta_{ki} R_{\beta\alpha jk},$$

where $\triangle^\perp$ is the Laplacian in the normal bundle NM. Thus

$$\begin{aligned}\frac{1}{2}\triangle\Phi &= \sum_{\alpha,i,j,k}(\phi^\alpha_{ijk})^2 + \sum_{\alpha,i,j}\phi^\alpha_{ij}\triangle^\perp\phi^\alpha_{ij}\\ &= \sum_{\alpha,i,j,k}(\phi^\alpha_{ijk})^2 + 2\sum_{\alpha,i,j}\phi^\alpha_{ij}H^\alpha_{ij} + 2K\Phi + \sum_{\alpha,\beta,i,j}\phi^\alpha_{ij}H^\beta R_{\beta\alpha ji} + \sum_{\alpha,\beta,i,j,k}\phi^\alpha_{ij}\phi^\beta_{ki}R_{\beta\alpha jk}.\end{aligned}$$

From (2.3) and $\Phi = S - 2H^2$,

$$\begin{aligned}\frac{1}{2}\triangle\Phi =& \sum_{\alpha,i,j,k}(\phi^\alpha_{ijk})^2 + 2\sum_{\alpha,i,j}\phi^\alpha_{ij}H^\alpha_{ij} + (2c + 2H^2 - \Phi)\Phi\\ &+ \sum_{\alpha,\beta} H^\beta(\phi^\alpha_{12}R_{\beta\alpha 21} + \phi^\alpha_{21}R_{\beta\alpha 12}) + \sum_{\alpha,\beta,i}(\phi^\alpha_{i1}\phi^\beta_{2i} - \phi^\alpha_{i2}\phi^\beta_{1i})R_{\beta\alpha 12}.\end{aligned}$$

By a direct computation, we have

$$R_{\alpha\beta kl} = \sum_i (h^\alpha_{ki}h^\beta_{il} - h^\alpha_{li}h^\beta_{ik}) = \sum_i(\phi^\alpha_{ki}\phi^\beta_{il} - \phi^\alpha_{li}\phi^\beta_{ik}).$$

This implies that $R_{\alpha\beta 12} = \sum_i(\phi^\alpha_{1i}\phi^\beta_{i2} - \phi^\alpha_{2i}\phi^\beta_{i1})$. Thus

$$\frac{1}{2}\triangle\Phi = \sum_{\alpha,i,j,k}(\phi^\alpha_{ijk})^2 + 2\sum_{\alpha,i,j}\phi^\alpha_{ij}H^\alpha_{ij} + (2c + 2H^2 - \Phi)\Phi - \sum_{\alpha,\beta} R^2_{\alpha\beta 12}. \qquad \blacksquare$$

Lemma 2.2. *Let M^2 be a surface in the n-dimensional space form $\mathbb{N}^n(c)$ of constant curvature c. Then $\sum_{\alpha,i,j}\phi^\alpha_{ijj}H^\alpha_i = |\nabla^\perp\mathbf{H}|^2$, where $|\nabla^\perp\mathbf{H}|^2 = \sum_{\alpha,i}(H^\alpha_i)^2$.*

Proof. It is an immediate consequence of the fact that

$$\sum_j \phi^\alpha_{ijj} = \sum_j \phi^\alpha_{jij} = \sum_j(\phi^\alpha_{jji} + H^\alpha_i\delta_{jj} - H^\alpha_j\delta_{ji}) = H^\alpha_i. \qquad \blacksquare$$

Lemma 2.3. *Let M^2 be a surface in the n-dimensional space form $\mathbb{N}^n(c)$ of constant curvature c. Then*

$$\sum_{\alpha,i,j,k}(\phi^\alpha_{ijk})^2 \geq |\nabla^\perp\mathbf{H}|^2.$$

The equality holds if and only if $\phi^\alpha_{111} = \phi^\alpha_{122} = \frac{H^\alpha_1}{2}$ and $\phi^\alpha_{222} = \phi^\alpha_{211} = \frac{H^\alpha_2}{2}$.

Proof. Since $0 = \sum_k \phi^\alpha_{kk} = \phi^\alpha_{11} + \phi^\alpha_{22}$, we therefore have $\phi^\alpha_{111} = -\phi^\alpha_{221}$ and $\phi^\alpha_{112} = -\phi^\alpha_{222}$, which implies

$$\begin{aligned}\sum_{\alpha,i,j,k} (\phi^\alpha_{ijk})^2 &= \sum_\alpha \Big[(\phi^\alpha_{111})^2 + (\phi^\alpha_{112})^2 + 2(\phi^\alpha_{121})^2 + 2(\phi^\alpha_{122})^2 + (\phi^\alpha_{221})^2 + (\phi^\alpha_{222})^2\Big]\\ &= \sum_\alpha 2\Big[(\phi^\alpha_{111})^2 + (\phi^\alpha_{222})^2 + (\phi^\alpha_{211})^2 + (\phi^\alpha_{122})^2\Big]\\ &\geq \sum_\alpha \Big[(\phi^\alpha_{111} + \phi^\alpha_{122})^2 + (\phi^\alpha_{222} + \phi^\alpha_{211})^2\Big]\\ &= \sum_\alpha \Big[(H^\alpha_1)^2 + (H^\alpha_2)^2\Big] = |\nabla^\perp \mathbf{H}|^2\end{aligned}$$

The equality holds if and only if $\phi^\alpha_{111} = \phi^\alpha_{122} = \frac{H^\alpha_1}{2}$ and $\phi^\alpha_{222} = \phi^\alpha_{211} = \frac{H^\alpha_2}{2}$. ■

Lemma 2.4. *Let M^2 be a surface in the n-dimensional space form $\mathbb{N}^n(c)$ of constant curvature c.*

(1) *If* $n = 3$ *then* $\sum_{\alpha,\beta} R^2_{\alpha\beta 12} = 0$.

(2) *If* $n \geq 4$ *then* $\sum_{\alpha,\beta} R^2_{\alpha\beta 12} \leq \dfrac{\Phi^2}{2}$. *Equality holds if and only if* $\sum_\alpha \phi^\alpha_{11}\phi^\alpha_{12} = 0$ *and* $\sum_\alpha (\phi^\alpha_{11})^2 = \sum_\alpha (\phi^\alpha_{12})^2$.

