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MILD WELL-POSEDNESS OF SECOND ORDER DIFFERENTIAL EQUATIONS ON THE REAL LINE

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Abstract. We study the $(W^{2,p}, W^{1,p})$-mild well-posedness of the second order differential equation $(P_2): u'' = Au + f$ on the real line $\mathbb{R}$, where A is a densely defined closed operator on a Banach space X. We completely characterize the $(W^{2,p}, W^{1,p})$-mild well-posedness of (P_2) by L^p-Fourier multipliers defined by the resolvent of A.

1. INTRODUCTION

Recently, Bu considered the $(W^{1,p}, L^p)$-mild well-posedness of the following problem:

$$(P_1): \quad u'(t) = Au(t) + f(t)$$

on the real line $\mathbb{R}$, where A is a closed operator on a complex Banach space X and $1 \le p < \infty$ [6]. He has shown that (P_1) is $(W^{1,p}, L^p)$-mildly well-posed if and only if $i\mathbb{R} \subset \rho(A)$ and the function m given by $m(x) = (ix - A)^{-1}$ defines an L^p-Fourier multiplier, where $\rho(A)$ denotes the resolvent set of A. On the other hand, the corresponding mild well-posedness for the periodic problem:

$$(P_{1,\text{per}}): \quad \begin{cases} u'(t) = Au(t) + f(t), & 0 \le t \le 2\pi, \\ u(0) = u(2\pi), \end{cases}$$

has been studied by Keyantuo and Lizama, where $f \in L^p(0, 2\pi; X), 1 \le p < \infty$ [8]. They have shown that $(P_{1,\text{per}})$ is $(W^{1,p}, L^p)$-mild well-posed if and only if $i\mathbb{Z} \subset \rho(A)$ and $((in - A)^{-1})_{n\in\mathbb{Z}}$ is an L^p-Fourier multiplier. In the same paper, they also considered the second order inhomogeneous problem of the form:

$$(P_{2,\text{per}}): \quad \begin{cases} u''(t) = Au(t) + f(t), & 0 \le t \le 2\pi, \\ u(0) = u(2\pi), \\ u'(0) = u'(2\pi), \end{cases}$$

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in the space $L^p(0, 2\pi; X), 1 \leq p < \infty$. They introduced two notions of mild well-posedness for $(P_{2,\text{per}})$ and they completely characterized the mild well-posedness of $(P_{2,\text{per}})$ by L^p-Fourier multipliers. More precisely, they proved that $(P_{2,\text{per}})$ is $(W^{2,p}, L^p)$-mildly well-posed if and only if $\{-k^2 : k \in \mathbb{Z}\} \subset \rho(A)$ and $((k^2 + A)^{-1})_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier; $(P_{2,\text{per}})$ is $(W^{2,p}, W^{1,p})$-mildly well-posed if and only if $\{-k^2 : k \in \mathbb{Z}\} \subset \rho(A)$ and $(ik(k^2 + A)^{-1})_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier. We note that the mild well-posedness of $(P_{1,\text{per}})$ was initially studied by Staffans in the special case when X is a Hilbert space and $p = 2$ [11].

In this paper, we study the $(W^{2,p}, W^{1,p})$-mild well-posedness of the following problem:

$$(P_2): \quad u''(t) = Au(t) + f(t)$$

on the real line $\mathbb{R}$, where A is a closed operator in a complex Banach space X and $1 \leq p < \infty$. Our main result is a characterization of the $(W^{2,p}, W^{1,p})$-mild well-posedness for (P_2): when A is densely defined, then (P_2) is $(W^{2,p}, W^{1,p})$-mild well-posed if and only if $(-\infty, 0] \subset \rho(A)$ and the functions m_1, m_2 given by $m_1(x) = -(x^2 + A)^{-1}$ and $m_2(x) = -ix(x^2 + A)^{-1}$ define L^p-Fourier multipliers. We also introduce and study the $(W^{2,p}, W^{1+\theta,p})$-mild well-posedness for (P_2) when $0 \leq \theta \leq 1$. When $\theta = 0$, we recover our main result.

We recall that the regularity of the problems (P_1) and (P_2) have been extensively studied in recent years. See e.g. [4-11] and references therein. Weis obtained a characterization of L^p-well-posedness for (P_1) using his operator-valued Fourier multiplier theorem on $L^p(\mathbb{R}; X)$ when X is a UMD Banach space and $1 < p < \infty$ [12]. Arendt and Bu studied L^p-well-posedness in interpolation spaces between X and $D(A)$ and mild well-poseness for (P_1) using the method of sum of bisectorial operators [4]. Schweiker studied the L^p-mild well-posedness and the well-posedness in the space $\text{BUC}(\mathbb{R}; X)$ of X-valued bounded and uniformly continuous functions for (P_1) and (P_2) [10]. Arendt, Batty and Bu obtained a characterization of the well-posedness of (P_1) in Hölder continuous function space [2] (see also [1] for a systematic study of (P_1) and (P_2)).

2. Mild-Well-Posedness and L^p-Fourier Multipliers

Let X be a complex Banach space and $1 \leq p < \infty$, we define as usual the first order Sobolev spaces by

$$W^{1,p}(\mathbb{R}; X) := \{f \in L^p(\mathbb{R}; X) : f' \in L^p(\mathbb{R}; X)\} \tag{1}$$

where f' is the distributional derivative of f, equipped with the norm

$$\|f\|_{W^{1,p}} := \|f\|_{L^p} + \|f'\|_{L^p}$$

and the second order Sobolev spaces by

$$W^{2,p}(\mathbb{R}; X) := \{f \in L^p(\mathbb{R}; X) : f', f'' \in L^p(\mathbb{R}; X)\} \tag{2}$$

equipped with the norm

$$\|f\|_{W^{2,p}} := \|f\|_{L^p} + \|f'\|_{L^p} + \|f''\|_{L^p}.$$

It is well known that $W^{1,p}(\mathbb{R}; X)$ and $W^{2,p}(\mathbb{R}; X)$ are Banach spaces.

Let A be a densely defined closed operator on X, we will always consider $D(A)$ as a Banach space equipped with its graph norm and we will consider the $D(A)$-valued Sobolev space $W^{2,p}(\mathbb{R}; D(A))$ which is a dense subspace of $L^p(\mathbb{R}; X)$ (see Lemma 2.3).

If $f \in L^p(\mathbb{R}; X)$, $u \in W^{2,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A))$ is called a strong L^p-solution of (P_2), if the equation (P_2) is satisfied a.e. on $\mathbb{R}$. We say that (P_2) is L^p-well-posed if for each $f \in L^p(\mathbb{R}; X)$, there exists a unique strong L^p-solutuion of (P_2). When (P_2) is L^p-well-posed, we let $\mathcal{B}f := u$, then $\mathcal{B}$ is linear and $\mathcal{B}$ maps continuously $L^p(\mathbb{R}; X)$ into $W^{2,p}(\mathbb{R}; X)$ by the Closed Graph Theorem. Therefore the image of $L^p(\mathbb{R}; X)$ by $\mathcal{B}$ is contained in $W^{1,p}(\mathbb{R}; X)$. On the other hand, it is easy to verify that $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$ when $u \in W^{2,p}(\mathbb{R}; D(A))$ by the L^p-well-posedness of (P_2), where $\mathcal{A}$ is defined by $\mathcal{A}u = u'' - Au$ with domain $D(\mathcal{A}) := W^{2,p}(\mathbb{R}; D(A))$.

For the characterization of the L^p-well-posedness of (P_2), strong conditions on the geometry of the underlying Banach space X and the Rademacher boundedness of the resolvent of A are needed [5]. This is the reason we consider in this paper a mild well-posedness for (P_2): besides other conditions on the closed operator A, we assume that there exists a strong L^p-solution of (P_2) only for f in a dense subspace (namely $W^{1,p}(\mathbb{R}; D(A))$) of $L^p(\mathbb{R}; X)$ (see [8] for a similar notion for $(P_{2,per})$).

Definition 2.1. Let $1 \le p < \infty$ and let A be a densely defined closed operator on X with domain $D(A)$. We say that (P_2) is $(W^{2,p}, W^{1,p})$-mildly well-posed, if there exists a bounded linear operator $\mathcal{B}$ that maps $L^p(\mathbb{R}; X)$ continuously into itself with range contained in $W^{1,p}(\mathbb{R}; X)$, $\mathcal{B}(W^{1,p}(\mathbb{R}; D(A))) \subset W^{2,p}(\mathbb{R}; D(A))$ and $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$ when $u \in W^{2,p}(\mathbb{R}; D(A))$, where $\mathcal{A}u = u'' - Au$ when $u \in W^{2,p}(\mathbb{R}; D(A))$. We call $\mathcal{B}$ the solution operator of the problem (P_2).

