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ON STABILITY OF A CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. We give some simple criteria for uniform asymptotic stability and exponential asymptotic stability of linear Volterra-Stieltjes differential equations. These criteria are given in terms of the matrix measure or the spectral abscissa of certain matrices derived from the coefficient matrices. An application of obtained results to linear integro-differential equations with delay is presented.

1. INTRODUCTION

In this paper, we are concerned with stability of linear Volterra-Stieltjes differential equations of the form

$$\dot{x}(t) = Ax(t) + \int_0^t d[B(s)]x(t-s), \quad a.e. \quad t \in \mathbb{R}_+ := [0, \infty), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a given matrix and $B(\cdot) \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, is a given matrix-valued function of locally bounded variation on $\mathbb{R}_+$.

Note that (1) encompasses linear Volterra integro-differential equations

$$\dot{x}(t) = Ax(t) + \int_0^t B(t-s)x(s)ds, \quad t \in \mathbb{R}_+, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ and $B(\cdot) \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and there is a considerable literature devoted to the study of asymptotic behavior of solutions of linear Volterra integro-differential equations (2), see [1, 2, 5, 7, 13, 15, 18, 20, 22-25] and references therein. A comprehensive theory of stability of linear Volterra integro-differential equations (2) can be found in [5]. In particular, the uniform asymptotic stability of (2) has been studied in [13, 15] and the exponential asymptotic stability of (2) has been considered in [17, 18].

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Recently, problems of stability and robust stability of linear Volterra integro-differential equations have attracted much attention from researchers, see e.g. [1, 2, 4, 16, 19, 20, 22-25].

In general, stability problems of the linear Volterra-Stieltjes differential equation (1) are more difficult than those of the linear Volterra integro-differential equation (2), see [11]. This is due to the Volterra-Stieltjes integral term $\int_0^t d[B(s)]x(t-s)$ which leads one to work with Riemann-Stieltjes integrals and Laplace-Stieltjes transforms instead of Riemann integrals and Laplace transforms.

Stability analysis of the linear Volterra-Stieltjes differential equation (1) has been done by ourselves in very recent papers [24, 25]. The present paper is a direct continuation of the preceding studies on this issue. We will present below some simple criteria for the uniform asymptotic stability and the exponential asymptotic stability of the linear Volterra-Stieltjes differential equation (1). These criteria may be used easily and quickly to show that the zero solution of (1) is uniformly asymptotically stable or/and exponentially asymptotically stable.

Furthermore, it is worth noticing that a linear Volterra integro-differential equation with finite delay of the form

$$(3) \qquad \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t-h_i) + \int_0^t B(t-s)x(s)ds, \quad t \in \mathbb{R}_+,$$

($A, A_i \in \mathbb{R}^{n\times n}$ ($i \in \underline{m}$) and $B(\cdot) \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$) or a linear Volterra integro-differential equation with infinite delay of the form

$$(4) \qquad \dot{x}(t) = Ax(t) + \sum_{i=1}^{\infty} A_i x(t-h_i) + \int_{-\infty}^t B(t-s)x(s)ds, \quad t \in \mathbb{R}_+,$$

($A, A_i \in \mathbb{R}^{n\times n}$, $i \in \mathbb{N}$, $\sum_{i=1}^{\infty} \|A_i\| < \infty$ and $B(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$), can be converted into a nonhomogeneous linear Volterra-Stieltjes differential equation of the form

$$(5) \qquad \dot{x}(t) = Ax(t) + \int_0^t d[C(s)]x(t-s) + f(t), \quad a.e.\ t \in \mathbb{R}_+,$$

where $C(\cdot) \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and $f \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$. Thus, stability criteria of (1) can be applied to (3) and (4), see for example, Example 3.14 and Section 4 of the present paper, [25, Section 4]. Equation (4) has appeared in optimal control problems with quadratic cost [9] and in models of nuclear reaction dynamics [8, page 307]. For further information on applications of Volterra-Stieltjes equations, we refer to [7].

The organization of the paper is as follows. In the next section, we give some notations and preliminary results which will be used in what follows. The main results of the paper are presented in Section 3. We first give some simple criteria for the uniform asymptotic stability of the linear Volterra-Stieltjes differential equation (1).

Then combining the obtained results with a recent result in [25, Theorem 3.4], we get explicit criteria for the exponential asymptotic stability of (1). An example is given to illustrate the obtained results. Finally, we apply results given in Section 3 to study asymptotic behavior of solutions of Volterra integro-differential equations with infinite delay.

2. Preliminaries

Let $\mathbb{N}$ be the set of all natural numbers. For a given $n \in \mathbb{N}$, we denote $\underline{n} := \{1, 2, ..., n\}$. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ where $\mathbb{C}$ and $\mathbb{R}$ denote the sets of all complex and all real numbers, respectively. For given $\gamma \in \mathbb{R}$, let us denote $\mathbb{C}_\gamma := \{z \in \mathbb{C} : \Re z \geq \gamma\}$ and let $\mathring{\mathbb{C}}_\gamma$ be the interior of $\mathbb{C}_\gamma$. For an integer $l, q \geq 1, \mathbb{K}^l$ denotes the $l-$dimensional vector space over $\mathbb{K}$ and $\mathbb{K}^{l\times q}$ stands for the set of all $l \times q$-matrices with entries in $\mathbb{K}$. Inequalities between real matrices or vectors will be understood componentwise, i.e. for two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{l\times q}$, we write $A \geq B$ if $a_{ij} \geq b_{ij}$ for $i = 1, \cdots, l,\ j = 1, \cdots, q$. In particular, if $a_{ij} > b_{ij}$ for $i = 1, \cdots, l,\ j = 1, \cdots, q$, then we write $A \gg B$ instead of $A \geq B$. We denote by $\mathbb{R}^{l\times q}_+$ the set of all nonnegative matrices $A \geq 0$. Similar notations are adopted for vectors.