Proof. In the case of $n = 3$, since $R_{\alpha\beta 12} = R_{3312} = 0$, we have $\sum_{\alpha,\beta} R^2_{\alpha\beta 12} = 0$. For $n \geq 4$,

$$\begin{aligned}\sum_{\alpha,\beta} R^2_{\alpha\beta 12} &= \sum_{\alpha,\beta}\Big[\sum_i (h^\alpha_{1i}h^\beta_{i2} - h^\alpha_{2i}h^\beta_{i1})\Big]^2 = \sum_{\alpha,\beta}\Big[\sum_i (\phi^\alpha_{1i}\phi^\beta_{i2} - \phi^\alpha_{2i}\phi^\beta_{i1})\Big]^2\\ &= 4\sum_{\alpha,\beta}(\phi^\alpha_{11}\phi^\beta_{12} - \phi^\beta_{11}\phi^\alpha_{12})^2\\ &= 4\sum_{\alpha,\beta}\Big[(\phi^\alpha_{11})^2(\phi^\beta_{12})^2 + (\phi^\beta_{11})^2(\phi^\alpha_{12})^2 - 2\phi^\alpha_{11}\phi^\beta_{11}\phi^\alpha_{12}\phi^\beta_{12}\Big]\\ &= 8\sum_\alpha(\phi^\alpha_{11})^2\sum_\alpha(\phi^\alpha_{12})^2 - 8\Big(\sum_\alpha \phi^\alpha_{11}\phi^\alpha_{12}\Big)^2.\end{aligned}$$

Since $0 = \sum_i \phi^\alpha_{ii} = \phi^\alpha_{11} + \phi^\alpha_{22}$, and

$$\Phi = \sum_\alpha(\phi^\alpha_{11})^2 + 2\sum_\alpha(\phi^\alpha_{12})^2 + \sum_\alpha(\phi^\alpha_{22})^2 = 2\Big[\sum_\alpha(\phi^\alpha_{11})^2 + \sum_\alpha(\phi^\alpha_{12})^2\Big],$$

we obtain

$$\begin{aligned}\sum_{\alpha,\beta} R^2_{\alpha\beta12} &= 8\Big[\sum_\alpha(\phi^\alpha_{11})^2\sum_\alpha(\phi^\alpha_{12})^2 - \Big(\sum_\alpha\phi^\alpha_{11}\phi^\alpha_{12}\Big)^2\Big]\\ &\le 8\sum_\alpha(\phi^\alpha_{11})^2\sum_\alpha(\phi^\alpha_{12})^2\\ &\le 2\Big[\sum_\alpha(\phi^\alpha_{11})^2+\sum_\alpha(\phi^\alpha_{12})^2\Big]^2\\ &= \frac{\Phi^2}{2}.\end{aligned}$$

Equality holds if and only if the inequalities become equalities. i.e., $\sum_\alpha \phi^\alpha_{11}\phi^\alpha_{12} = 0$ and $\sum_\alpha(\phi^\alpha_{11})^2 = \sum_\alpha(\phi^\alpha_{12})^2$. ■

Lemma 2.5. *Let M^2 be a surface in the n-dimensional space form $\mathbb{N}^n(c)$ of constant curvature c. For $n = 3$,*

$$\Phi\sum_{i,j,k}\phi^2_{ijk} = \frac{|\nabla\Phi|^2}{2} + 2\Phi|\nabla H|^2 - 2\sum_{i,j}\phi_{ij}H_i\Phi_j.$$

Proof. Since $0 = \phi_{11}+\phi_{22}$, we therefore have $\phi_{111} = -\phi_{221}$ and $\phi_{112} = -\phi_{222}$. By using

$$\phi_{ijj} = \phi_{jij} = \phi_{jji} + H_i\delta_{jj} - H_j\delta_{ji},$$

we have

$$\begin{aligned}\sum_{i,j,k}\phi^2_{ijk} &= 4\phi^2_{111} + 4\phi^2_{222} - 4\phi_{111}H_1 - 4\phi_{222}H_2 + 2|\nabla H|^2,\\ \frac{|\nabla\Phi|^2}{2} &= 4\Phi\phi^2_{111} + 4\Phi\phi^2_{222} - 16\phi^2_{12}\phi_{111}H_1 - 16\phi^2_{12}\phi_{222}H_2\\ &\quad +16\phi_{12}(\phi_{11}\phi_{111}H_2 + \phi_{22}\phi_{222}H_1) + 8\phi^2_{12}|\nabla H|^2,\\ \sum_{i,j}\phi_{ij}H_i\Phi_j &= -4\phi^2_{12}\phi_{111}H_1 - 4\phi^2_{12}\phi_{222}H_2 + 4\phi^2_{11}\phi_{111}H_1 + 4\phi^2_{22}\phi_{222}H_2\\ &\quad +8\phi_{12}(\phi_{11}\phi_{111}H_2 + \phi_{22}\phi_{222}H_1) + 4\phi^2_{12}|\nabla H|^2.\end{aligned}$$

By a direct computation, the proof is then straightforward. ■

Lemma 2.6. ([7]). *Let M^2 be a compact minimal surface in the n-dimensional unit sphere S^n. If $0 \le S \le \frac{4}{3}$ then either $S = 0$ and M^2 is totally geodesic, or $S = \frac{4}{3}$, $n = 4$ and M^2 is the Veronese surface.*

3. The First Variation of the Willmore Functional

Let $x : M^2 \to \mathbb{N}^n(c)$ be an isometric immersion, and let

$$X : M^2 \times (-\varepsilon, \varepsilon) \to \mathbb{N}^n(c)$$

be a variation of x such that $X(\cdot, t) = x_t$ and $x_0 = x$. Along $X : M^2 \times (-\varepsilon, \varepsilon) \to \mathbb{N}^n(c)$, we choose a local orthonormal basis $\{e_A\}$ for $T\mathbb{N}^n(c)$ with dual basis $\{\omega_A\}$, such that $\{e_i(\cdot, t)\}$ forms a local orthonormal basis for $x_t : M^2 \times \{t\} \to \mathbb{N}^n(c)$. Since $T^*(M^2 \times (-\varepsilon, \varepsilon)) = T^*M^2 \oplus T^*(-\varepsilon, \varepsilon)$, the pullback of $\{\omega_A\}$ and $\{\omega_{AB}\}$ on $\mathbb{N}^n(c)$ through $X : M^2 \times (-\varepsilon, \varepsilon) \to \mathbb{N}^n(c)$ have the following decomposition

$$X^*\omega_i = \theta_i + V_i dt, \qquad X^*\omega_\alpha = \theta_\alpha + V_\alpha dt = V_\alpha dt,$$
$$X^*\omega_{ij} = \theta_{ij} + L_{ij} dt, \qquad X^*\omega_{i\alpha} = \theta_{i\alpha} + L_{i\alpha} dt,$$
$$X^*\omega_{\alpha\beta} = \theta_{\alpha\beta} + L_{\alpha\beta} dt,$$

where $\{V_i, V_\alpha, L_{ij}, L_{i\alpha}, L_{\alpha\beta}\}$ are local functions on $M^2 \times (-\varepsilon, \varepsilon)$ with $L_{ij} + L_{ji} = 0$, $L_{\alpha\beta} + L_{\beta\alpha} = 0$ and

$$V = \frac{d}{dt}\Big|_{t=0} x_t = \sum_i V_i dx_0(e_i) + \sum_\alpha V_\alpha e_\alpha,$$

is the variation vector field of $x_t : M^2 \times \{t\} \to \mathbb{N}^n(c)$. The differential operator d on $T^*(M^2 \times (-\varepsilon, \varepsilon))$ can be written by $d = d_M + dt\frac{\partial}{\partial t}$, where d_M is the differential operator on T^*M^2.