Remarks 2.1.

1. When (P_2) is $(W^{2,p}, W^{1,p})$-mildly well-posed, if $\mathcal{B}$ is the solution operator, for each $u \in W^{2,p}(\mathbb{R}; D(A))$, we have $(\mathcal{B}u)'' - A(\mathcal{B}u) = u$ by assumption. Suppose that $v \in W^{2,p}(\mathbb{R}; D(A))$ also satisfies $v'' - Av = u$, i.e., $\mathcal{A}v = u$. Then $\mathcal{B}\mathcal{A}v = \mathcal{B}u = v$ by assumption. This shows that for each $u \in W^{2,p}(\mathbb{R}; D(A))$, there exists a unique solution $v \in W^{2,p}(\mathbb{R}; D(A))$ satisfying $v'' - Av = u$ and this solution is given by $\mathcal{B}u$.
2. When (P_2) is $(W^{2,p}, W^{1,p})$-mildly well-posed, if $\mathcal{B}$ is the solution operator, then $\mathcal{B}$ is a bounded linear operator from $L^p(\mathbb{R}; X)$ into $W^{1,p}(\mathbb{R}; X)$. Indeed, if $u_n, u \in L^p(\mathbb{R}; X)$, $u_n \to u$ in $L^p(\mathbb{R}; X)$ and $\mathcal{B}u_n \to v$ in $W^{1,p}(\mathbb{R}; X)$, then $\mathcal{B}u_n \to v$ in $L^p(\mathbb{R}; X)$ as $W^{1,p}(\mathbb{R}; X) \subset L^p(\mathbb{R}; X)$ and the inclusion is

obviously continuous, therefore $v = \mathcal{B}u$ by the boundedness of $\mathcal{B}$ on $L^p(\mathbb{R}; X)$. This implies that $\mathcal{B}$ is a bounded linear operator from $L^p(\mathbb{R}; X)$ into $W^{1,p}(\mathbb{R}; X)$ by the Closed Graph Theorem. A similar argument shows that $\mathcal{B}$ is a bounded linear operator from $W^{1,p}(\mathbb{R}; D(A))$ into $W^{2,p}(\mathbb{R}; D(A))$. This implies that $\mathcal{B}$ acts also boundedly on $W^{2,p}(\mathbb{R}; D(A))$ by the Closed Graph Theorem.

In this paper, we will show that (P_2) is $(W^{2,p}, W^{1,p})$-mild well-posed if and only if $(-\infty, 0] \subset \rho(A)$ and the functions m_1, m_2 given by $m_1(x) = -(x^2 + A)^{-1}$ and $m_2(x) = -ix(x^2 + A)^{-1}$ define L^p-Fourier multipliers. This may be considered as the parallel result for (P_2) of Keyantuo and Lizama's result obtained in [8] for the periodic problem $(P_{2,\text{per}})$.

In order to study the $(W^{2,p}, W^{1,p})$-mild well-posedness, we need to introduce the Fourier transform for vector-valued functions. Let X be a complex Banach space, we denote by $\mathcal{S}(\mathbb{R}; X)$ the Schwartz class consisting of all X-valued rapidly decreasing smooth functions on $\mathbb{R}$, more precisely an X-valued function ϕ on $\mathbb{R}$ is in $\mathcal{S}(\mathbb{R}; X)$ if ϕ is infinitely differentiable and for all $m, n \in \mathbb{N} \cup \{0\}$, we have

$$\sup_{s \in \mathbb{R}} (1 + |s|)^m \left\| \phi^{(n)}(s) \right\| < \infty.$$

It is well-known that the Fourier transform $\mathcal{F}$ defined on $L^1(\mathbb{R}; X)$ by

$$(\mathcal{F}\phi)(t) := \int_{\mathbb{R}} e^{-its} \phi(s) ds, \quad (t \in \mathbb{R})$$

is an isomorphism on $\mathcal{S}(\mathbb{R}; X)$ and its inverse on $\mathcal{S}(\mathbb{R}; X)$ is given by

$$(\mathcal{F}^{-1}\phi)(t) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{its} \phi(s) ds, \quad (t \in \mathbb{R}).$$

It is well known that $\mathcal{S}(\mathbb{R}; X)$ is dense in $L^p(\mathbb{R}; X)$, $W^{1,p}(\mathbb{R}; X)$ and $W^{2,p}(\mathbb{R}; X)$ when $1 \le p < \infty$ (see Lemma 2.3). Thus $W^{1,p}(\mathbb{R}; X)$ (resp. $W^{2,p}(\mathbb{R}; X)$) is the completion of $\mathcal{S}(\mathbb{R}; X)$ under the norm $\| \cdot \|_{W^{1,p}}$ (resp. $\| \cdot \|_{W^{2,p}}$).

Let $m : \mathbb{R} \to \mathcal{L}(X)$ be a bounded measurable function and $1 \le p < \infty$, where $\mathcal{L}(X)$ is the space of all bounded linear operators on X. We say that m defines an L^p-Fourier multiplier, if there exists a constant $C > 0$ such that

$$\left\| \mathcal{F}^{-1}(m\mathcal{F}f) \right\|_{L^p} \le C \left\| f \right\|_{L^p}$$

whenever $f \in \mathcal{S}(\mathbb{R}; X)$ [1, 12]. We note that when $f \in \mathcal{S}(\mathbb{R}; X)$, the function $m\mathcal{F}f$ is in $L^1(\mathbb{R}; X)$, therefore the term $\mathcal{F}^{-1}(m\mathcal{F}f)$ in the left hand side makes sense. When m is an L^p-Fourier multiplier, there exists a unique bounded linear operator B on $L^p(\mathbb{R}; X)$ satisfying $\mathcal{F}(Bf) = m\mathcal{F}f$ when $f \in \mathcal{S}(\mathbb{R}; X)$. This follows easily from the density of $\mathcal{S}(\mathbb{R}; X)$ in $L^p(\mathbb{R}; X)$ [5].

Next we introduce the weighted L^p-spaces $L^p_{\alpha,\omega}(\mathbb{R};X)$, first order weighted Sobolev spaces $W^{1,p}_{\alpha,\omega}(\mathbb{R};X)$ and second order weighted Sobolev spaces $W^{2,p}_{\alpha,\omega}(\mathbb{R};X)$. We let ω be a fixed C^2-function on $\mathbb{R}$ such that $\omega(t)\geq 1$ for $t\in\mathbb{R}$ and $\omega(t)=|t|$ when $|t|\geq 2$. For fixed $\alpha>0$, we let $L^p_{\alpha,\omega}(\mathbb{R};X)$ be the space of all measurable functions $f:\mathbb{R}\to X$ such that

$$\|f\|_{L^p_{\alpha,\omega}} := \left(\int_{\mathbb{R}} e^{-p\alpha\omega(t)}\|f(t)\|^p\,dt\right)^{1/p} < \infty.$$

$L^p_{\alpha,\omega}(\mathbb{R};X)$ equipped with the norm $\|\cdot\|_{L^p_{\alpha,\omega}}$ becomes a Banach space. We define first weighted Sobolev spaces $W^{1,p}_{\alpha,\omega}(\mathbb{R};X)$ as the space of all functions $f\in L^p_{\alpha,\omega}(\mathbb{R};X)$ such that $f'\in L^p_{\alpha,\omega}(\mathbb{R};X)$. Here f' is understood in the sense of distributions. $W^{1,p}_{\alpha,\omega}(\mathbb{R};X)$ equipped with the norm

$$\|f\|_{W^{1,p}_{\alpha,\omega}} := \|f\|_{L^p_{\alpha,\omega}} + \|f'\|_{L^p_{\alpha,\omega}}$$

is a Banach space. In a similar way, we define the second order weighted Sobolev spaces $W^{2,p}_{\alpha,\omega}(\mathbb{R};X)$ as the space of all functions $f\in L^p_{\alpha,\omega}(\mathbb{R};X)$ such that $f',f''\in L^p_{\alpha,\omega}(\mathbb{R};X)$, where f',f'' are also understood in the sense of distributions. $W^{2,p}_{\alpha,\omega}(\mathbb{R};X)$ equipped with the norm

$$\|f\|_{W^{2,p}_{\alpha,\omega}} := \|f\|_{L^p_{\alpha,\omega}} + \|f'\|_{L^p_{\alpha,\omega}} + \|f''\|_{L^p_{\alpha,\omega}}$$

is a Banach space. We need the following preparation.