For $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l\times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be *monotonic* if $\|x\| \leq \|y\|$ whenever $x, y \in \mathbb{K}^n, |x| \leq |y|$. Every p-norm on $\mathbb{K}^n, 1 \leq p \leq \infty$, is monotonic. Throughout the paper, if otherwise not stated, the norm of a matrix $P \in \mathbb{K}^{l\times q}$ is understood as its operator norm associated with a given pair of monotonic vector norms on $\mathbb{K}^l$ and $\mathbb{K}^q$, that is $\|P\| = \max\{\|Py\| : \ \|y\| = 1\}$. Note that, one has

$$P \in \mathbb{K}^{l\times q}, Q \in \mathbb{R}^{l\times q}_+, |P| \leq Q \quad \Rightarrow \quad \|P\| \leq \|\,|P|\,\| \leq \|Q\|, \tag{6}$$

see, e.g. [26].

For any matrix $A \in \mathbb{K}^{n\times n}$ the *spectral abscissa* of A is denoted by $s(A) = \max\{\Re\lambda : \lambda \in \sigma(A)\}$, where $\sigma(A) := \{s \in \mathbb{C} : \det(sI_n - A) = 0\}$ is the spectrum of A. For an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$, the matrix measure of $A \in \mathbb{R}^{n\times n}$ is defined by

$$\mu(A) := \lim_{t\to 0^+} \frac{\|I_n + tA\| - 1}{t}, \tag{7}$$

where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. Then the following holds

$$s(A) \leq \mu(A) \leq \|A\|, \tag{8}$$

$$\mu(A + B) \leq \mu(A) + \mu(B) \qquad A, B \in \mathbb{R}^{n\times n}, \tag{9}$$

see e.g. [27]. Especially, for an operator norm on $\mathbb{R}^{n\times n}$ associated with a given monotonic vector norm on $\mathbb{R}^n$, it follows from (6)-(7) that

(10) $$A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}_+^{n\times n} \quad |A| \le B \quad \Rightarrow \quad \mu(A) \le \mu(|A|) \le \mu(B).$$

A matrix $A \in \mathbb{R}^{n\times n}$ is called a *Metzler matrix* if all off-diagonal elements of A are nonnegative. We now summarize some properties of Metzler matrices which will be used in what follows.

Theorem 2.1. [26]. *Suppose that $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix. Then*

(i) *$($Perron-Frobenius$)$ $s(A)$ is an eigenvalue of A and there exists a nonnegative eigenvector $x \neq 0$ such that $Ax = s(A)x$.*

(ii) *Given $\alpha \in \mathbb{R}$, there exists a nonzero vector $x \ge 0$ such that $Ax \ge \alpha x$ if and only if $s(A) \ge \alpha$.*

(iii) *$(tI_n - A)^{-1}$ exists and is nonnegative if and only if $t > s(A)$.*

(iv) *Given $B \in \mathbb{R}_+^{n\times n}$, $C \in \mathbb{C}^{n\times n}$. Then*

(11) $$|C| \le B \quad \Longrightarrow \quad s(A+C) \le s(A+B).$$

The following is immediate from Theorem 2.1.

Theorem 2.2. *Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Then the following statements are equivalent*

(i) $s(A) < 0$;

(ii) $-A^{-1}$ *exists and is nonnegative;*

(iii) $Ap \ll 0$ *for some* $p \ge 0$;

(iv) For given $b \in \mathbb{R}^n, b \gg 0$, there exists $x \in \mathbb{R}_+^n$ such that $Ax + b = 0$.

(v) For any $x \in \mathbb{R}_+^n \setminus \{0\}$, the row vector $x^T A$ has at least one negative entry.

To make the presentation self-contained we present here some basic facts on matrix-valued functions of bounded variation.
Let $\mathbb{K}^{m\times n}$ be endowed with the norm $\|\cdot\|$ and $C([\alpha,\beta],\mathbb{K}^{m\times n})$ be the Banach space of all continuous functions on $[\alpha,\beta]$ with values in $\mathbb{K}^{m\times n}$ normed by the maximum norm $\|\varphi\| = \max_{\theta\in[\alpha,\beta]} \|\varphi(\theta)\|$.

A matrix function $\eta(\cdot) : [\alpha,\beta] \to \mathbb{K}^{m\times n}$ is said to be of bounded variation if

$$\mathrm{Var}(\eta;\alpha,\beta) := \sup_{P\in\mathcal{P}} \sum_{k=0}^{n_P-1} \|\eta(\theta_{k+1}) - \eta(\theta_k)\| < +\infty,$$

where the supremum is taken over the set

$$\mathcal{P} = \{P = \{\theta_0,\ldots,\theta_{n_P}\} | P \text{ is a partition of } [\alpha,\beta]\}$$

of all partitions of the interval $[\alpha, \beta]$. The set $BV([\alpha,\beta],\mathbb{K}^{m\times n})$ of all matrix functions $\eta(\cdot)$ of bounded variation on $[\alpha,\beta]$ satisfying $\eta(\alpha)=0$ is a Banach space endowed with the norm $\|\eta\| = \mathrm{Var}(\eta;\alpha,\beta)$.

One says that $\eta(\cdot)\in BV_{loc}(\mathbb{R}_+,\mathbb{R}^{l\times q})$ if it is of bounded variation on any compact interval of $\mathbb{R}_+$ and $\eta(0)=0$. Then $NBV_{loc}(\mathbb{R}_+,\mathbb{R}^{l\times q})$ is the set of functions in $BV_{loc}(\mathbb{R}_+,\mathbb{R}^{l\times q})$ that are continuous from the right on $\mathbb{R}_+$.

Let $\eta(\cdot)\in BV_{loc}(\mathbb{R}_+,\mathbb{R}^{m\times n})$ be given. Set $V_\eta(T) := \mathrm{Var}(\eta;0,T)$, for $T>0$. It is clear that $0\le V_\eta(T_1)\le V_\eta(T_2) < +\infty,\ 0<T_1<T_2$. If the limit $\lim_{T\to+\infty} V_\eta(T)$ exists and is finite then $\eta(\cdot)$ is said to be of bounded variation on $\mathbb{R}_+$ and $V_\eta := \lim_{T\to+\infty} V_\eta(T)$ is called the total variation of $\eta(\cdot)$. One often writes $\int_0^{+\infty} |d[\eta(s)]|$ instead of V_η.