Lemma 3.1. ([11]). *Under the above notations, we have*

$$(1)\ \frac{\partial \theta_i}{\partial t} = \sum_j (V_{i,j} + L_{ij} - \sum_\alpha h^\alpha_{ij} V_\alpha)\theta_j,$$
$$(2)\ L_{i\alpha} = V_{\alpha,i} + \sum_j h^\alpha_{ij} V_j,$$
$$(3)\ \frac{\partial \theta_{i\alpha}}{\partial t} = \sum_j (L_{i\alpha,j} + L_{ik} h^\alpha_{jk} - \sum_\beta L_{\beta\alpha} h^\beta_{ij} + c\delta_{ij})\theta_j,$$
$$(4)\ \frac{\partial h^\alpha_{ij}}{\partial t} = V_{\alpha,ij} + \sum_k (L_{ik} h^\alpha_{kj} + L_{jk} h^\alpha_{ki} + h^\alpha_{ijk} V_k) + \sum_\beta L_{\alpha\beta} h^\beta_{ij} + \sum_{\beta,k} h^\alpha_{ik} h^\beta_{kj} V_\beta + c\delta_{ij} V_\alpha.$$

where h_{ij}^{β} *and the covariant derivatives* $V_{i,j}$*,* $V_{\alpha,i}$ *and* $L_{i\alpha,j}$ *are defined on* $M^2 \times \{t\}$ *by*

$$\begin{aligned} \theta_{i\alpha} &= \sum_j h_{ij}^{\alpha}\theta_j, \\ \sum_j V_{i,j}\theta_j &= d_M V_i + \sum_j V_j\theta_{ji}, \\ \sum_i V_{\alpha,i}\theta_i &= d_M V_\alpha + \sum_\beta V_\beta\theta_{\beta\alpha}, \\ \sum_j L_{i\alpha,j}\theta_j &= d_M L_{i\alpha} + \sum_j L_{j\alpha}\theta_{ji} + \sum_\beta L_{j\beta}\theta_{\beta\alpha}. \end{aligned}$$

Lemma 3.2. ([11]). *With the same notations as above, we have*

$$\sum_{\alpha,\beta,i,j} L_{\alpha\beta}h_{ij}^{\alpha}h_{ij}^{\beta} = 0, \qquad \sum_{\alpha,i,j,k} L_{ij}h_{ik}^{\alpha}h_{kj}^{\alpha} = 0,$$

$$\sum_{i,j} L_{ij}h_{ij}^{\alpha} = 0, \qquad \sum_{\alpha,\beta} L_{\alpha\beta}H^{\alpha}H^{\beta} = 0.$$

Lemma 3.3. *Let* M^2 *be a surface in the* n*-dimensional space form* $\mathbb{N}^n(c)$*. Under the above notations, we have*

(1) $\dfrac{\partial H^\alpha}{\partial t} = \dfrac{1}{2}\triangle^{\perp}V_\alpha + \sum_k H_k^{\alpha}V_k + \sum_\beta L_{\alpha\beta}H^\beta + \dfrac{1}{2}\sum_{\beta,i,k} h_{ik}^{\alpha}h_{ki}^{\beta}V_\beta + cV_\alpha$

(2) $\dfrac{\partial S}{\partial t} = 2\sum_{\alpha,i,j} h_{ij}^{\alpha}V_{\alpha,ij} + 2\sum_{\alpha,i,j,k} h_{ij}^{\alpha}h_{ijk}^{\alpha}V_k + 2\sum_{\alpha,\beta,i,j,k} h_{ij}^{\alpha}h_{ik}^{\alpha}h_{kj}^{\beta}V_\beta + 4c\sum_\alpha H^\alpha V_\alpha.$

Proof. Set $i = j$ in Lemma 3.1, since $\sum_{i,j} L_{ij}h_{ij}^{\alpha} = 0$, we get

$$\begin{aligned} \frac{\partial H^\alpha}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial t}\sum_i h_{ii}^{\alpha} \\ &= \frac{1}{2}\triangle^{\perp}V_\alpha + \sum_k H_k^{\alpha}V_k + \sum_\beta L_{\alpha\beta}H^\beta + \frac{1}{2}\sum_{\beta,i,k} h_{ik}^{\alpha}h_{ki}^{\beta}V_\beta + cV_\alpha. \end{aligned}$$

By using Lemma 3.1 and Lemma 3.2,

$$\begin{aligned} &\frac{\partial S}{\partial t} = \frac{\partial}{\partial t}\sum_{\alpha,i,j}(h_{ij}^{\alpha})^2 = 2\sum_{\alpha,i,j} h_{ij}^{\alpha}\frac{\partial h_{ij}^{\alpha}}{\partial t} \\ &= 2\sum_{\alpha,i,j} h_{ij}^{\alpha}V_{\alpha,ij} + 2\sum_{\alpha,i,j,k} h_{ij}^{\alpha}h_{ijk}^{\alpha}V_k + 2\sum_{\alpha,\beta,i,j,k} h_{ij}^{\alpha}h_{ik}^{\alpha}h_{kj}^{\beta}V_\beta + 4c\sum_\alpha H^\alpha V_\alpha. \quad \blacksquare \end{aligned}$$

Lemma 3.4. *Let M^2 be a surface in the n-dimensional space form $\mathbb{N}^n(c)$. Under the above notations, we have*

(1) $\frac{\partial}{\partial t}(\theta_1 \wedge \theta_2) = \Big(\sum_i V_{i,i} - 2\sum_\alpha H^\alpha V_\alpha\Big)\theta_1 \wedge \theta_2$

(2) $\frac{\partial \Phi}{\partial t} = 2\sum_{\alpha,i,j} h^\alpha_{ij} V_{\alpha,ij} - 2\sum_\alpha H^\alpha \triangle^\perp V_\alpha + \sum_k \Phi_k V_k + 2\sum_{\alpha,\beta,i,j,k} h^\alpha_{ij} h^\alpha_{ik} h^\beta_{kj} V_\beta$
$-2\sum_{\alpha,\beta,i,k} h^\alpha_{ik} h^\beta_{ki} H^\alpha V_\beta.$

Proof. From Lemma 3.1 and $L_{11} = L_{22} = 0$, we have

$$\begin{aligned}\frac{\partial}{\partial t}(\theta_1 \wedge \theta_2) &= \frac{\partial \theta_1}{\partial t} \wedge \theta_2 + \theta_1 \wedge \frac{\partial \theta_2}{\partial t}\\ &= (V_{1,1} + L_{11} - \sum_\alpha h^\alpha_{11} V_\alpha)\theta_1 \wedge \theta_2 + (V_{2,2} + L_{22} - \sum_\alpha h^\alpha_{22} V_\alpha)\theta_1 \wedge \theta_2\\ &= \Big(\sum_i V_{i,i} - 2\sum_\alpha H^\alpha V_\alpha\Big)\theta_1 \wedge \theta_2.\end{aligned}$$