Lemma 2.1. *The mapping $f\mapsto\Phi(f):=e^{-\alpha\omega}f$ is an isomorphism from $L^p_{\alpha,\omega}(\mathbb{R};X)$ into $L^p(\mathbb{R};X)$, from $W^{1,p}_{\alpha,\omega}(\mathbb{R};X)$ into $W^{1,p}(\mathbb{R};X)$ and from $W^{2,p}_{\alpha,\omega}(\mathbb{R};X)$ into $W^{2,p}(\mathbb{R};X)$.*

Proof. Follows the same lines as the proof in Bu [6], we have that the mapping $f\mapsto\Phi(f)$ is an isomorphism from $L^p_{\alpha,\omega}(\mathbb{R};X)$ into $L^p(\mathbb{R};X)$ and from $W^{1,p}_{\alpha,\omega}(\mathbb{R};X)$ into $W^{1,p}(\mathbb{R};X)$. Next we prove that the mapping $f\mapsto\Phi(f)$ is also an isomorphism from $W^{2,p}_{\alpha,\omega}(\mathbb{R};X)$ into $W^{2,p}(\mathbb{R};X)$. Indeed, we note that when $f\in W^{2,p}_{\alpha,\omega}(\mathbb{R};X)$,

$$(e^{-\alpha\omega}f)' = -\alpha w' e^{-\alpha\omega}f + e^{-\alpha\omega}f',$$

and

$$\begin{aligned}(e^{-\alpha\omega}f)'' &= -\alpha\omega'' e^{-\alpha\omega}f - \alpha w'(-\alpha w' e^{-\alpha\omega}f + e^{-\alpha\omega}f') - \alpha\omega' e^{-\alpha\omega}f' + e^{-\alpha\omega}f''\\ &= (-\alpha\omega'' + \alpha^2(\omega')^2)e^{-\alpha\omega}f - 2\alpha\omega' e^{-\alpha\omega}f' + e^{-\alpha\omega}f''.\end{aligned}$$

observe that ω',ω'' are bounded on $\mathbb{R}$. Thus $\Phi(f)\in W^{2,p}(\mathbb{R};X)$ whenever $f\in W^{2,p}_{\alpha,\omega}(\mathbb{R};X)$ and $\|\Phi(f)\|_{W^{2,p}}\leq C\,\|f\|_{W^{2,p}_{\alpha,\omega}}$ for some constant $C\geq 0$ depending only on α, ω and p. The map Φ is clearly injective from $W^{2,p}_{\alpha,\omega}(\mathbb{R}; X)$ into $W^{2,p}(\mathbb{R}; X)$, it remains to show that Φ is surjective. To this end we let $g \in W^{2,p}(\mathbb{R}; X)$ and $f = e^{\alpha\omega} g$. We observe that

$$f' = \alpha w' e^{\alpha\omega} g + e^{\alpha\omega} g',$$

and

$$\begin{aligned} f'' &= \alpha\omega'' e^{\alpha\omega} g + \alpha w'(\alpha w' e^{\alpha\omega} g + e^{\alpha\omega} g') + \alpha\omega' e^{\alpha\omega} g' + e^{\alpha\omega} g'' \\ &= (\alpha\omega'' + \alpha^2(\omega')^2) e^{\alpha\omega} g + 2\alpha\omega' e^{\alpha\omega} g' + e^{\alpha\omega} g'', \end{aligned}$$

which implies that $f \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; X)$ and $\Phi(f) = g$. Here we have also used the fact that ω', ω'' are bounded on $\mathbb{R}$. This completes the proof. ■

We will transform the $(W^{2,p}, W^{1,p})$-mild well-posedness of (P_2) into a similar mild well-posedness in weighted function spaces. This idea was firstly used by Mielke in the study of L^p-well-posedness for (P_1) [9] (see also [6] and [10]).

Definition 2.2. Let X be a Banach space, $1 \le p < \infty,\ \alpha > 0$ and let $A : D(A) \to X$ be a densely defined closed operator on X. We say that (P_2) is $(W^{2,p}_{\alpha,\omega}, W^{1,p}_{\alpha,\omega})$-mildly well-posed, if there exists a bounded linear operator $\mathcal{B}_\alpha$ that maps boundedly from $L^p_{\alpha,\omega}(\mathbb{R}; X)$ into $W^{1,p}_{\alpha,\omega}(\mathbb{R}; X)$, $\mathcal{B}_\alpha(W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))) \subset W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$, $\mathcal{B}_\alpha$ also satisfies $\mathcal{B}_\alpha \mathcal{A}_\alpha u = \mathcal{A}_\alpha \mathcal{B}_\alpha u = u$ when $u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$, where $\mathcal{A}_\alpha = u'' - Au$ when $u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$.

Remark 2.1. When (P_2) is $(W^{2,p}_{\alpha,\omega}, W^{1,p}_{\alpha,\omega})$-mildly well-posed, for each $u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$, we have $(\mathcal{B}_\alpha u)'' - A(\mathcal{B}_\alpha u) = u$ by assumption. Suppose that $v \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$ also satisfies $v'' - Av = u$, i.e., $\mathcal{A}_\alpha v = u$. Then $\mathcal{B}_\alpha \mathcal{A}_\alpha v = \mathcal{B}_\alpha u = v$ by assumption. This shows that for each $u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$, there exists a unique solution $v \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$ satisfying $v'' - Av = u$ and this solution is given by $\mathcal{B}_\alpha u$.

The following lemma will be useful for proving the main results of this paper.

Lemma 2.2. *Let X be a Banach space, $1 \le p < \infty$ and let $A : D(A) \to X$ be a densely defined closed operator on X. We assume that (P_2) is $(W^{2,p}, W^{1,p})$-mildly well-posed. Then it is $(W^{2,p}_{\alpha,\omega}, W^{1,p}_{\alpha,\omega})$-mildly well-posed when $\alpha > 0$ is small enough.*

Proof. Let $\Phi_{\alpha,\omega}(t) = e^{-\alpha\omega(t)}$ and $\Phi_{-\alpha,\omega}(t) = e^{\alpha\omega(t)}$ when $t \in \mathbb{R}$. Since (P_2) is $(W^{2,p}, W^{1,p})$-mildly well-posed, there exists a bounded linear operator $\mathcal{B}$ that maps $L^p(\mathbb{R}; X)$ continuously into itself with range in $W^{1,p}(\mathbb{R}; X)$, $\mathcal{B}(W^{1,p}(\mathbb{R}; D(A))) \subset W^{2,p}(\mathbb{R}; D(A))$ and $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$ when $u \in W^{2,p}(\mathbb{R}; D(A))$. Let $u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$ and let $u_1 = \Phi_{\alpha,\omega} u$. It follows from Lemma 2.1 that $u_1 \in W^{2,p}(\mathbb{R}; D(A))$. We have $u_1'' - Au_1 \in L^p(\mathbb{R}; X)$ and $\mathcal{B}(u_1'' - Au_1) = u_1$ by assumption and Remarks 2.1. We observe that

$$u_1' = -\alpha\omega' \Phi_{\alpha,\omega} u + \Phi_{\alpha,\omega} u',$$

and

$$\begin{aligned}u_1'' &= -\alpha\omega''\Phi_{\alpha,\omega}u - \alpha\omega'(-\alpha\omega'\Phi_{\alpha,\omega}u + \Phi_{\alpha,\omega}u') - \alpha\omega'\Phi_{\alpha,\omega}u' + \Phi_{\alpha,\omega}u'' \\ &= (-\alpha\omega'' + \alpha^2(w')^2)\Phi_{\alpha,\omega}u - 2\alpha\omega'\Phi_{\alpha,\omega}u' + \Phi_{\alpha,\omega}u''.\end{aligned}$$

It follows that

$$\mathcal{B}(u_1''-Au_1)=-\alpha\mathcal{B}[(\omega''-\alpha(w')^2)\Phi_{\alpha,\omega}u+2\omega'\Phi_{\alpha,\omega}u']+\mathcal{B}\Phi_{\alpha,\omega}u''-\mathcal{B}A\Phi_{\alpha,\omega}u=\Phi_{\alpha,\omega}u,$$

which implies

(3) $$\mathcal{B}\Phi_{\alpha,\omega}(u'' - Au) = \Phi_{\alpha,\omega}u + \alpha\mathcal{B}[(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}u + 2\omega'\Phi_{\alpha,\omega}u'].$$

For $u \in W^{1,p}(\mathbb{R}; X)$, we define

$$Du := \mathcal{B}[(\omega'' + \alpha(w')^2)u + 2\omega'u'].$$

By Remarks 2.1, $\mathcal{B}$ is a bounded linear operator from $W^{1,p}(\mathbb{R}; D(A))$ into $W^{2,p}(\mathbb{R}; D(A))$, it follows that D is bounded and linear on $W^{2,p}(\mathbb{R}; D(A))$. Since $\mathcal{B}$ maps boundedly $L^p(\mathbb{R}; X)$ into $W^{1,p}(\mathbb{R}; X)$ by assumption, D is also bounded and linear on $W^{1,p}(\mathbb{R}; X)$. By (3), we have

$$\mathcal{B}\Phi_{\alpha,\omega}(u'' - Au) = (I + \alpha D)\Phi_{\alpha,\omega}u$$

when $u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$. We note that the bounded linear operator $I + \alpha D$ is invertible on $W^{1,p}(\mathbb{R}; X)$ and $W^{2,p}(\mathbb{R}; D(A))$ when $\alpha > 0$ is small enough. For such α, we obtain