Given $\eta(\cdot)\in BV([\alpha,\beta],\mathbb{K}^{m\times n})$ then for any continuous functions $\gamma\in C([\alpha,\beta],\mathbb{K})$ and $\varphi\in C([\alpha,\beta],\mathbb{K}^n)$, the integrals $\int_\alpha^\beta \gamma(\theta)d[\eta(\theta)]$ and $\int_\alpha^\beta d[\eta(\theta)]\varphi(\theta)$ exist and one has

$$\left\|\int_\alpha^\beta \gamma(\theta)d[\eta(\theta)]\right\| \le \max_{\theta\in[\alpha,\beta]} |\gamma(\theta)|\,\|\eta\|, \tag{12}$$
$$\left\|\int_\alpha^\beta d[\eta(\theta)]\varphi(\theta)\right\| \le \max_{\theta\in[\alpha,\beta]} \|\varphi(\theta)\|\,\|\eta\|,$$

see e.g. [3, page 49].

3. Stability of Linear Volterra-Stieltjes Differential Equations

Consider a linear Volterra-Stieltjes differential equation of the form (1), where $A\in\mathbb{R}^{n\times n}$ and $B(\cdot)\in NBV_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times n})$ are given. From the theory of integro-differential equations (see e.g. [11, Ch. 3]), it is well-known that there exists a unique locally absolutely continuous matrix-valued function $R(\cdot):\mathbb{R}_+\to\mathbb{R}^{n\times n}$ such that

$$\dot R(t) = AR(t) + \int_0^t d[B(s)]R(t-s), \quad a.e.\ t\in\mathbb{R}_+, \quad R(0)=I_n. \tag{13}$$

Then $R(\cdot)$ is called the *resolvent* of (1). Moreover, for a given $f\in L^1_{loc}(\mathbb{R}_+,\mathbb{R}^n)$, the following nonhomogeneous equation

$$\dot x(t) = Ax(t) + \int_0^t d[B(s)]x(t-s) + f(t), \quad a.e.\ t\in\mathbb{R}_+, \tag{14}$$

has a unique solution $x(\cdot)$ satisfying the initial condition $x(0)=x_0\in\mathbb{R}^n$. This solution is locally absolutely continuous on $\mathbb{R}_+$ and is given by the *variation of constants formula*

$$x(t) = R(t)x_0 + \int_0^t R(t-s)f(s)ds, \quad t\in\mathbb{R}_+, \tag{15}$$

see e.g. [11, page 81].

Definition 3.1. Let $\sigma \in \mathbb{R}_+$ and $\varphi \in C([0,\sigma],\mathbb{R}^n)$ be given. A vector-valued function $x(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ is called a solution of (1) through (σ, φ) if $x(\cdot)$ is absolutely continuous on any compact subinterval of $[\sigma, +\infty)$ and satisfies (1) almost everywhere on $[\sigma, +\infty)$ and $x(t) = \varphi(t)\ \forall t \in [0, \sigma]$. We denote it by $x(\cdot; \sigma, \varphi)$.

3.1. Explicit criteria for uniform asymptotic stability

First we adapt here the standard notions of stability of linear integro-differential equations (see e.g. [5]) to (1).

Definition 3.2. The zero solution of (1) is said to be:

(i) uniformly stable (US) if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\varphi \in C([0,\sigma],\mathbb{R}^n), \|\varphi\| < \delta \Rightarrow \|x(t;\sigma,\varphi)\| < \epsilon, \forall t \geq \sigma.$$

(ii) uniformly asymptotically stable (UAS) if it is US and if there exists $\delta_0 > 0$ such that $\forall \epsilon > 0,\ \exists\, T(\epsilon) > 0$:

$$\varphi \in C([0,\sigma],\mathbb{R}^n), \|\varphi\| < \delta_0 \Rightarrow \|x(t;\sigma,\varphi)\| < \epsilon, \forall t \geq T(\epsilon) + \sigma.$$

If the zero solution of (1) is US (UAS) then we also say that (1) is US (UAS), respectively.

Recall that the *Laplace-Stieltjes transform* of $F(\cdot) \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, is defined by

$$\tilde{F}(z) := \int_0^{+\infty} e^{-zs} d[F(s)],$$

on a set $\mathcal{U} \subset \mathbb{C}$ where it exists, see [3, p. 56]. For example, if the total variation of $B(\cdot)$ on $\mathbb{R}_+$ is finite, that is

$$\int_0^{+\infty} |dB(s)| < +\infty, \tag{16}$$

then the Laplace-Stieltjes transform $\tilde{B}(z)$ of $B(\cdot)$ is well-defined on $\mathbb{C}_0$. Let us define

$$\mathcal{H}(z) := zI_n - A - \tilde{B}(z), \tag{17}$$

for appropriate $z \in \mathbb{C}$. Then, $\mathcal{H}(\cdot)$ is called the *characteristic matrix* of (1).

Proposition 3.3. [24]. *Let (16) hold. Then the following statements are equivalent*

(i) $\det \mathcal{H}(z) \neq 0, \forall z \in \mathbb{C}_0$;

(ii) *the resolvent* $R(\cdot)$ *of (1) belongs to* $L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$*;*

(iii) *(1) is UAS.*

We are now in the position to state the main results of this paper.

Theorem 3.4. *Let* $A \in \mathbb{R}^{n\times n}$ *and* $B(\cdot) = (b_{ij}(\cdot)) \in NBV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ *be given. Let* $\|\cdot\|$ *be an operator norm on* $\mathbb{R}^{n\times n}$ *associated with a monotone vector norm on* $\mathbb{R}^n$ *and let* $\mu(\cdot)$ *be the matrix measure induced by* $\|\cdot\|$. *Assume that* (16) *holds and* $\mathcal{B}$ *is defined by*

$$\mathcal{B} := \left(\int_0^{+\infty} \left|d[b_{ij}(s)]\right|\right) \in \mathbb{R}^{n\times n}. \tag{18}$$