By using Lemma 3.2 and Lemma 3.3, we have

$$\sum_\alpha H^\alpha \frac{\partial H^\alpha}{\partial t} = \frac{1}{2}\sum_\alpha H^\alpha \triangle^\perp V_\alpha + \sum_{\alpha,k} H^\alpha H^\alpha_k V_k + \frac{1}{2}\sum_{\alpha,\beta,i,k} h^\alpha_{ik} h^\beta_{ki} H^\alpha V_\beta + c\sum_\alpha H^\alpha V_\alpha.$$

Since

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t}(S - 2H^2) = \frac{\partial S}{\partial t} - 2\frac{\partial H^2}{\partial t} = \frac{\partial S}{\partial t} - 4\sum_\alpha H^\alpha \frac{\partial H^\alpha}{\partial t},$$

by Lemma 3.3, we get

$$\begin{aligned}\frac{\partial \Phi}{\partial t} &= 2\sum_{\alpha,i,j} h^\alpha_{ij} V_{\alpha,ij} - 2\sum_\alpha H^\alpha \triangle^\perp V_\alpha + \sum_k (S - 2H^2)_k V_k\\ &\quad +2\sum_{\alpha,\beta,i,j,k} h^\alpha_{ij} h^\alpha_{ik} h^\beta_{kj} V_\beta - 2\sum_{\alpha,\beta,i,k} h^\alpha_{ik} h^\beta_{ki} H^\alpha V_\beta\\ &= 2\sum_{\alpha,i,j} h^\alpha_{ij} V_{\alpha,ij} - 2\sum_\alpha H^\alpha \triangle^\perp V_\alpha + \sum_k \Phi_k V_k\\ &\quad +2\sum_{\alpha,\beta,i,j,k} h^\alpha_{ij} h^\alpha_{ik} h^\beta_{kj} V_\beta - 2\sum_{\alpha,\beta,i,k} h^\alpha_{ik} h^\beta_{ki} H^\alpha V_\beta.\end{aligned}$$ ∎

Now, we calculate the first variation of the Willmore functional $W(x_0)$ for a surface M^2 in space form $\mathbb{N}^n(c)$.

Theorem 3.5. *Let M^2 be a surface in the n-dimensional space form $\mathbb{N}^n(c)$. Then M^2 is a Willmore surface if and only if*

$$\triangle^{\perp} H^{\alpha} + \sum_{\beta,i,j} h_{ij}^{\alpha} h_{ij}^{\beta} H^{\beta} - 2H^2 H^{\alpha} = 0, \qquad 3 \le \alpha \le n.$$

Proof. For $x_t : M_t = M \times \{t\} \to \mathbb{N}^n(c)$, we consider the Willmore functional

$$W(x_t) = \int_{M_t} \Phi dv = \int_{M_t} \Phi\, \theta_1 \wedge \theta_2.$$

From Lemma 3.4, we have

$$\begin{aligned}
\frac{\partial W(x_t)}{\partial t} &= \int_{M_t} \frac{\partial \Phi}{\partial t}\, \theta_1 \wedge \theta_2 + \int_{M_t} \Phi \frac{\partial}{\partial t}(\theta_1 \wedge \theta_2) \\
&= \int_{M_t} 2\Big[\sum_{\alpha,i,j} h_{ij}^{\alpha} V_{\alpha,ij} - \sum_{\alpha,i} H^{\alpha} V_{\alpha,ii} + \frac{1}{2} \sum_k \Phi_k V_k + \sum_{\alpha,\beta,i,j,k} h_{ij}^{\alpha} h_{ik}^{\alpha} h_{kj}^{\beta} V_{\beta} \\
&\quad - \sum_{\alpha,\beta,i,k} h_{ik}^{\alpha} h_{ki}^{\beta} H^{\alpha} V_{\beta} \Big] \theta_1 \wedge \theta_2 + \int_{M_t} \Phi (\sum_i V_{i,i} - 2 \sum_{\alpha} H^{\alpha} V_{\alpha})\, \theta_1 \wedge \theta_2.
\end{aligned}$$

By Stokes' theorem, we obtain

$$\begin{aligned}
\frac{\partial W(x_t)}{\partial t} &= 2 \int_{M_t} \Big[- \sum_{\alpha,i,j} h_{ijj}^{\alpha} V_{\alpha,i} + \sum_{\alpha,i} H_i^{\alpha} V_{\alpha,i} + \sum_{\alpha,\beta,i,j,k} h_{ij}^{\alpha} h_{ik}^{\alpha} h_{kj}^{\beta} V_{\beta} \\
&\qquad - \sum_{\alpha,\beta,i,k} h_{ik}^{\alpha} h_{ki}^{\beta} H^{\alpha} V_{\beta} - \Phi \sum_{\alpha} H^{\alpha} V_{\alpha} \Big] \theta_1 \wedge \theta_2 \\
&= 2 \int_{M_t} \Big[- \sum_{\alpha,i} H_i^{\alpha} V_{\alpha,i} + \sum_{\alpha,\beta,i,j,k} h_{ij}^{\beta} h_{ik}^{\beta} h_{kj}^{\alpha} V_{\alpha} - \sum_{\alpha,\beta,i,k} h_{ik}^{\beta} h_{ki}^{\alpha} H^{\beta} V_{\alpha} \\
&\qquad - \Phi \sum_{\alpha} H^{\alpha} V_{\alpha} \Big] \theta_1 \wedge \theta_2 \\
&= 2 \int_{M_t} \sum_{\alpha} \Big[\triangle^{\perp} H^{\alpha} + \sum_{\beta,i,j,k} h_{ij}^{\beta} h_{ik}^{\beta} h_{kj}^{\alpha} - \sum_{\beta,i,k} h_{ik}^{\beta} h_{ki}^{\alpha} H^{\beta} - \Phi H^{\alpha} \Big] V_{\alpha} \theta_1 \wedge \theta_2.
\end{aligned}$$

From (2.2),

$$\sum_{\beta,i} h_{ij}^{\beta} h_{ik}^{\beta} = c\delta_{jk} + 2 \sum_{\beta} H^{\beta} h_{jk}^{\beta} - R_{jk},$$

we have

$$\sum_{\beta,i,j,k} h_{ij}^{\beta} h_{ik}^{\beta} h_{kj}^{\alpha} = 2cH^{\alpha} + 2 \sum_{\beta,j,k} H^{\beta} h_{jk}^{\beta} h_{kj}^{\alpha} - \sum_{j,k} R_{jk} h_{kj}^{\alpha}.$$

Since

$$R_{jk} = \frac{R}{2}\delta_{jk} = K\delta_{jk},$$

by (2.3), we get

$$\sum_{j,k} R_{jk}h^{\alpha}_{kj} = \sum_{j,k} K\delta_{jk}h^{\alpha}_{kj} = 2KH^{\alpha} = (2c + 4H^2 - S)H^{\alpha} = (2c + 2H^2 - \Phi)H^{\alpha}.$$