(4) $$\Phi_{-\alpha,\omega}(I + \alpha D)^{-1}\mathcal{B}\Phi_{\alpha,\omega}(u'' - Au) = u,$$

whenever $u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$. Let

$$\mathcal{B}_\alpha := \Phi_{-\alpha,\omega}(I + \alpha D)^{-1}\mathcal{B}\Phi_{\alpha,\omega}.$$

If $u \in L^p_{\alpha,\omega}(\mathbb{R}; X)$, then $\mathcal{B}\Phi_{\alpha,\omega}u \in W^{1,p}(\mathbb{R}; X)$ by assumption and Lemma 2.1, it follows that $\mathcal{B}_\alpha u \in W^{1,p}_{\alpha,\omega}(\mathbb{R}; X)$ as we have shown that $1 + \alpha D$ is invertible on $W^{1,p}(\mathbb{R}; X)$. Thus $\mathcal{B}_\alpha$ is bounded and linear from $L^p_{\alpha,\omega}(\mathbb{R}; X)$ into $W^{1,p}_{\alpha,\omega}(\mathbb{R}; X)$.

We notice that when $u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$, we have $\mathcal{B}\Phi_{\alpha,\omega}u \in W^{2,p}(\mathbb{R}; D(A))$ by assumption and Lemma 2.1. Since $(I + \alpha D)^{-1}$ is bounded on $W^{2,p}(\mathbb{R}; D(A))$, it follows that $\mathcal{B}_\alpha u = \Phi_{-\alpha,\omega}(I + \alpha D)^{-1}\mathcal{B}\Phi_{\alpha,\omega}u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$ by Lemma 2.1. We have shown that $\mathcal{B}_\alpha(W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))) \subset W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$. It is clear from the definition of $\mathcal{B}_\alpha$ and (4) that $\mathcal{B}_\alpha\mathcal{A}_\alpha u = u$ when $u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$.

Next we show that $\mathcal{A}_\alpha\mathcal{B}_\alpha u = u$ when $u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$. Let $v = \mathcal{B}_\alpha u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$. We claim that $v'' = Av + u$. In fact, from the definition of v, we see that

$$\Phi_{\alpha,\omega}v + \alpha D\Phi_{\alpha,\omega}v = \mathcal{B}\Phi_{\alpha,\omega}u,$$

which implies

$$\begin{aligned}\Phi_{\alpha,\omega}v &= \mathcal{B}\Phi_{\alpha,\omega}u - \alpha D\Phi_{\alpha,\omega}v \\ &= \mathcal{B}\Phi_{\alpha,\omega}u - \alpha\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v - 2\alpha\mathcal{B}\omega'\Phi_{\alpha,\omega}v'.\end{aligned}$$

Thus we obtain

$$v = \Phi_{-\alpha,\omega}\mathcal{B}\Phi_{\alpha,\omega}u - \alpha\Phi_{-\alpha,\omega}\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v - 2\alpha\Phi_{-\alpha,\omega}\mathcal{B}\omega'\Phi_{\alpha,\omega}v'. \tag{5}$$

Since $\Phi_{\alpha,\omega}u,\ (\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v \in W^{2,p}(\mathbb{R}; D(A))$, it follows that $\Phi_{-\alpha,\omega}\mathcal{B}\Phi_{\alpha,\omega}u$ and $\Phi_{-\alpha,\omega}\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v$ belong to $W^{2,p}_{\alpha,w}(\mathbb{R}; D(A))$ by Lemma 2.1. This implies that $\Phi_{-\alpha,\omega}\mathcal{B}\omega'\Phi_{\alpha,\omega}v' \in W^{2,p}_{\alpha,w}(\mathbb{R}; D(A))$ by (5) as $v \in W^{2,p}_{\alpha,w}(\mathbb{R}; D(A))$. Thus $\mathcal{B}\omega'\Phi_{\alpha,\omega}v' \in W^{2,p}(\mathbb{R}; D(A))$ by Lemma 2.1. It is clear that $\mathcal{B}\Phi_{\alpha,\omega}u$ and $\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v$ belong to $W^{2,p}(\mathbb{R}; D(A))$ by assumption and Lemma 2.1. Therefore

$$[\mathcal{B}\Phi_{\alpha,\omega}u]'' = A\mathcal{B}\Phi_{\alpha,\omega}u + \Phi_{\alpha,\omega}u,$$

$$[\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v]'' = A\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v + (\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v,$$

and

$$[\mathcal{B}\omega'\Phi_{\alpha,\omega}v']'' = A\mathcal{B}\omega'\Phi_{\alpha,\omega}v' + \omega'\Phi_{\alpha,\omega}v'.$$

by the assumption that $\mathcal{A}\mathcal{B}u = u$ when $u \in W^{2,p}(\mathbb{R}; D(A))$. By (5), we have that

$$\begin{aligned}v' = {}& \alpha\omega'\Phi_{-\alpha,\omega}\mathcal{B}\Phi_{\alpha,\omega}u + \Phi_{-\alpha,\omega}[\mathcal{B}\Phi_{\alpha,\omega}u]' - \alpha^2\omega'\Phi_{-\alpha,\omega}\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v \\ & - \alpha\Phi_{-\alpha,\omega}[\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v]' - 2\alpha^2\omega'\Phi_{-\alpha,\omega}\mathcal{B}\omega'\Phi_{\alpha,\omega}v' \\ & - 2\alpha\Phi_{-\alpha,\omega}[\mathcal{B}\omega'\Phi_{\alpha,\omega}v']',\end{aligned}$$

which implies

$$\begin{aligned}v'' = {}& \alpha\omega''\Phi_{-\alpha,\omega}\mathcal{B}\Phi_{\alpha,\omega}u + \alpha\omega'\{\alpha\omega'\Phi_{-\alpha,\omega}\mathcal{B}\Phi_{\alpha,\omega}u + \Phi_{-\alpha,\omega}[\mathcal{B}\Phi_{\alpha,\omega}u]'\} \\ & + \alpha\omega'\Phi_{-\alpha,\omega}[\mathcal{B}\Phi_{\alpha,\omega}u]' + \Phi_{-\alpha,\omega}[\mathcal{B}\Phi_{\alpha,\omega}u]'' - \alpha^2\omega''\Phi_{-\alpha,\omega}\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v \\ & + -\alpha^2\omega'\{\alpha\omega'\Phi_{-\alpha,\omega}\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v\Phi_{-\alpha,\omega}[\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v]'\} \\ & - \alpha^2\omega'\Phi_{-\alpha,\omega}[\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v]' - \alpha\Phi_{-\alpha,\omega}[\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v]'' \\ & - 2\alpha^2\omega''\Phi_{-\alpha,\omega}\mathcal{B}\omega'\Phi_{\alpha,\omega}v' - 2\alpha^2\omega'\{\alpha\omega'\Phi_{-\alpha,\omega}\mathcal{B}\omega'\Phi_{\alpha,\omega}v' + \Phi_{-\alpha,\omega}[\mathcal{B}\omega'\Phi_{\alpha,\omega}v']'\} \\ & - 2\alpha^2\omega'\Phi_{-\alpha,\omega}[\mathcal{B}\omega'\Phi_{\alpha,\omega}v']' - 2\alpha\Phi_{-\alpha,\omega}[\mathcal{B}\omega'\Phi_{\alpha,\omega}v']'' \\ = {}& \alpha\omega''\{\Phi_{-\alpha,\omega}\mathcal{B}\Phi_{\alpha,\omega}u - \alpha\Phi_{-\alpha,\omega}\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v - 2\alpha\Phi_{-\alpha,\omega}\mathcal{B}\omega'\Phi_{\alpha,\omega}v'\} \\ & + \alpha^2(\omega')^2\Phi_{-\alpha,\omega}\mathcal{B}\Phi_{\alpha,\omega}u + 2\alpha\omega'\Phi_{-\alpha,\omega}[\mathcal{B}\Phi_{\alpha,\omega}u]' + \Phi_{-\alpha,\omega}A\mathcal{B}\Phi_{\alpha,\omega}u + u \\ & - \alpha^3(\omega')^2\Phi_{-\alpha,\omega}\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v - 2\alpha^2\omega'\Phi_{-\alpha,\omega}[\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v]' \\ & - \alpha\Phi_{-\alpha,\omega}A\mathcal{B}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v - \alpha\Phi_{-\alpha,\omega}(\omega'' - \alpha(w')^2)\Phi_{\alpha,\omega}v \\ & - -2\alpha^3(\omega')^2\Phi_{-\alpha,\omega}\mathcal{B}\omega'\Phi_{\alpha,\omega}v'4\alpha^2\omega'\Phi_{-\alpha,\omega}[\mathcal{B}\omega'\Phi_{\alpha,\omega}v']' \\ & - -2\alpha\Phi_{-\alpha,\omega}A\mathcal{B}\omega'\Phi_{\alpha,\omega}v' - 2\alpha\Phi_{-\alpha,\omega}\omega'\Phi_{\alpha,\omega}v'\end{aligned}$$