If

$$\mu(A) + \mu(\mathcal{B}) < 0 \tag{19}$$

then (1) is UAS. In particular, (1) is UAS provided

$$\mu(A) + \|\mathcal{B}\| < 0. \tag{20}$$

Proof. Suppose (19) holds. We show that (1) is UAS. Since (16) holds, the Laplace-Stieltjes transform $\tilde{B}(z)$ of $B(\cdot)$ is well-defined on $\mathbb{C}_0$. By Proposition 3.3, it remains to show that $\det \mathcal{H}(z) \neq 0, \forall z \in \mathbb{C}_0$. Assume on the contrary that $\det \mathcal{H}(z) = 0$ for some $z_0 \in \mathbb{C}_0$. In particular, z_0 is an eigenvalue of $A + \int_0^{+\infty} e^{-z_0 s} d[B(s)]$. It follows that $0 \leq \Re z_0 \leq s(A + \int_0^{+\infty} e^{-z_0 s} d[B(s)])$. Taking into account the properties (8)-(9) of the matrix measure, we get

$$\begin{aligned} 0 &\leq s(A + \int_0^{+\infty} e^{-z_0 s} d[B(s)]) \leq \mu(A + \int_0^{+\infty} e^{-z_0 s} d[B(s)]) \\ &\leq \mu(A) + \mu\biggl(\int_0^{+\infty} e^{-z_0 s} d[B(s)]\biggr). \end{aligned} \tag{21}$$

On the other hand, since $\Re z_0 \geq 0$ and (16) holds, it follows that

$$\left|\int_0^{+\infty} e^{-z_0 s} d[b_{ij}(s)]\right| \leq \int_0^{+\infty} \left|d[b_{ij}(s)]\right|, \quad \forall i, j \in \underline{n}.$$

Thus

$$\left|\int_0^{+\infty} e^{-z_0 s} d[B(s)]\right| \leq \left(\int_0^{+\infty} \left|d[b_{ij}(s)]\right|\right) = \mathcal{B}. \tag{22}$$

By (10), we have

$$\mu\biggl(\int_0^{+\infty} e^{-z_0 s} d[B(s)]\biggr) \leq \mu(\mathcal{B}). \tag{23}$$

Then (21)-(23) imply that $\mu(A) + \mu(\mathcal{B}) \geq 0$. However, this conflicts with (19).

Finally, it follows from (8) that if (20) holds, so does (19). This completes the proof. ■

The following theorem gives a sharper condition for the uniform asymptotic stability of (1).

Theorem 3.5. *Let* $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ *and* $B(\cdot) = (b_{ij}(\cdot)) \in NBV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ *be given. Suppose that* (16) *holds and* $\mathcal{B}$ *is defined by* (18) *and*

$$A_d := diag(a_{11}, ..., a_{nn}) \in \mathbb{R}^{n\times n}. \tag{24}$$

If

$$s(A_d + |A - A_d| + \mathcal{B}) < 0, \tag{25}$$

then (1) *is UAS.*

Remark 3.6. It is worth noticing that (25) is, in general, less conservative than (19). For example, for the operator norm on $\mathbb{R}^{n\times n}$ associated with l_1-norm $\|x\|_1 := \sum_{i=1}^{n} |x_i|$ on $\mathbb{R}^n$, the induced matrix measure $\mu(\cdot)$ is given by

$$\mu(A) = max_{j\in \underline{n}}\{a_{jj} + \sum_{i=1, i\neq j}^{n} |a_{ij}|\},$$

see e.g. [27]. This yields $\mu(A) = \mu(A_d + |A - A_d|)$. Furthermore, it follows from (8)-(9) that

$$\mu(A)+\mu(\mathcal{B}) = \mu(A_d+|A-A_d|)+\mu(\mathcal{B}) \geq \mu(A_d+|A-A_d|+\mathcal{B}) \geq s(A_d+|A-A_d|+\mathcal{B}).$$

Proof of Theorem 3.5. Since (16) holds, the Laplace-Stieltjes transform $\tilde{B}(z)$ of $B(\cdot)$ is well-defined on $\mathbb{C}_0$. Suppose (25) holds. We show that

$$\det \mathcal{H}(z) \neq 0, \forall z \in \mathbb{C}_0,$$

and then (1) is UAS, by Proposition 3.3. Assume on the contrary that $\det \mathcal{H}(z) = 0$ for some $z_0 \in \mathbb{C}_0$. This gives

$$0 \leq \Re z_0 \leq s(A + \int_0^{+\infty} e^{-z_0 s} d[B(s)]). \tag{26}$$

We claim that

$$s(A + \int_0^{+\infty} e^{-z_0 s} d[B(s)]) \leq s(A_d + |A - A_d| + \mathcal{B}).$$

Let $\alpha := \max_{i\in\underline{n}} |a_{ii}|$. This yields $A_d + \alpha I_n \geq 0$. Let λ be an arbitrary eigenvalue of $A + \int_0^{+\infty} e^{-z_0 s} d[B(s)]$. Then there exists a nonzero vector $x \in \mathbb{C}^n$ such that

$$\left(A + \int_0^{+\infty} e^{-z_0 s} d[B(s)]\right) x = \lambda x.$$

Hence,

$$\begin{aligned}&\alpha x + \left(A + \int_0^{+\infty} e^{-z_0 s} d[B(s)]\right) x \\ &= (\alpha I_n + A_d)x + (A - A_d + \int_0^{+\infty} e^{-z_0 s} d[B(s)])x = (\lambda + \alpha)x.\end{aligned}$$

Taking (22) into account, we get

$$\begin{aligned}(\Re\lambda + \alpha)|x| \leq |(\lambda + \alpha)x| &\leq |\alpha I_n + A_d||x| + |A - A_d||x| + \left|\int_0^{+\infty} e^{-z_0 s} d[B(s)]\right||x| \\ &\leq (\alpha I_n + A_d)|x| + |A - A_d||x| + \mathcal{B}|x|.\end{aligned}$$

Therefore

$$(A_d + |A - A_d| + \mathcal{B})|x| \geq \Re\lambda\ |x|.$$

Since $A_d + |A - A_d| + \mathcal{B}$ is a Metzler matrix, $s(A_d + |A - A_d| + \mathcal{B}) \geq \Re\lambda$, by Theorem 2.1 (ii). As λ is an arbitrary eigenvalue of $A + \int_0^{+\infty} e^{-z_0 s} d[B(s)]$, we get $s(A_d + |A - A_d| + \mathcal{B}) \geq s(A + \int_0^{+\infty} e^{-z_0 s} d[B(s)])$. Hence, $s(A_d + |A - A_d| + \mathcal{B}) \geq 0$, by (26). However, this conflicts with (25). This completes the proof. ■