Thus

$$\sum_{\beta,i,j,k} h^{\beta}_{ij}h^{\beta}_{ik}h^{\alpha}_{kj} = 2\sum_{\beta,j,k} H^{\beta}h^{\beta}_{jk}h^{\alpha}_{kj} - 2H^2H^{\alpha} + \Phi H^{\alpha}.$$

This implies that

$$\frac{\partial W(x_t)}{\partial t} = 2\int_{M_t}\sum_{\alpha}\Big[\triangle^{\perp}H^{\alpha} + \sum_{\beta,j,k} H^{\beta}h^{\beta}_{jk}h^{\alpha}_{kj} - 2H^2H^{\alpha}\Big]V_{\alpha}\theta_1\wedge\theta_2.$$

By the definition, M^2 is a Willmore surface if and only if $\frac{\partial W(x_t)}{\partial t} = 0$. That is,

$$\triangle^{\perp}H^{\alpha} + \sum_{\beta,j,k} H^{\beta}h^{\beta}_{jk}h^{\alpha}_{kj} - 2H^2H^{\alpha} = 0, \qquad 3 \le \alpha \le n.$$ ■

By using $\phi^{\alpha}_{ij} = h^{\alpha}_{ij} - H^{\alpha}\delta_{ij}$, we have the following theorem.

Theorem 3.6. *Let M^2 be a surface in the n-dimensional space form $\mathbb{N}^n(c)$. Then M^2 is a Willmore surface if and only if*

$$\triangle^{\perp}H^{\alpha} + \sum_{\beta,i,j}\phi^{\alpha}_{ij}\phi^{\beta}_{ij}H^{\beta} = 0, \qquad 3 \le \alpha \le n.$$

Lemma 3.7. *Let M^2 be a surface in the n-dimensional space form $\mathbb{N}^n(c)$. Then*

$$\int_M |\nabla^{\perp}\mathbf{H}|^2 dv \le \int_M \Phi H^2 dv.$$

Equality holds if and only if either $n = 3$ or $\phi^{\alpha}_{ij} = C_{ij}H^{\alpha}$ for some functions C_{ij} at the points where $\Phi \ne 0$ and $H \ne 0$ when $n \ge 4$.

Proof. By use of Theorem 3.6 and the Cauchy-Schwarz inequality, we have

$$\int_M |\nabla^{\perp}\mathbf{H}|^2 dv = -\int_M\sum_{\alpha} H^{\alpha}\triangle^{\perp}H^{\alpha}dv = \int_M\sum_{\alpha,\beta,i,j}\phi^{\alpha}_{ij}\phi^{\beta}_{ij}H^{\alpha}H^{\beta}dv$$

$$= \int_M\sum_{i,j}\Big(\sum_{\alpha}\phi^{\alpha}_{ij}H^{\alpha}\Big)^2 dv \le \int_M\Big(\sum_{\alpha,i,j}(\phi^{\alpha}_{ij})^2\Big)\Big(\sum_{\alpha}(H^{\alpha})^2\Big)dv = \int_M \Phi H^2 dv.$$

Equality holds if and only if either $n = 3$ or $\phi^{\alpha}_{ij} = C_{ij}H^{\alpha}$ for some functions C_{ij} at the points where $\Phi \ne 0$ and $H \ne 0$ when $n \ge 4$. ■

4. The Proof of Theorem 1.3

Now, we prove the Theorem 1.3. Integrating both sides of the Lemma 2.1 over M^2, by Stokes' theorem, we have

$$\begin{aligned} 0 &= \int_M \frac{1}{2}\triangle\Phi dv \\ &= \int_M \Big(\sum_{\alpha,i,j,k} (\phi^\alpha_{ijk})^2 + 2\sum_{\alpha,i,j} \phi^\alpha_{ij} H^\alpha_{ij} + \Phi(2c + 2H^2 - \Phi) - \sum_{\alpha,\beta} R^2_{\alpha\beta 12}\Big) dv \\ &= \int_M \Big(\sum_{\alpha,i,j,k} (\phi^\alpha_{ijk})^2 - 2\sum_{\alpha,i,j} \phi^\alpha_{ijj} H^\alpha_{i} + \Phi(2c + 2H^2 - \Phi) - \sum_{\alpha,\beta} R^2_{\alpha\beta 12}\Big) dv. \end{aligned}$$

By using Lemma 2.2 and Lemma 2.3,

$$\begin{aligned} 0 &\geq \int_M \Big(|\nabla^\perp \mathbf{H}|^2 - 2|\nabla^\perp \mathbf{H}|^2 + \Phi(2c + 2H^2 - \Phi) - \sum_{\alpha,\beta} R^2_{\alpha\beta 12}\Big) dv \\ &= \int_M \Big(- |\nabla^\perp \mathbf{H}|^2 + \Phi(2c + 2H^2 - \Phi) - \sum_{\alpha,\beta} R^2_{\alpha\beta 12}\Big) dv. \end{aligned}$$

From Lemma 2.4 and Lemma 3.7, if $n = 3$, we get

$$0 \geq \int_M \Big[- \Phi H^2 + \Phi(2c + 2H^2 - \Phi)\Big] dv = \int_M \Phi(2c + H^2 - \Phi) dv,$$

if $n \geq 4$, we get

$$0 \geq \int_M \Big[- \Phi H^2 + \Phi(2c + 2H^2 - \Phi) - \frac{\Phi^2}{2}\Big] dv = \int_M \Phi(2c + H^2 - \frac{3}{2}\Phi) dv.$$

Thus

$$\int_M \Phi\Big(C(n)(c + \frac{H^2}{2}) - \Phi\Big) dv \leq 0,$$

where $C(n) = 2$ when $n = 3$ and $C(n) = \frac{4}{3}$ when $n \geq 4$.
(1)For $n = 3$, if $0 \leq \Phi \leq 2c + H^2$, we have

$$0 \geq \int_M \Big[\Phi(2c + H^2 - \Phi)\Big] dv \geq 0.$$

Then either $\Phi = 0$ and M^2 is totally umbilical sphere or $\Phi = 2c + H^2$. In the latter case, all the integral inequalities become equalities. If $\Phi > 0$, it follows from Lemma