$$
\begin{aligned}
&= \alpha\omega''v + \alpha^2(\omega')^2 v + 2\alpha\omega'\Phi_{-\alpha,\omega}[\mathcal{B}\Phi_{\alpha,\omega}u \\
&\quad - \alpha\mathcal{B}(\omega''-\alpha(w')^2)\Phi_{\alpha,\omega}v - 2\alpha\mathcal{B}\omega'\Phi_{\alpha,\omega}v']' + Av + u - \alpha\omega''v + \alpha^2(\omega')^2v - 2\alpha\omega'v' \\
&= Av + u + 2\alpha^2(\omega')^2v - 2\alpha\omega'v' + 2\alpha\omega'\Phi_{-\alpha,\omega}[\Phi_{\alpha,\omega}v]' \\
&= Av + u + 2\alpha^2(\omega')^2v - 2\alpha\omega'v' + 2\alpha\omega'\Phi_{-\alpha,\omega}[-\alpha\omega'\Phi_{\alpha,\omega}v + \Phi_{\alpha,\omega}v'] \\
&= Av + u + 2\alpha^2(\omega')^2v - 2\alpha\omega'v' - 2\alpha^2(\omega')^2v + 2\alpha\omega'v' \\
&= Av + u.
\end{aligned}
$$

Thus $\mathcal{A}_\alpha\mathcal{B}_\alpha u = u$ when $u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$. We have shown that (P_2) is $(W^{2,p}_{\alpha,\omega}, W^{1,p}_{\alpha,\omega})$-mildly well-posed. This completes the proof. ■

Lemma 2.3. *Let X be a Banach space and $1 \le p < \infty$, then $\mathcal{S}(\mathbb{R}; X)$ is dense in $L^p(\mathbb{R}; X)$, $W^{1,p}(\mathbb{R}; X)$ and $W^{2,p}(\mathbb{R}; X)$. If $A : D(A) \to X$ is a densely defined closed operator on X, then $\mathcal{S}(\mathbb{R}; D(A))$ is dense in $L^p(\mathbb{R}; X)$.*

Proof. The proof is a modification of the proof of Lemma 3 of Bu [6]. We omit it. ■

Now we are going to prove the following result which characterizes $(W^{2,p}, W^{1,p})$-mildly well-posedness in terms of operator-valued L^p-Fourier multipliers defined by the resolvent of A.

Theorem 2.1. *Let X be a Banach space, $1 \le p < \infty$ and let $A : D(A) \to X$ be a densely defined closed operator on X. Then the following assertions are equivalent.*

(i) (P_2) *is $(W^{2,p}, W^{1,p})$-mildly well-posed;*

(ii) $(-\infty, 0] \subset \rho(A)$ *and the functions m_1, m_2 defined on $\mathbb{R}$ by $m_1(x) = -(x^2 + A)^{-1}$ and $m_2(x) = -ix(x^2 + A)^{-1}$ are L^p-Fourier multipliers.*

Proof. (i)⇒ (ii): Suppose that (P_2) is $(W^{2,p}, W^{1,p})$-mildly well-posed, then (P_2) is $(W^{2,p}_{\alpha,\omega}, W^{1,p}_{\alpha,\omega})$-mildly well posed when $\alpha > 0$ is small enough by Lemma 2.2. By the Closed Graph Theorem, there exists a constant $C > 0$ satisfying

$$\|\mathcal{B}_\alpha f\|_{W^{1,p}_{\alpha,\omega}} \le C\,\|f\|_{L^p_{\alpha,\omega}} \tag{6}$$

when $f \in L^p_{\alpha,\omega}(\mathbb{R}; X)$. Firstly, we show that $(-\infty, 0] \subset \rho(A)$. Let $\xi \in \mathbb{R}$ and $y \in X$ be fixed. Then there exits $y_n \in D(A)$ such that $y_n \to y$ when $n \to \infty$ as $D(A)$ is dense in X by assumption. We define $f(t) = e^{i\xi t}y$ and $f_n(t) = e^{i\xi t}y_n$ for $t \in \mathbb{R}$. Then $f \in L^p_{\alpha,\omega}(\mathbb{R}; X)$, $f_n \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$ and $f_n \to f$ in $L^p_{\alpha,\omega}(\mathbb{R}; X)$ when $n \to \infty$. Let $u_n := \mathcal{B}_\alpha f_n$, then $u_n \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$ by the $(W^{2,p}_{\alpha,\omega}, W^{1,p}_{\alpha,\omega})$-mild well-posedness of (P_2). We have

$$u_n''(t) - Au_n(t) = f_n(t)$$

a.e. on $\mathbb{R}$ by the equality $\mathcal{A}_\alpha \mathcal{B}_\alpha u = u$ when $u \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$.

Since $f_n(s+t) = e^{i\xi s} f_n(t)$ when $t \in \mathbb{R}$, both functions $u_n(s+\cdot)$ and $e^{i\xi s}u_n$ in $W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$ are strong L^p-solutions of

$$u'' - Au = e^{i\xi s} f_n.$$

We deduce that $u_n(s+t) = e^{i\xi s}u_n(t)$ when $s, t \in \mathbb{R}$ by Remark 2.1. Therefore there exists $x_n \in D(A)$ such that $u_n(t) = e^{i\xi t}x_n$ when $t \in \mathbb{R}$. Thus

$$-\xi^2 e^{i\xi t}x_n - e^{i\xi t}Ax_n = e^{i\xi t}y_n$$

when $t \in \mathbb{R}$ or equivalently

$$-\xi^2 x_n - Ax_n = y_n. \tag{7}$$

Since $f_n \to f$ in $L^p_{\alpha,\omega}(\mathbb{R}; X)$, it follows that $u_n \to \mathcal{B}_\alpha f$ in $L^p_{\alpha,\omega}(\mathbb{R}; X)$ when $n \to \infty$. Hence there exists $x \in X$ such that $(\mathcal{B}_\alpha f)(t) = e^{i\xi t}x$ when $t \in \mathbb{R}$ and $x_n \to x$ when $n \to \infty$. We conclude from (7) and the closedness of A that $x \in D(A)$ and

$$-\xi^2 x - Ax = y, \tag{8}$$

which implies that $-\xi^2 - A$ is surjective.

To show that $-\xi^2 - A$ is also injective, we assume that $Ax_0 = -\xi^2 x_0$ for some $x_0 \in D(A)$. Then $u_0 \in W^{2,p}_{\alpha,\omega}(\mathbb{R}; D(A))$ defined by $u_0(t) = e^{i\xi t}x_0$ solves the equation $u'' - Au = 0$. We deduce that $x_0 = 0$ by Remark 2.1. Thus $-\xi^2 - A$ is injective. We have shown that $-\xi^2 \in \rho(A)$ since A is closed. Since $\xi \in \mathbb{R}$ is arbitrary, we conclude that $(-\infty, 0] \subset \rho(A)$.

It follows from (8) that $x = (-\xi^2 - A)^{-1}y$. We note that $\|f\|_{L^p_{\alpha,\omega}} = c_{\alpha,\omega,p}\|y\|$, $\|\mathcal{B}_\alpha f\|_{L^p_{\alpha,\omega}} = c_{\alpha,\omega,p}\|x\|$ and $\|(\mathcal{B}_\alpha f)'\|_{L^p_{\alpha,\omega}} = c_{\alpha,\omega,p}\|i\xi x\|$ for some constant $c_{\alpha,\omega,p} > 0$ depending only on α, ω and p. By (6), we have

$$\|x\| \le C\|y\|, \quad \|i\xi x\| \le C\|y\|,$$

or equivalently

$$\left\|(-\xi^2 - A)^{-1}\right\| \le C, \quad \left\|i\xi(-\xi^2 - A)^{-1}\right\| \le C$$

when $\xi \in \mathbb{R}$.