Corollary 3.7. *Assume that* (16) *holds. Then (1) is UAS provided one of the following conditions holds*

(i)

(27) $$s(A_d + |A - A_d| + \mathcal{B}) < 0;$$

(ii)

(28) $$(A_d + |A - A_d| + \mathcal{B})p \ll 0, \qquad \textit{for some} \quad p \in \mathbb{R}^n_+;$$

(iii) $A_d + |A - A_d| + \mathcal{B}$ *is invertible and*

(29) $$(A_d + |A - A_d| + \mathcal{B})^{-1} \leq 0;$$

(iv) *For given* $b \in \mathbb{R}^n, b \gg 0$, *there exists* $y \in \mathbb{R}^n_+$ *such that*

$$(A_d + |A - A_d| + \mathcal{B})y + b = 0. \tag{30}$$

(v) *For any* $x \in \mathbb{R}^n_+ \setminus \{0\}$, *the row vector* $x^T(A_d + |A - A_d| + \mathcal{B})$ *has at least one negative entry.*

Proof. Since $A_d + |A - A_d| + \mathcal{B}$ is a Metzler matrix, the assertions are immediate from Theorems 2.2 and 3.5. ■

Corollary 3.8. *Assume that* $A \in \mathbb{R}^{n\times n}$ *is a Metzler matrix and* $B(\cdot) = (b_{ij}(\cdot)) \in NBV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ *is an increasing matrix function on* $\mathbb{R}_+$ $(i.e.\ b_{ij}(\theta_2) \geq b_{ij}(\theta_1), 0 \leq \theta_1 < \theta_2$ *for* $i, j \in \underline{n})$ *and* (16) *holds. Then* (1) *is UAS if and only if*

$$s(A + B(\infty)) < 0, \tag{31}$$

where

$$B(\infty) := \lim_{\theta\to+\infty} B(\theta). \tag{32}$$

Proof. Since (16) holds, the Laplace-Stieltjes transform $\tilde{B}(z)$ of $B(\cdot)$ is well-defined on $\mathbb{C}_0$. Since $B(\cdot)$ is increasing on $\mathbb{R}_+$ and $B(0) = 0$, it follows that

$$\mathcal{B} := \left(\int_0^{+\infty} \left|d[b_{ij}(s)]\right|\right) = \left(\int_0^{+\infty} d[b_{ij}(s)]\right) = \left(\lim_{\theta\to+\infty} b_{ij}(\theta)\right) = B(\infty) \in \mathbb{R}^{n\times n}_+.$$

Moreover, we have $A = A_d + |A - A_d|$, provided A is a Metzler matrix. Then, (25) reduces to (31). It remains to show that $s(A + B(\infty)) < 0$ provided (16) holds and (1) is UAS. Assume on the contrary that $s(A+B(\infty)) \geq 0$. Consider the real function

$$f(\theta) = \theta - s(A + \int_0^{+\infty} e^{-\theta s} d[B(s)]) \qquad \theta \in [0, +\infty).$$

Clearly, f is continuous and $\lim_{\theta\to+\infty} f(\theta) = +\infty$. By the assumption, $f(0) = -s(A + \int_0^{+\infty} d[B(s)]) = -s(A + B(\infty)) \leq 0$. Then there is a $\theta_1 \geq 0$ such that $f(\theta_1) = 0$, or equivalently, $\theta_1 = s(A+\int_0^{+\infty} e^{-\theta_1 s} d[B(s)])$. Consequently, by Theorem 2.1(i), we have $\theta_1 \in \sigma(A + \int_0^{+\infty} e^{-\theta_1 s} d[B(s))])$. Hence, $\det \mathcal{H}(\theta_1) = 0$ with $\theta_1 \geq 0$. So (1) is not UAS, by Theorem 3.3. This is a contradiction and it completes the proof. ■

Remark 3.9. In particular, if $B(\cdot)$ is locally absolutely continuous on $\mathbb{R}_+$, that is,

$$B(t) = \int_0^t C(s)ds, \qquad t \geq 0; \quad C(\cdot) \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n}) \tag{33}$$

then (1) reduces to the linear Volterra integro-differential equation

$$\dot{x}(t) = Ax(t) + \int_0^t C(t-s)x(s)ds, \ t \in \mathbb{R}_+. \tag{34}$$

Then (16) becomes

$$\int_0^{+\infty} \|C(s)\| ds < +\infty \tag{35}$$

and $\mathcal{B}$ is now given by

$$\mathcal{B} := \int_0^{+\infty} \big|C(s)\big| ds \in \mathbb{R}^{n\times n}.$$

Thus (19) and (25) reduce to

$$\mu(A) + \mu\left(\int_0^{+\infty} |C(s)| ds\right) < 0, \tag{36}$$

and

$$s(A_d + |A - A_d| + \int_0^{+\infty} |C(s)| ds) < 0, \tag{37}$$

respectively. Thus, the linear Volterra integro-differential equation (34) is UAS provided either (35) and (36) hold or (35) and (37) hold.

3.2. Explicit criteria for exponential asymptotic stability

In this subsection, we deal with the exponential asymptotic stability of the Volterra-Stieltjes differential equation (1) which is defined as follows.

Definition 3.10. The zero solution of (1) is said to be exponentially asymptotically stable (EAS) if there exist $M, \alpha > 0$ such that

$$\forall \sigma \geq 0, \forall \varphi \in C([0,\sigma], \mathbb{R}^n), \forall t \geq \sigma : \qquad \|x(t;\sigma,\varphi)\| \leq Me^{-\alpha(t-\sigma)}\|\varphi\|.$$

If the zero solution of (1) is EAS then we also say that (1) is EAS.

By definition, it is easy to see that the exponential asymptotic stability of (1) implies its uniform asymptotic stability. However, the converse of this statement does not hold even for linear Volterra integro-differential equations (2), see e.g. [18].