2.1 and Lemma 2.5 that

$$\begin{aligned}
\int_M (2c+2H^2-\Phi)dv &= \int_M (\frac{1}{2}\frac{\Delta\Phi}{\Phi} - \frac{\sum_{i,j,k}\phi_{ijk}^2}{\Phi} - \frac{2\sum_{i,j}\phi_{ij}H_{ij}}{\Phi})dv \\
&= \int_M (\frac{1}{2}\frac{\Delta\Phi}{\Phi} - \frac{|\nabla\Phi|^2}{2\Phi^2} - \frac{2|\nabla H|^2}{\Phi} + \frac{2\sum_{i,j}\phi_{ij}H_i\Phi_j}{\Phi^2} - \frac{2\sum_{i,j}\phi_{ij}H_{ij}}{\Phi})dv \\
&= \int_M (\frac{1}{2}\Delta\ln\Phi - \frac{2|\nabla H|^2}{\Phi} + \frac{2\sum_{i,j}\phi_{ij}H_i\Phi_j}{\Phi^2} - \frac{2\sum_{i,j}\phi_{ij}H_{ij}}{\Phi})dv \\
&= \int_M (-\frac{2|\nabla H|^2}{\Phi} + \frac{2\sum_{i,j}\phi_{ij}H_i\Phi_j}{\Phi^2} + 2\sum_{i,j}(\frac{\phi_{ij}}{\Phi})_j H_i)dv \\
&= \int_M (-\frac{2|\nabla H|^2}{\Phi} + \frac{2\sum_{i,j}\phi_{ij}H_i\Phi_j}{\Phi^2} + 2\sum_{i,j}\frac{\Phi\phi_{ijj}-\phi_{ij}\Phi_j}{\Phi^2}H_i)dv \\
&= \int_M (-\frac{2|\nabla H|^2}{\Phi} + \frac{2|\nabla H|^2}{\Phi})dv = 0.
\end{aligned}$$

By using $\Phi = 2c + H^2$, we get

$$0 = \int_M (2c+2H^2-\Phi)dv = \int_M H^2 dv.$$

Then M^2 is a minimal surface in $\mathbb{N}^n(c)$. Since $\Phi = 2c + H^2 > 0$, we have $c > 0$. Thus M^2 is a minimal surface in $S^3(c)$ with $S = \Phi = 2c$, we can conclude that M^2 is the Clifford torus (see [3]).

(2)For $n \geq 4$, if $0 \leq \Phi \leq \frac{4}{3}c + \frac{2}{3}H^2$, we have

$$0 \geq \int_M \Big[\Phi(2c + H^2 - \frac{3}{2}\Phi)\Big]dv \geq 0.$$

Then either $\Phi = 0$ and M^2 is totally umbilical or $\Phi = \frac{4}{3}c + \frac{2}{3}H^2$. In the latter case, all the integral inequalities become equalities. If $\Phi > 0$, assume that $H(p) \neq 0$ at some point $p \in M^2$, we shall derive a contradiction. Since by assumption $H(p) \neq 0$, $\Phi(p) > 0$, Lemma 3.7 gives $\phi_{ij}^\alpha = C_{ij}H^\alpha$ for some functions C_{ij}. Furthermore, Lemma 2.4 implies $C_{11}(p) = C_{12}(p) = 0$, and hence $C_{21}(p) = C_{22}(p) = 0$. This mean that $\Phi(p) = 0$, a contradiction. This contradiction shows that M^2 is a minimal surface in $\mathbb{N}^n(c)$. Since $\Phi = \frac{4}{3}c + \frac{2}{3}H^2 > 0$, we have $c > 0$. Thus M^2 is a minimal surface in $S^n(c)$, $n \geq 4$ with $S = \frac{4}{3}c$, we can conclude that $n = 4$ and M^2 is the Veronese surface (see [7]).

5. The F-Willmore Surfaces in Space Forms

In this section, we shall calculate the first variation of F-Willmore functional on M^2 and prove the Theorem 1.5.

Theorem 5.1. *Let M^2 be a compact surface in the n-dimensional space form $\mathbb{N}^n(c)$ of constant curvature c. Then M is an F-Willmore surface if and only if*

$$\sum_{i,j}(F'(\Phi))_{ji}\phi_{ij}^{\alpha}+2\sum_{i}(F'(\Phi))_i H_i^{\alpha}-F(\Phi)H^{\alpha}$$
$$+F'(\Phi)\Big[\triangle^{\perp}H^{\alpha}+H^{\alpha}\Phi+\sum_{\beta,i,j}\phi_{ij}^{\alpha}\phi_{ij}^{\beta}H^{\beta}\Big]=0,$$

for every $3\leq\alpha\leq n$.

Proof. For $x_t : M_t = M^2\times\{t\}\to\mathbb{N}^n(c)$, we consider the F-Willmore functional

$$W_F(x_t)=\int_{M_t}F(\Phi)dv=\int_{M_t}F(\Phi)\,\theta_1\wedge\theta_2.$$

From Lemma 3.4, we have

$$\frac{\partial W_F(x_t)}{\partial t}=\int_{M_t}F'(\Phi)\frac{\partial\Phi}{\partial t}\,\theta_1\wedge\theta_2+\int_{M_t}F(\Phi)\frac{\partial}{\partial t}(\theta_1\wedge\theta_2)$$
$$=\int_{M_t}2F'(\Phi)\Big[\sum_{\alpha,i,j}h_{ij}^{\alpha}V_{\alpha,ij}-\sum_{\alpha,i}H^{\alpha}V_{\alpha,ii}+\frac{1}{2}\sum_{k}\Phi_k V_k+\sum_{\alpha,\beta,i,j,k}h_{ij}^{\alpha}h_{ik}^{\alpha}h_{kj}^{\beta}V_{\beta}$$
$$-\sum_{\alpha,\beta,i,k}h_{ik}^{\alpha}h_{ki}^{\beta}H^{\alpha}V_{\beta}\Big]\theta_1\wedge\theta_2+\int_{M_t}F(\Phi)(\sum_i V_{i,i}-2\sum_{\alpha}H^{\alpha}V_{\alpha})\,\theta_1\wedge\theta_2.$$

By Stokes' theorem, we obtain

$$\frac{\partial W_F(x_t)}{\partial t}=2\int_{M_t}F'(\Phi)\sum_{\alpha}\Big[\triangle^{\perp}H^{\alpha}+\sum_{\beta,i,j,k}h_{ij}^{\beta}h_{ik}^{\beta}h_{kj}^{\alpha}-\sum_{\beta,i,k}h_{ik}^{\beta}h_{ki}^{\alpha}H^{\beta}\Big]V_{\alpha}\theta_1\wedge\theta_2$$
$$+2\int_{M_t}\sum_{\alpha}\Big[\sum_{i,j}(F'(\Phi))_{ji}h_{ij}^{\alpha}-\sum_{i}(F'(\Phi))_{ii}H^{\alpha}+2\sum_{i}(F'(\Phi))_i H_i^{\alpha}\Big]V_{\alpha}\theta_1\wedge\theta_2$$
$$-2\int_{M_t}F(\Phi)\sum_{\alpha}H^{\alpha}V_{\alpha}\theta_1\wedge\theta_2.$$