We have shown that $(-\infty, 0] \subset \rho(A)$ and the functions m_1, m_2 defined on $\mathbb{R}$ by $m_1(x) := (-x^2 - A)^{-1}$ and $m_2(x) := ix(-x^2 - A)^{-1}$ are uniformly bounded on $\mathbb{R}$. For fixed $f \in L^p(\mathbb{R}; X)$, there exists a sequence $(f_n)_{n\ge1} \subset \mathcal{S}(\mathbb{R}; D(A))$ such that $f_n \to f$ in $L^p(\mathbb{R}; X)$ when $n \to \infty$ by Lemma 2.3. Let $u_n := \mathcal{B}f_n \in W^{2,p}(\mathbb{R}; D(A))$. Then $(u_n)'' - Au_n = f_n$ and $u_n \to \mathcal{B}f$ in $L^p(\mathbb{R}; X)$ when $n \to \infty$ since $\mathcal{B}$ maps $L^p(\mathbb{R}; X)$ continuously into itself by assumption.

On the other hand, the function g_n given by $g_n(x) := (-x^2 - A)^{-1}\mathcal{F}f_n(x)$ is in $\mathcal{S}(\mathbb{R}; D(A))$. Here we have used the facts that for each $n \in \mathbb{N}$, $\mathcal{F}f_n \in \mathcal{S}(\mathbb{R}; D(A))$, m_1 is infinitely differentiable and $m_1^{(k)}(x) = \sum_{n=1}^{k+1} p_n(x)m_1(x)^n$ for all $k \in \mathbb{N}$, where p_n is a polynomial. Let $v_n := \mathcal{F}^{-1}g_n$, then $v_n \in \mathcal{S}(\mathbb{R}; D(A))$ and thus $v_n \in W^{2,p}(\mathbb{R}; D(A))$. Now we can see easily that $v_n'' - Av_n = f_n$. It follows from Remarks 2.1 that $u_n = v_n$. This shows that m_1 is an L^p-Fourier multiplier and the bounded linear operator on $L^p(\mathbb{R}; X)$ defined by m_1 is in fact $\mathcal{B}$. In a similar way, we show that m_2 is also an L^p-Fourier multiplier. Therefore the implication (i) $\Rightarrow$ (ii) is true.

(ii) $\Rightarrow$ (i): We assume that $(-\infty, 0] \subset \rho(A)$ and the functions m_1, m_2 given by $m_1(x) = -(x^2 + A)^{-1}$ and $m_2(x) = -ix(x^2 + A)^{-1}$ define L^p-Fourier multipliers. Then m_1 and m_2 are uniformly bounded on $\mathbb{R}$ [12]. Let $\mathcal{B}$ and $\mathcal{B}_1$ be the bounded linear operators on $L^p(\mathbb{R}; X)$ given by m_1 and m_2, respectively. Let $C := \|\mathcal{B}\|$ and $C_1 := \|\mathcal{B}_1\|$. For $f \in \mathcal{S}(\mathbb{R}; X)$, we have $\mathcal{F}(\mathcal{B}f)(x) = m_1(x)\mathcal{F}f(x)$ and

$$\mathcal{F}(\mathcal{B}_1 f)(x) = m_2(x)\mathcal{F}f(x) = ixm_1(x)\mathcal{F}f(x) = ix\mathcal{F}(\mathcal{B}f)(x).$$

It follows from the assumption that m_1, m_2 define L^p-Fourier multipliers that $\mathcal{B}f \in W^{1,p}(\mathbb{R}; X)$ and $[\mathcal{B}f]' = \mathcal{B}_1 f$. Furthermore we have $\|\mathcal{B}f\|_{W^{1,p}} \le (C + C_1)\|f\|_{L^p}$. This implies that the image of $L^p(\mathbb{R}; X)$ by $\mathcal{B}$ is contained in $W^{1,p}(\mathbb{R}; X)$ by Lemma 2.3.

Let $f \in \mathcal{S}(\mathbb{R}; D(A))$. Then $f,\ Af \in \mathcal{S}(\mathbb{R}; X)$, $\mathcal{F}(\mathcal{B}f)(x) = m_1(x)\mathcal{F}f(x)$ and $\mathcal{F}(A\mathcal{B}f)(x) = m_1(x)(Af)(x)$. It follows that $\mathcal{B}(Af) = A\mathcal{B}f$ and $\|\mathcal{B}f\|_{L^p(\mathbb{R};D(A))} \le C\|f\|_{L^p(\mathbb{R};D(A))}$. On the other hand, we have $[\mathcal{B}f]' = \mathcal{B}f'$, thus $\mathcal{F}([\mathcal{B}f]')(x) = ix\mathcal{F}(\mathcal{B}f)(x) = m_2(x)\mathcal{F}f(x)$ and $\mathcal{F}(A[\mathcal{B}f]')(x) = \mathcal{F}(A\mathcal{B}f')(x) = m_2(x)(Af)(x)$. We deduce that $\|[\mathcal{B}f]'\|_{L^p(\mathbb{R};D(A))} \le C_1\|f\|_{L^p(\mathbb{R};D(A))}$. It follows that

$$\|\mathcal{B}f\|_{W^{1,p}(\mathbb{R};D(A))} \le (C + C_1)\|f\|_{L^p(\mathbb{R};D(A))}.$$

Thus $\mathcal{B}$ maps boundedly $L^p(\mathbb{R}; D(A))$ into $W^{1,p}(\mathbb{R}; D(A))$ by Lemma 2.3. A similar argument shows that $\mathcal{B}$ also maps boundedly $W^{1,p}(\mathbb{R}; D(A))$ into $W^{2,p}(\mathbb{R}; D(A))$. This implies that $\mathcal{B}$ acts boundedly on $W^{2,p}(\mathbb{R}; D(A))$ by the Closed Graph Theorem.

Let $f \in \mathcal{S}(\mathbb{R}; D(A))$. Then

$$\mathcal{F}(A^i(\mathcal{B}f)^{(j)}) = m_1\mathcal{F}(A^i f^{(j)})$$

when $0 \le i,\ j \le 2$ as A is clearly commute with m_1. It follows that $\|\mathcal{B}f\|_{W^{2,p}(\mathbb{R};D(A))} \le C\|f\|_{W^{2,p}(\mathbb{R};D(A))}$ by the assumption that m_1 defines an L^p-Fourier multiplier. This shows that $\mathcal{B}$ maps boundedly from $W^{2,p}(\mathbb{R}; D(A))$ into itself by Lemma 2.3.

It remains to show that $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$ when $u \in W^{2,p}(\mathbb{R}; D(A))$. Let $f \in \mathcal{S}(\mathbb{R}; D(A))$. Then it is clear that we have

$$\mathcal{F}(\mathcal{B}\mathcal{A}f)(x) = m_1(x)\mathcal{F}(\mathcal{A}f)(x) = m_1(x)(-x^2 - A)\mathcal{F}f(x) = \mathcal{F}f(x)$$

$$\mathcal{F}(\mathcal{A}\mathcal{B}f)(x) = -(x^2 + A)\mathcal{F}(\mathcal{B}f)(x) = (-x^2 - A)m_1(x)\mathcal{F}f(x) = \mathcal{F}f(x).$$

Thus

$$\mathcal{B}\mathcal{A}f = \mathcal{A}\mathcal{B}f = f.$$

This equality remains true when $f \in W^{2,p}(\mathbb{R}; D(A))$ by the boundedness of $\mathcal{A}$ from $W^{2,p}(\mathbb{R}; D(A))$ into $L^p(\mathbb{R}; X)$, the boundedness of $\mathcal{B}$ on $L^p(\mathbb{R}; X)$ and $W^{2,p}(\mathbb{R}; D(A))$ and Lemma 2.3. This shows that the implication (ii) ⇒ (i) is true. The proof is complete. ■

Next we show that when X is a UMD Banach space and $1 < p < \infty$, one can give a simpler characterization of the $(W^{2,p}, W^{1,p})$-mild well-posedness for (P_2). For this we need to use the operator-valued Fourier multiplier theorem on $L^p(\mathbb{R}, X)$ obtained by Weis [12]. Weis' result involves the Rademacher boundedness for sets of bounded linear operators on Banach spaces. Let γ_j be the j-th Rademacher function on $[0, 1]$ given by $\gamma_j(t) = sgn(\sin(2^j t))$ when $j \geq 1$. For $x \in X$, we denote by $\gamma_j \otimes x$ the X-valued function $t \to r_j(t)x$ on $[0, 1]$.

Definition 2.3. Let X be a Banach space. A set $\mathbf{T} \subset \mathcal{L}(X)$ is said to be Rademacher bounded, if there exists $C > 0$ such that

$$\Big\| \sum_{j=1}^{n} \gamma_j \otimes T_j x_j \Big\|_{L^1} \leq C \Big\| \sum_{j=1}^{n} \gamma_j \otimes x_j \Big\|_{L^1}$$

for all $T_1, \ldots, T_n \in \mathbf{T}, x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$.