Let us introduce, for given $F(\cdot) \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{\ell\times q})$, a scalar function $V_F(\cdot)$ defined by

$$V_F \ : \ \mathbb{R}_+ \to \mathbb{R}_+; \ s \mapsto V_F(s) := Var(F; 0, s). \tag{38}$$

Then, it is easy to see that we have, for any $T \geq 0$ and any $g \in C([0,T], \mathbb{R})$,

$$\left\| \int_0^T g(s) d[F(s)] \right\| \leq \int_0^T |g(s)| d[V_F(s)]. \tag{39}$$

Proposition 3.11. [25]. *If (1) is UAS and*

$$\exists\alpha > 0 : \qquad \int_0^{+\infty} e^{\alpha s} d[V_B(s)] < +\infty, \tag{40}$$

then (1) is EAS.

Remark 3.12. (i) If $B(\cdot) = (b_{ij}(\cdot))$ is increasing on $\mathbb{R}_+$ then (40) is equivalent to

$$\exists\alpha > 0 : \left| \int_0^{+\infty} e^{\alpha s} d[b_{ij}(s)] \right| < +\infty, \forall i, j \in \underline{n}. \tag{41}$$

(ii) In particular, if $B(\cdot)$ is defined by (33) then (40) reduces to

$$\exists\alpha > 0 : \int_0^{+\infty} e^{\alpha t} \|C(s)\| ds < +\infty. \tag{42}$$

(iii) By (39), if (40) holds then (16) holds.

The following theorem is immediate from Proposition 3.11, Theorem 3.4 and Corollary 3.7.

Theorem 3.13. *Let $A \in \mathbb{R}^{n\times n}$ and $B(\cdot) \in NBV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ be given. Then (1) is EAS provided*
(a) (40) *and one of the conditions* (19)*,* (20) *hold*
or
(b) (40) *and one of the conditions (i), (ii), (iii), (iv), (v) of Corollary 3.7 hold.*

We illustrate the obtained results by an example.

Example 3.14. Consider a scalar linear Volterra integro-differential equation with delay given by

$$\dot{x}(t) = ax(t) + \sum_{i=1}^{m} a_i x(t - h_i) + \int_0^t e^{-\tau} x(t - \tau) d\tau, \quad x(t) \in \mathbb{R}, \quad t \geq 0, \tag{43}$$

where $0 < h_1 < h_2 < ... < h_m := h$ and $a, a_i \in \mathbb{R}$ for $i \in \underline{m}$.

For a given $\varphi \in C([-h, 0], \mathbb{R})$, (43) has a unique solution, denoted by $x(\cdot; \varphi)$, satisfying the initial condition,

$$x(s) = \varphi(s), \; s \in [-h, 0], \tag{44}$$

see e.g. [5].
We will show that (43) is exponentially asymptotically stable provided $a + \sum_{i=1}^m |a_i| < -1$. That is, there exist $M, \alpha > 0$ such that for any $\varphi \in C([-h, 0], \mathbb{R})$,

$$|x(t; \phi)| \leq Me^{-\alpha t} \|\varphi\|, \qquad \forall t \geq 0.$$

To this end, for each $\varphi \in C([-h,0],\mathbb{R})$, we extend φ to the interval $[-h,\infty)$ by setting $\varphi(s)=0$ for any $s>0$. Let us define

$$\eta_i(\cdot):\mathbb{R}_+\to\mathbb{R};\quad s\mapsto \eta_i(s):=\begin{cases}0 & \text{if } s\in[0,h_i)\\ a_i & \text{if } s\in[h_i,+\infty)\end{cases}$$

for every $i\in \underline{m}$, and

$$b(s):=1-e^{-s};\quad \eta(s):=\sum_{i=1}^m \eta_i(s);\quad f(s)=\sum_{i=1}^m a_i\varphi(s-h_i),\ s\geq 0.$$

It is clear that $b(\cdot)\in NBV(\mathbb{R}_+,\mathbb{R})$, is increasing on $\mathbb{R}_+$ and $\eta(\cdot)\in NBV(\mathbb{R}_+,\mathbb{R})$, $f\in L^1_{loc}(\mathbb{R}_+,\mathbb{R})$. Then $x(\cdot;\varphi)$ satisfies

$$\dot{x}(t)=ax(t)+\int_0^t d[b(\tau)+\eta(\tau)]x(t-\tau)+f(t),\quad t\geq 0;\quad x(0)=\varphi(0).$$

Furthermore, we have

$$\int_0^{+\infty}|d[b(t)+\eta(t)]|\leq 1+\sum_{i=1}^m|a_i|$$

and

$$\int_0^{+\infty}e^{\alpha t}dV_{\eta+b}(s)\leq \sum_{i=1}^m|a_i|e^{\alpha h_i}+\frac{1}{1-\alpha}$$

for any $\alpha\in(0,1)$. By Theorem 3.13, the equation

$$\dot{x}(t)=ax(t)+\int_0^t d[b(\tau)+\eta(\tau)]x(t-\tau),\ t\geq 0, \tag{45}$$

is EAS provided $a+\sum_{i=1}^m|a_i|<-1$. Let $R(\cdot)$ be the resolvent of (45). Since (45) is EAS, it follows that $|R(t)|\leq e^{-\alpha t},\ t\geq 0$. By the variation of constants formula (15),

$$x(t;\varphi)=R(t)\varphi(0)+\int_0^t R(t-s)f(s)ds,\quad t\geq 0.$$

This implies

$$\begin{aligned}|x(t;\varphi)|&\leq e^{-\alpha t}|\varphi(0)|+\sum_{i=1}^m|a_i|\int_0^t e^{-\alpha(t-s)}|\varphi(s-h_i)|ds\\ &=e^{-\alpha t}|\varphi(0)|+e^{-\alpha t}\sum_{i=1}^m|a_i|e^{\alpha h_i}\int_{-h_i}^{t-h_i}e^{\alpha s}|\varphi(s)|ds\\ &\leq e^{-\alpha t}|\varphi(0)|+e^{-\alpha t}\sum_{i=1}^m|a_i|e^{\alpha h_i}\int_{-h_i}^{0}e^{\alpha s}|\varphi(s)|ds\\ &\leq e^{-\alpha t}\Big(1+\sum_{i=1}^m|a_i|(\frac{e^{\alpha h_i}-1}{\alpha})\Big)\|\varphi\|.\end{aligned}$$