Thus M^2 is an F-Willmore surface if and only if

$$\sum_{i,j}(F'(\Phi))_{ji}h_{ij}^{\alpha}-\sum_{i}(F'(\Phi))_{ii}H^{\alpha}+2\sum_{i}(F'(\Phi))_i H_i^{\alpha}$$
$$+F'(\Phi)\Big[\triangle^{\perp}H^{\alpha}+\sum_{\beta,i,j,k}h_{ij}^{\beta}h_{ik}^{\beta}h_{kj}^{\alpha}-\sum_{\beta,i,k}h_{ik}^{\beta}h_{ki}^{\alpha}H^{\beta}\Big]-F(\Phi)H^{\alpha}=0,$$

for every $3 \leq \alpha \leq n$. By using $h^\alpha_{ij} = \phi^\alpha_{ij} + H^\alpha \delta_{ij}$ and $\sum_{\beta,i,j} \phi^\beta_{ij}\phi^\beta_{jk}\phi^\alpha_{ki} = 0$, we have that M^2 is an F-Willmore surface if and only if

$$\sum_{i,j}(F'(\Phi))_{ji}\phi^\alpha_{ij} + 2\sum_i (F'(\Phi))_i H^\alpha_i - F(\Phi)H^\alpha + F'(\Phi)\Big[\triangle^\perp H^\alpha + H^\alpha\Phi + \sum_{\beta,i,j}\phi^\alpha_{ij}\phi^\beta_{ij}H^\beta\Big] = 0,$$

for every $3 \leq \alpha \leq n$. ■

Theorem 5.2. *Let M^2 be a compact F-Willmore surface in the n-dimensional space form $\mathbb{N}^n(c)$. Then*

$$\int_M \Big\{F'(\Phi)\Big[\Phi H^2 - |\nabla^\perp \mathbf{H}|^2 + \sum_{\alpha,\beta,i,j}\phi^\alpha_{ij}\phi^\beta_{ij}H^\alpha H^\beta\Big] - \sum_{\alpha,i,j}F''(\Phi)\phi^\alpha_{ij}\Phi_j H^\alpha_i - F(\Phi)H^2\Big\}dv = 0.$$

Proof. By using Theorem 5.1 and Stokes' theorem, we have that

$$\begin{aligned} 0 &= \int_M \Big[\sum_{\alpha,i,j}(F'(\Phi))_{ji}\phi^\alpha_{ij}H^\alpha + 2\sum_{\alpha,i}(F'(\Phi))_i H^\alpha H^\alpha_i - F(\Phi)H^2\Big]dv \\ &\quad + \int_M F'(\Phi)\Big[\sum_\alpha H^\alpha \triangle^\perp H^\alpha + \Phi H^2 + \sum_{\alpha,\beta,i,j}\phi^\alpha_{ij}\phi^\beta_{ij}H^\alpha H^\beta\Big]dv \\ &= \int_M \Big\{\sum_{\alpha,i,j}(F'(\Phi))_{ji}\phi^\alpha_{ij}H^\alpha + \sum_{\alpha,i}(F'(\Phi))_i H^\alpha H^\alpha_i - F(\Phi)H^2 \\ &\quad + F'(\Phi)\Big[\Phi H^2 - |\nabla^\perp \mathbf{H}|^2 + \sum_{\alpha,\beta,i,j}\phi^\alpha_{ij}\phi^\beta_{ij}H^\alpha H^\beta\Big]\Big\}dv \\ &= \int_M \Big\{-\sum_{\alpha,i,j}F''(\Phi)\phi^\alpha_{ij}\Phi_j H^\alpha_i - F(\Phi)H^2 + F'(\Phi)\Big[\Phi H^2 - |\nabla^\perp \mathbf{H}|^2 \\ &\quad + \sum_{\alpha,\beta,i,j}\phi^\alpha_{ij}\phi^\beta_{ij}H^\alpha H^\beta\Big]\Big\}dv. \end{aligned}$$ ■

Proof of Theorem 1.5. Now, we prove the Theorem 1.5. If $\Phi = 0$, this theorem holds obviously. We consider the case $\Phi \neq 0$. Since

$$\frac{1}{2}\triangle F(\Phi) = \frac{1}{2}F''(\Phi)|\nabla\Phi|^2 + \frac{1}{2}F'(\Phi)\triangle\Phi,$$

by Lemma 2.1, we have

$$\begin{aligned} \frac{1}{2}\triangle F(\Phi) &= F'(\Phi)\Big[\sum_{\alpha,i,j,k}(\phi^\alpha_{ijk})^2 + 2\sum_{\alpha,i,j}\phi^\alpha_{ij}H^\alpha_{ij} + (2c + 2H^2 - \Phi)\Phi - \sum_{\alpha,\beta}R^2_{\alpha\beta12}\Big] \\ &\quad + \frac{1}{2}F''(\Phi)|\nabla\Phi|^2 \end{aligned}$$

Integrating both sides of this equation over M^2, by Stokes' theorem and Lemma 2.2, we have

$$\begin{aligned}
0 &= \int_M \frac{1}{2}\triangle F(\Phi)dv \\
&= \int_M \Big\{\frac{1}{2}F''(\Phi)|\nabla\Phi|^2 + F'(\Phi)\Big[\sum_{\alpha,i,j,k}(\phi^\alpha_{ijk})^2 + 2\sum_{\alpha,i,j}\phi^\alpha_{ij}H^\alpha_{ij} \\
&\quad +(2c+2H^2-\Phi)\Phi - \sum_{\alpha,\beta}R^2_{\alpha\beta 12}\Big]\Big\}dv \\
&= \int_M \Big\{F''(\Phi)\Big[\frac{1}{2}|\nabla\Phi|^2 - 2\sum_{\alpha,i,j}\phi^\alpha_{ij}\Phi_j H^\alpha_i\Big] + F'(\Phi)\Big[\sum_{\alpha,i,j,k}(\phi^\alpha_{ijk})^2 - 2|\nabla^\perp\mathbf{H}|^2 \\
&\quad +(2c+2H^2-\Phi)\Phi - \sum_{\alpha,\beta}R^2_{\alpha\beta 12}\Big]\Big\}dv.
\end{aligned}$$

(1) If $F'(\Phi) \geq 0$, by using Lemma 2.3 and Lemma 2.4, we obtain

$$\begin{aligned}
0 \geq \int_M &\Big\{F''(\Phi)\Big[\frac{1}{2}|\nabla\Phi|^2 - 2\sum_{\alpha,i,j}\phi^\alpha_{ij}\Phi_j H^\alpha_i\Big] \\
&+F'(\Phi)\Big[-|\nabla^\perp\mathbf{H}|^2 + (2c+2H^2-K(n)\Phi)\Phi\Big]\Big\}dv
\end{aligned}$$

where $K(n) = 1$ when $n = 3$ and $K(n) = \frac{3}{2}$ when $n \geq 4$. From Theorem 5.2,

$$\begin{aligned}
&\int_M F'(\Phi)\Big[\Phi H^2 - |\nabla^\perp\mathbf{H}|^2\Big]dv \\
&= \int_M \Big\{\sum_{\alpha,i,j}F''(\Phi)\phi^\alpha_{ij}\Phi_j H^\alpha_i + F(\Phi)H^2 - F'(\Phi)\sum_{\alpha,\beta,i,j}\phi^\alpha_{ij}\phi^\beta_{ij}H^\alpha H^\beta\Big\}dv.
\end{aligned}$$