Let $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ be Rademacher bounded sets. Then it can be seen easily from the definition that the product set $\mathbf{ST} := \{ST : S \in \mathbf{S}, T \in \mathbf{T}\}$, the union set $\mathbf{S} \cup \mathbf{T}$ and the sum set $\mathbf{S} + \mathbf{T} := \{S + T : S \in \mathbf{S}, T \in \mathbf{T}\}$ are still Rademacher bounded. It was shown by Weis that when X is a UMD Banach space and $1 < p < \infty$, if $m : \mathbb{R} \to \mathcal{L}(X)$ is a C^1-function such that both sets $\{m(x) : x \in \mathbb{R}\}$ and $\{xm'(x) : x \in \mathbb{R}\}$ are Rademacher bounded, then m is an L^p-Fourier multiplier [12, Theorem 3.4]. This result together with Theorem 2.1 gives the following characterization of the $(W^{2,p}, W^{1,p})$-mild well-posedness (P_2) when X is a UMD Banach space and $1 < p < \infty$.

Corollary 2.2. *Let X be a* UMD *Banach space, $1 < p < \infty$ and let $A : D(A) \to X$ be a densely defined closed operator on X. Then the following assertions are equivalent.*

(i) (P_2) *is $(W^{2,p}, W^{1,p})$-mildly well-posed;*

(ii) $(-\infty, 0] \subset \rho(A)$ *and the function m given by $m(x) = -ix(x^2 + A)^{-1}$ is an L^p-Fourier multiplier.*

Proof. The implication (i) ⇒ (ii) is clearly true by Theorem 2.1, we only need to show that the implication (ii) ⇒ (i) is true. We assume that $(-\infty, 0] \subset \rho(A)$ and m

given by $m(x) = ix\eta(x)$ defines an L^p-Fourier multiplier, where $\eta(x) = -(x^2+A)^{-1}$ when $x \in \mathbb{R}$. By Theorem 2.1, it will suffice to show that the function η defines an L^p-Fourier multiplier. By [12, Theorem 3.4], we only need to show that both sets $\{\eta(x) : x \in \mathbb{R}\}$ and $\{x\eta'(x) : x \in \mathbb{R}\}$ are Rademacher bounded as X is a UMD Banach space and $1 < p < \infty$. Since η is analytic, we deduce that the set $\{\eta(x) : |x| \leq 1\}$ is Rademacher bounded [12, Proposition 2.6]. The assumption that m defines an L^p-Fourier multiplier implies that the set $\{ix\eta(x) :\in \mathbb{R}\}$ is Rademacher bounded [7], we deduce that the set $\{\eta(x) : |x| \geq 1\}$ is Rademacher bounded. Here we have used the fact that the set $\{\frac{I_X}{ix} : |x| \geq 1\}$ is Rademacher bounded and the easy fact that the product set of two Rademacher bounded sets is still Rademacher bounded [12], where I_X denotes the identity operator on X. We have shown that the set $\{\eta(x) : x \in \mathbb{R}\}$ is Rademacher bounded as the union of two Rademacher bounded sets is still Rademacher bounded [3, 7, 12].

On the other hand $\eta'(x) = 2x\eta(x)^2$, thus $x\eta'(x) = 2x^2\eta(x)^2 = -2m(x)^2$. The function $2m(x)^2$ is analytic, therefore the set $\{x\eta'(x) : |x| \leq 1\}$ is Rademacher bounded [12, Proposition 2.6]. We deduce from the assumption that m defines an L^p-Fourier multiplier that the set $\{x\eta'(x) : |x| \geq 1\}$ is also Rademacher bounded [7]. It follows that the set $\{x\eta'(x) : x \in \mathbb{R}\}$ is Rademacher bounded. The proof is complete. ■

The next result gives a sufficient condition involved Rademacher boundedness of the resolvent of A for the problem (P_2) to be $(W^{2,p}, W^{1,p})$-mildly well-posed when X is a UMD Banach space and $1 < p < \infty$.

Corollary 2.3. *Let X be a* UMD *Banach space, $1 < p < \infty$ and let $A : D(A) \to X$ be a densely defined closed operator on X. We assume that $(-\infty, 0] \subset \rho(A)$ and the set $\{x^{3/4}(x+A)^{-1} : x \geq 0\}$ is Rademacher bounded. Then (P_2) is $(W^{2,p}, W^{1,p})$-mildly well-posed.*

Proof. Let $m(x) = -ix(x^2 + A)^{-1}$ when $x \in \mathbb{R}$. It will suffice to show that both sets $\{m(x) : x \in \mathbb{R}\}$ and $\{xm'(x) : x \in \mathbb{R}\}$ are Rademacher bounded by Corollary 2.2 and [12, Theorem 3.4]. The set $\{m(x) : |x| \leq 1\}$ is Rademacher bounded as m is analytic [12, Proposition 2.6]. The set $\{m(x) : |x| > 1\}$ is also Rademacher bounded as $\{|x|^{3/2}(x^2 + A)^{-1} : |x| > 1\}$ is Rademacher bounded by assumption. Here we have used the fact that the set $\{\frac{I_X}{\sqrt{|x|}} : |x| > 1\}$ is Rademacher bounded and the easy fact that the product set of two Rademacher bounded sets is still Rademacher bounded [12]. Thus $\{m(x) : x \in \mathbb{R}\}$ is Rademacher bounded as the union of two Rademacher bounded sets is still Rademacher bounded [3, 7, 12]. We have $xm'(x) = m(x) + 2sgn(x)i[|x|^{3/2}(x^2 + A)^{-1}]^2$. Therefore $\{xm'(x) : x \in \mathbb{R}\}$ is Rademacher bounded by assumption as the product set of two Rademacher bounded sets is still Rademacher bounded [12]. The proof is complete. ■

Let $0 \leq \theta \leq 1$ be fixed, we define the fractional Sobolev space $W^{1+\theta,p}(\mathbb{R};X)$ of order $1+\theta$ as the completion of $\mathcal{S}(\mathbb{R};X)$ under the norm

$$\|f\|_{W^{1+\theta,p}} := \|f\|_{L^p} + \|f'\|_{L^p} + \|\mathcal{F}^{-1}\xi\mathcal{F}f\|_{L^p},$$

where

$$(9) \qquad \xi(x) := (ix)^{1+\theta} = \begin{cases} |x|^{1+\theta}e^{\frac{(1+\theta)i\pi}{2}}, & x \geq 0, \\ |x|^{1+\theta}e^{\frac{-(1+\theta)i\pi}{2}}, & x < 0. \end{cases}$$

Here f' is understood in the sense of distributions. It is clear that when $\theta = 1$, $\xi(x) = -x^2$, this implies that when $\theta = 1$, the above definition coincides with the definition (2) of $W^{2,p}(\mathbb{R};X)$. It is also clear that when $\theta = 0$, the above definition coincides with the definition (1) of $W^{1,p}(\mathbb{R};X)$. It is also clear from the definition that $W^{1+\theta,p}(\mathbb{R};X) \subset W^{1,p}(\mathbb{R};X)$ and the embedding is continuous. Now we are ready to introduce a mild well-posedness for (P_2) which will generalize the $(W^{2,p}, W^{1,p})$-mild well-posedness for (P_2).

Definition 2.4. Let $1 \leq p < \infty,\ 0 \leq \theta \leq 1$ and let A be a densely defined closed operator on a Banach space X with domain $D(A)$. We say that (P_2) is $(W^{2,p}, W^{1+\theta,p})$-mildly well-posed, if there exists a bounded linear operator $\mathcal{B}$ that maps $L^p(\mathbb{R};X)$ continuously into itself with range contained in $W^{1+\theta,p}(\mathbb{R};X)$, $\mathcal{B}(W^{1,p}(\mathbb{R};D(A))) \subset W^{2,p}(\mathbb{R};D(A)))$ and $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$ when $u \in W^{2,p}(\mathbb{R};D(A))$, where $\mathcal{A}u = u'' - Au$ when $u \in W^{2,p}(\mathbb{R};D(A))$. We call $\mathcal{B}$ the solution operator of the problem (P_2).

It is clear from the definition that when (P_2) is $(W^{2,p}, W^{1+\theta,p})$-mildly well-posed, then it is $(W^{2,p}, W^{1,p})$-mildly well-posed. It is also clear that the $(W^{2,p}, W^{1+\theta,p})$-mild well-posedness of (P_2) coincides with the $(W^{2,p}, W^{1,p})$-mild well-posednees of (P_2) when $\theta = 0$. We have actually the following characterization of the $(W^{2,p}, W^{1+\theta,p})$-mild well-posedness of (P_2) which may be considered as a generalization of Theorem 2.1.