4. An Application

In this section, we apply the obtained results in the previous section to study asymptotic behavior of solutions of integro-differential equations with delay of the form

$$\begin{aligned}\dot{x}_i(t) = a_{ii}x_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij}x_j(t-\tau_{ij}) \\ + \sum_{j=1}^{n}\int_{-\infty}^{t} k_{ij}(t-s)x_j(s)ds + f_i(t),\ t \geq 0.\end{aligned} \tag{46}$$

Here a_{ij} $(i, j \in \underline{n})$ are given real numbers, τ_{ij} $(i, j \in \underline{n}, i \neq j)$ are given positive numbers and $k_{ij}(\cdot), f_i(\cdot)\ : \mathbb{R}_+ \to \mathbb{R}$ $(i, j \in \underline{n})$ are given continuous functions.

Let $BC((-\infty, 0], \mathbb{R}^n)$ be the Banach space of all bounded continuous functions on $(-\infty, 0]$, endowed with the standard supremum norm. For given $\varphi \in BC((-\infty, 0], \mathbb{R}^n)$, we consider for (46) the initial condition

$$x(t) = \varphi(t), \qquad t \in (-\infty, 0]. \tag{47}$$

Set

$$A_d := diag(a_{11}, ..., a_{nn}) \in \mathbb{R}^{n\times n},$$

and define for each $i, j \in \underline{n}, i \neq j$,

$$A_{ij} = (\delta_{pq}) \in \mathbb{R}^{n\times n} \quad \text{where } \delta_{pq} := a_{ij} \quad \text{if } p = i \text{ and } q = j, \text{ otherwise } \delta_{pq} = 0,$$

and

$$K(\cdot) = (k_{ij}(\cdot)) \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n}).$$

Then (46) can be rewritten as

$$\dot{x}(t) = A_d x(t) + \sum_{i,j=1, i\neq j}^{n} A_{ij}x(t-\tau_{ij}) + \int_{-\infty}^{t} K(t-s)x(s)ds + f(t),\ t \geq 0, \tag{48}$$

where $x(t) := (x_1(t), ..., x_n(t))^T \in \mathbb{R}^n$ and $f(t) := (f_1(t), ..., f_n(t))^T \in \mathbb{R}^n$, for $t \geq 0$.

In what follows, we always assume that

$$\int_0^{+\infty} |k_{ij}(s)|ds < +\infty, \qquad i, j \in \underline{n}. \tag{49}$$

Note that under the assumption (49), the initial value problem (46)-(47) has a unique solution $x(\cdot; \varphi)$, see e.g. [10].

Theorem 4.1. *Let* (49) *hold and let* $f(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ *be given. If there exist nonnegative numbers* $p_1, p_2, ..., p_n$ *such that*

$$p_i(-a_{ii} - \int_0^{+\infty} |k_{ii}(s)|ds) > \sum_{j=1, j\neq i}^{n} (|a_{ij}| + \int_0^{+\infty} |k_{ij}(s)|ds)p_j, \qquad \forall i \in \underline{n}, \tag{50}$$

then $x(t; \varphi) \to 0$ *as* $t \to +\infty$ *for any* $\varphi \in BC((-\infty, 0], \mathbb{R}^n)$.

Proof. Let us define for every $i, j \in \underline{n}, i \neq j$,

$$\eta_{ij}(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}; \quad s \mapsto \eta_{ij}(s) := \begin{cases} 0 & \text{if } s \in [0, \tau_{ij}) \\ A_{ij} & \text{if } s \in [\tau_{ij}, +\infty) \end{cases}$$

and for $s \geq 0$,

$$\gamma_1(s) := \sum_{i,j=1, i\neq j}^{n} \eta_{ij}(s); \ \gamma_2(s) := \int_0^s K(\tau)d\tau; \ \gamma(s) := \gamma_1(s) + \gamma_2(s).$$

Clearly, $\gamma(\cdot) := (\gamma_{ij}(\cdot)) \in NBV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$. Furthermore, we have

$$\int_0^{+\infty} |d[\gamma(s)]| \leq \sum_{i,j=1, i\neq j}^{n} \|A_{ij}\| + \int_0^{+\infty} \|K(s)\|ds < +\infty. \tag{51}$$

Extending φ to the whole real line by setting $\varphi(s) := 0, \ s > 0$ and then we define

$$f_1(s) := \sum_{i,j=1, i\neq j}^{n} A_{ij}\varphi(s-\tau_{ij}) \quad s \geq 0; \qquad f_2(s) := \int_{-\infty}^{0} K(s-\tau)\varphi(\tau)d\tau, \quad s \geq 0.$$

Let $F(s) := f_1(s) + f_2(s) + f(s), \ s \geq 0$. Then $F(\cdot)$ is locally integrable on $\mathbb{R}_+$. It is easy to check that the solution $x(\cdot) := x(\cdot; \varphi)$ of the initial value problem (48)-(47) (so (46)-(47)) satisfies the Volterra-Stieltjes differential equation

$$\dot{x}(t) = A_d x(t) + \int_0^t d[\gamma(s)]x(t-s) + F(t) \quad a.e. \ t \in \mathbb{R}_+, \tag{52}$$

and fulfills the initial condition $x(0) = \varphi(0)$.

Let $R_0(\cdot)$ be the resolvent of the homogeneous linear Volterra-Stieltjes differential equation associated with (52)

$$\dot{x}(t) = A_d x(t) + \int_0^t d[\gamma(s)]x(t-s), \quad a.e. \ t \in \mathbb{R}_+. \tag{53}$$

Then the solution $x(\cdot; \varphi)$ of (48)-(47) is represented as

$$x(t; \varphi) = R_0(t)\varphi(0) + \int_0^t R_0(t-s)F(s)ds, \tag{54}$$

by the variation of constants formula (15).