Then

$$\begin{aligned}
0 \geq \int_M &\Big\{F''(\Phi)\Big[\frac{1}{2}|\nabla\Phi|^2 - \sum_{\alpha,i,j}\phi^\alpha_{ij}\Phi_j H^\alpha_i\Big] + F(\Phi)H^2 \\
&+F'(\Phi)\Big[(2c+H^2-K(n)\Phi)\Phi - \sum_{\alpha,\beta,i,j}\phi^\alpha_{ij}\phi^\beta_{ij}H^\alpha H^\beta\Big]\Big\}dv.
\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
\sum_{\alpha,\beta,i,j}\phi^\alpha_{ij}\phi^\beta_{ij}H^\alpha H^\beta &= \sum_{i,j}(\sum_\alpha \phi^\alpha_{ij}H^\alpha)^2 \\
&\leq \sum_{\alpha,i,j}(\phi^\alpha_{ij})^2\sum_\alpha (H^\alpha)^2 = \Phi H^2,
\end{aligned}$$

and the equality holds if and only if either $n = 3$ or $n \geq 4$ and $\phi^\alpha_{ij} = C_{ij}H^\alpha$ for some constant functions C_{ij}. Hence we can conclude that

$$0 \geq \int_M \Big\{F''(\Phi)\Big[\frac{1}{2}|\nabla\Phi|^2 - \sum_{\alpha,i,j}\phi^\alpha_{ij}\Phi_j H^\alpha_i\Big] + F(\Phi)H^2 + F'(\Phi)(2c - K(n)\Phi)\Phi\Big\}dv.$$

The equality holds if all the integral inequalities become equalities. If $\Phi \neq 0$, then $\phi_{111} = \phi_{122} = \frac{H_1}{2}$ and $\phi_{222} = \phi_{211} = \frac{H_2}{2}$ when $n = 3$. If $n \geq 4$, we have $\sum_\alpha \phi^\alpha_{11}\phi^\alpha_{12} = 0$ and $\sum_\alpha(\phi^\alpha_{11})^2 = \sum_\alpha(\phi^\alpha_{12})^2$ by Lemma 2.4. Since $\phi^\alpha_{ij} = C_{ij}H^\alpha$ for some constant functions C_{ij}, we get $C_{11}C_{12}H^2 = 0$ and $C^2_{11}H^2 = C^2_{12}H^2$. Since $\Phi \neq 0$, this implies that $H = 0$.

(2)If $F'(\Phi) \leq 0$, by using Lemma 2.3, Lemma 2.4 and Theorem 5.2, we obtain

$$\begin{aligned} 0 \leq & \int_M \Big\{F''(\Phi)\Big[\frac{1}{2}|\nabla\Phi|^2 - 2\sum_{\alpha,i,j}\phi^\alpha_{ij}\Phi_j H^\alpha_i\Big] \\ & +F'(\Phi)\Big[-|\nabla^\perp \mathbf{H}|^2 + (2c + 2H^2 - K(n)\Phi)\Phi\Big]\Big\}dv \\ = & \int_M \Big\{F''(\Phi)\Big[\frac{1}{2}|\nabla\Phi|^2 - \sum_{\alpha,i,j}\phi^\alpha_{ij}\Phi_j H^\alpha_i\Big] + F(\Phi)H^2 \\ & +F'(\Phi)\Big[(2c + H^2 - K(n)\Phi)\Phi - \sum_{\alpha,\beta,i,j}\phi^\alpha_{ij}\phi^\beta_{ij}H^\alpha H^\beta\Big]\Big\}dv. \end{aligned}$$

By Cauchy-Schwarz inequality, we can conclude that

$$0 \leq \int_M \Big\{F''(\Phi)\Big[\frac{1}{2}|\nabla\Phi|^2 - \sum_{\alpha,i,j}\phi^\alpha_{ij}\Phi_j H^\alpha_i\Big] + F(\Phi)H^2 + F'(\Phi)(2c - K(n)\Phi)\Phi\Big\}dv.$$

The equality holds if $\phi_{111} = \phi_{122} = \frac{H_1}{2}$ and $\phi_{222} = \phi_{211} = \frac{H_2}{2}$ when $n = 3$, $H = 0$ when $n \geq 4$. This completes the proof. ■

In order to get more information about the F-Willmore surface in the n-dimensional space form $\mathbb{N}^n(c)$. We try to find some suitable functions $F(\Phi)$ and apply them to the integral inequalities in Theorem 1.5.

Example 5.3. If M^2 is a F-Willmore surface and $F(\Phi)$ is a constant function, then $F''(\Phi) = F' = 0$. By the integral inequalities in Theorem 1.5, we have that

$$0 = \int_M F(\Phi)H^2 dv.$$

This implies that M is a minimal surface. The Willmore surface and the p-Willmore surface are two special cases of the F-Willmore surface in space form $\mathbb{N}^n(c)$, if we choose $F(\Phi) = \Phi$ and $F(\Phi) = \Phi^p$ respectively. Thus we can conclude that a F-Willmore surface in space form is a minimal surface if $F(\Phi)$ is a constant function, a Willmore surface if $F(\Phi) = \Phi$ and a p-Willmore surface if $F(\Phi) = \Phi^p$.

Example 5.4. Let M^2 be a constant mean curvature F-Willmore surface in the n-dimensional space form $\mathbb{N}^n(c)$. Choose $F(\Phi) = e^{\Phi}$, by the integral inequality in Theorem 1.5, we have that

$$0 \geq \int_M e^{\Phi}\Big\{\Big[\frac{1}{2}|\nabla\Phi|^2 - \sum_{\alpha,i,j}\phi_{ij}^{\alpha}\Phi_j H_i^{\alpha}\Big] + H^2 + (2c - K(n)\Phi)\Phi\Big\}dv.$$

If Φ is constant, then this integral inequality can be written by

$$0 \geq \int_M e^{\Phi}\Big[-K(n)\Phi^2 + 2c\Phi + H^2\Big]dv.$$

Assume that $0 \leq \Phi \leq \frac{c}{K(n)} + \frac{\sqrt{c^2+K(n)H^2}}{K(n)}$. Then we have $-K(n)\Phi^2+2c\Phi+H^2 \geq 0$ and

$$\int_M e^{\Phi}\Big[-K(n)\Phi^2 + 2c\Phi + H^2\Big]dv = 0.$$

This implies that either $\Phi = 0$ or $\Phi = \frac{c}{K(n)} + \frac{\sqrt{c^2+K(n)H^2}}{K(n)}$. Thus we can conclude that a constant mean curvature F-Willmore surface in space form with $\Phi = constant$ and $0 \leq \Phi \leq \frac{c}{K(n)} + \frac{\sqrt{c^2+K(n)H^2}}{K(n)}$ is a surface either $\Phi = 0$ or $\Phi = \frac{c}{K(n)} + \frac{\sqrt{c^2+K(n)H^2}}{K(n)}$.

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