Theorem 2.4. *Let X be a Banach space, $1 \leq p < \infty$, $0 \leq \theta \leq 1$ and let $A : D(A) \to X$ be a densely defined closed operator on X. Then the following assertions are equivalent.*

(i) (P_2) *is* $(W^{2,p}, W^{1+\theta,p})$*-mildly well-posed;*

(ii) $(-\infty, 0] \subset \rho(A)$ *and the functions* m_1, m_2 *and* m_3 *defined on* $\mathbb{R}$ *by* $m_1(x) = -(x^2+A)^{-1}$, $m_2(x) = -ix(x^2+A)^{-1}$ *and* $m_3(x) = -(ix)^{1+\theta}(x^2+A)^{-1}$ *define* L^p*-Fourier multipliers.*

Proof. (i) ⇒ (ii): Assume that (P_2) is $(W^{2,p}, W^{1+\theta,p})$-mildly well-posed and let $\mathcal{B}$ be the solution operator. Then it is $(W^{2,p}, W^{1,p})$-mildly well-posed. Thus $(-\infty, 0] \subset$

$\rho(A)$ and the functions m_1 and m_2 defined on $\mathbb{R}$ given by $m_1(x) = -(x^2 + A)^{-1}$, $m_2(x) = -ix(x^2 + A)^{-1}$ define L^p-Fourier multipliers by Theorem 2.1, moreover the bounded linear operator defined by the L^p-Fourier multiplier m_1 is $\mathcal{B}$ by the proof of Theorem 2.1. Since $\mathcal{B}$ is bounded and linear from $L^p(\mathbb{R}; X)$ into itself with range contained in $W^{1+\theta,p}(\mathbb{R}; X)$ by assumption, it follows easily from the Closed Graph Theorem that $\mathcal{B}$ is a bounded linear operator from $L^p(\mathbb{R}; X)$ into $W^{1+\theta,p}(\mathbb{R}; X)$. Here we have used the fact that the embedding $W^{1+\theta,p}(\mathbb{R}; X) \subset W^{1,p}(\mathbb{R}; X)$ is continuous. This implies clearly that m_3 defined by $m_3(x) = -(ix)^{1+\theta}(x^2 + A)^{-1}$ defines an L^p-Fourier multiplier.

(ii) $\Rightarrow$ (i): Assume that $(-\infty, 0] \subset \rho(A)$ and the functions m_1, m_2 and m_3 defined on $\mathbb{R}$ given by $m_1(x) = -(x^2 + A)^{-1}$, $m_2(x) = -ix(x^2 + A)^{-1}$ and $m_3(x) = -(ix)^{1+\theta}(x^2+A)^{-1}$ define L^p-Fourier multipliers. Then (P_2) is $(W^{2,p}, W^{1,p})$-mildly well-posed by Theorem 2.1. This means that there exists a bounded linear operator $\mathcal{B}$ that maps $L^p(\mathbb{R}; X)$ continuously into itself with range contained in $W^{1,p}(\mathbb{R}; X)$, $\mathcal{B}(W^{1,p}(\mathbb{R}; D(A))) \subset W^{2,p}(\mathbb{R}; D(A))$ and $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$ when $u \in W^{2,p}(\mathbb{R}; D(A))$. The bounded linear operator defined by the L^p-Fourier multiplier m_1 is $\mathcal{B}$ by the proof of Theorem 2.1. Since m_3 defines an L^p-Fourier multiplier, we deduce that the image of $L^p(\mathbb{R}; X)$ by $\mathcal{B}$ is contained in $W^{1+\theta,p}(\mathbb{R}; X)$. The proof is complete. ∎

Proposition 2.1. *Let X be a Banach space, $1 \le p < \infty$ and let $A : D(A) \to X$ be a densely defined closed operator on X. If (P_2) is $(W^{2,p}, W^{2,p})$-mildly well-posed, then it is L^p-well-posed.*

Proof. We assume that (P_2) is $(W^{2,p}, W^{2,p})$-mildly well-posed and $\mathcal{B}$ is the solution operator. Then $\mathcal{B}$ maps $L^p(\mathbb{R}; X)$ continuously into itself with range contained in $W^{2,p}(\mathbb{R}; X)$, $\mathcal{B}(W^{1,p}(\mathbb{R}; D(A))) \subset W^{2,p}(\mathbb{R}; D(A)))$ and $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$ when $u \in W^{2,p}(\mathbb{R}; D(A))$. It follows from the boundedness of $\mathcal{B}$ on $L^p(\mathbb{R}; X)$ and the Closed Graph Theorem that $\mathcal{B}$ is a bounded linear operator from $L^p(\mathbb{R}; X)$ into $W^{2,p}(\mathbb{R}; X)$.

Let $f \in L^p(\mathbb{R}; X)$, then there exists $f_n \in W^{2,p}(\mathbb{R}; D(A))$ such that $f_n \to f$ in $L^p(\mathbb{R}; X)$ by Lemma 2.3. We deduce that $\mathcal{B}f_n \to \mathcal{B}f$ in $W^{2,p}(\mathbb{R}; X)$. Since $(\mathcal{B}f_n)'' \to (\mathcal{B}f)''$ and $\mathcal{B}f_n \to \mathcal{B}f$ in $L^p(\mathbb{R}; X)$, there exists a subsequence f_{n_k} of f_n such that $(\mathcal{B}f_{n_k})'' \to (\mathcal{B}f)''$ and $\mathcal{B}f_{n_k} \to \mathcal{B}f$ a.e. on $\mathbb{R}$. Using the equality $(\mathcal{B}f_{n_k})'' = A\mathcal{B}f_{n_k} + \mathcal{B}f_{n_k}$ and the closedness of A, we deduce that $\mathcal{B}f(t) \in D(A)$ and $(\mathcal{B}f)''(t) = A\mathcal{B}f(t) + \mathcal{B}f(t)$ for almost all $t \in \mathbb{R}$. This implies that $\mathcal{B}f \in L^p(\mathbb{R}; D(A))$ and $(\mathcal{B}f)'' = A\mathcal{B}f + \mathcal{B}f$. Thus $\mathcal{B}f \in W^{2,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A))$ is a strong L^p-solution of (P_2).

To show the uniqueness of the strong L^p-solution of (P_2), we let $u \in W^{2,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A))$ be such that $u'' = Au$. Then there exist $u_n \in W^{2,p}(\mathbb{R}; D(A))$ such that $u_n \to u$ in $W^{2,p}(\mathbb{R}; X)$ as well as in $L^p(\mathbb{R}; D(A))$ by the density of $D(A)$ in X. We have $\mathcal{B}\mathcal{A}u_n = u_n$ by assumption. Letting $n \to \infty$, we obtain that $\mathcal{B}(u'' - Au) = u$, here we have used the boundedness of $\mathcal{B}$ on $L^p(\mathbb{R}; X)$. It follows that $u = 0$ as $u'' - Au = 0$. We have shown that (P_2) is L^p-well-posed. The proof is complete. ∎

Remark 2.2. We do not know whether the inverse implication of Proposition 2.1 remains true: when (P_2) is L^p-well-posed, if $\mathcal{B}$ is the solution operator, then $\mathcal{B}$ maps $L^p(\mathbb{R};X)$ continuously into itself with range contained in $W^{2,p}(\mathbb{R};X)$, and $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$ when $u \in W^{2,p}(\mathbb{R};D(A))$, but we do not know whether the inclusion $\mathcal{B}(W^{1,p}(\mathbb{R};D(A))) \subset W^{2,p}(\mathbb{R};D(A)))$ is true. Meanwhile, Theorem 2.4 gives a sufficient condition for the L^p-well-posedness of (P_2): if $(-\infty,0] \subset \rho(A)$ and the functions m_1, m_2 and m_3 defined on $\mathbb{R}$ given by $m_1(x) = -(x^2+A)^{-1}$, $m_2(x) = -ix(x^2+A)^{-1}$ and $m_3(x) = x^2(x^2+A)^{-1}$ define L^p-Fourier multipliers, then (P_2) is L^p-well-posed.

When X is a UMD Banach space, we have the following characterization of the $(W^{2,p}, W^{1+\theta,p})$-mild well-posedness when $1 < p < \infty$. The proof is similar to the proof of Corollary 2.2, we omit it.

Corollary 2.5. *Let X be a* UMD *Banach space,* $1 < p < \infty$, $\frac{1}{2} \le \theta \le 1$ *and let $A : D(A) \to X$ be a densely defined closed operator on X. Then the following assertions are equivalent.*

(i) (P_2) *is $(W^{2,p}, W^{1+\theta,p})$-mildly well-posed;*

(ii) $(-\infty,0] \subset \rho(A)$ *and the function m given by* $m(x) = -(ix)^{1+\theta}(x^2+A)^{-1}$ *is an L^p-Fourier multiplier, where $(ix)^{1+\theta}$ is defined by* (9).

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