Let $\gamma^* := (\int_0^{+\infty} |d[\gamma_{ij}(s)]|) \in \mathbb{R}^{n\times n}$. Note that

$$\int_0^{+\infty} |d[\gamma_{ii}(s)]| = \int_0^{+\infty} |k_{ii}(s)|ds, \qquad \forall i \in \underline{n}, \tag{55}$$

and

$$(56) \qquad \int_0^{+\infty} |d[\gamma_{ij}(s)]| \le |a_{ij}| + \int_0^{+\infty} |k_{ij}(s)|ds, \qquad \forall i,j \in \underline{n}, i \neq j.$$

Let $p := (p_1, p_2, ..., p_n)^T \in \mathbb{R}^n_+$. Then (50), (55), (56) imply that

$$(A_d + \gamma^*)\, p \ll 0.$$

By Corollary 3.7, (53) is UAS. Thus, $R_0(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$, by Theorem 3.3. Furthermore, as shown in the beginning of the proof of Proposition 3.3, $R_0(t) \to 0$ as $t \to +\infty$. Taking into account (54), it remains to show that

$$\begin{aligned}&\int_0^t R_0(t-s)F(s)ds \\ &= \int_0^t R_0(t-s)f_1(s)ds + \int_0^t R_0(t-s)f_2(s)ds + \int_0^t R_0(t-s)f(s)ds \to 0\end{aligned}$$

as $t \to +\infty$. Since $f(\cdot), f_1(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ and $R_0(t) \to 0$ as $t \to +\infty$, it follows that $\int_0^t R_0(t-s)f_1(s)ds$ and $\int_0^t R_0(t-s)f(s)ds$ tend to 0 as $t \to +\infty$, by a standard property of the convolution, see e.g. [3, page 22]. Finally,

$$\begin{aligned}\|\int_0^t R_0(t-s)f_2(s)ds\| &\le \int_0^t \|R_0(t-s)\| \int_{-\infty}^0 \|K(s-\tau)\|\|\varphi(\tau)\|d\tau ds \\ &\le \|\varphi\| \int_0^t \|R_0(t-s)\| \int_s^{+\infty} \|K(\tau)\|d\tau ds.\end{aligned}$$

By $\int_s^{+\infty} \|K(\tau)\|d\tau \to 0$ as $s \to +\infty$ and $\|R_0(\cdot)\| \in L^1(\mathbb{R}_+, \mathbb{R})$, we have $\int_0^t \|R_0(t-s)\| \int_s^{+\infty} \|K(\tau)\|d\tau ds \to 0$, as $t \to +\infty$. This completes the proof. ■

Remark 4.2. The result of Theorem 4.1 has been proven in [14, Th. 2.1] by the Lyapunov's method under the stronger hypotheses that

$$\int_0^{+\infty} |k_{ij}(s)|ds < +\infty, \qquad \int_0^{+\infty} s|k_{ij}(s)|ds < +\infty \quad \forall i,j \in \underline{n}$$

and

$$f(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^n) \cap BC(\mathbb{R}_+, \mathbb{R}^n).$$

In Theorem 4.1, we do not assume that $\int_0^{+\infty} s|k_{ij}(s)|ds < +\infty,\ \forall i,j \in \underline{n}$ and $f(\cdot) \in BC(\mathbb{R}_+, \mathbb{R}^n)$.

We conclude the paper with a result on exponential decay of solutions of (46).

Theorem 4.3. *If there exists a real number* $\alpha > 0$ *such that*

$$\int_0^{+\infty} e^{\alpha s}|k_{ij}(s)|ds < +\infty, \qquad \forall i,j \in \underline{n} \tag{57}$$

and (50) *holds then* (46) *(with* $f_i = 0\ \forall i \in \underline{n}$*) is exponentially asymptotically stable. That is, there exist* $K, \beta > 0$ *such that*

$$\forall \varphi \in BC((-\infty,0],\mathbb{R}^n), \forall t \geq 0: \qquad \|x(t;\varphi)\| \leq Ke^{-\beta t}\|\varphi\|. \tag{58}$$

Proof. In the proof of Theorem 4.1, we have shown that (53) is UAS. Then (57) implies that (53) is EAS, by Proposition 3.11. In particular, there exist $M > 0, \beta > 0$ such that

$$\|R_0(t)\| \leq Me^{-\beta t}, \quad t \geq 0, \tag{59}$$

where $R_0(\cdot)$ is the resolvent of (53). It follows from (54) that

$$x(t;\varphi) = R_0(t)\varphi(0) + \int_0^t R_0(t-s)(f_1(s)+f_2(s))ds.$$

Without loss of generality, we assume that $\beta < \alpha$. Taking this and (57) into account, we have

$$\|x(t;\varphi)\| \leq Me^{-\beta t}\|\varphi\| + \int_0^t Me^{-\beta(t-s)}(\|f_1(s)\| + \|f_2(s)\|)ds \leq Me^{-\beta t}\|\varphi\|$$
$$+Me^{-\beta t}\left(\sum_{i,j=1,i\neq j}^{n} \|A_{ij}\|\int_0^{\tau_{ij}} e^{\beta s}\|\varphi(s-\tau_{ij})\|ds + \|\varphi\|\int_0^t e^{\beta s}\int_{-\infty}^0 \|K(s-\tau)\|d\tau ds\right)$$
$$\leq Me^{-\beta t}\|\varphi\|\left(1+\sum_{i,j=1,i\neq j}^{n}\|A_{ij}\|\left(\frac{e^{\beta\tau_{ij}}-1}{\beta}\right)+\int_0^t e^{-(\alpha-\beta)s}\int_{-\infty}^0 e^{\alpha(s-\tau)}\|K(s-\tau)\|d\tau ds\right)$$
$$\leq Me^{-\beta t}\|\varphi\|\left(1+\sum_{i,j=1,i\neq j}^{n}\|A_{ij}\|\left(\frac{e^{\beta\tau_{ij}}-1}{\beta}\right)+\left(\int_0^\infty e^{\alpha\tau}\|K(\tau)\|d\tau\right)\left(\int_0^\infty e^{-(\alpha-\beta)s}ds\right)\right).$$

This completes the proof. ■

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References

1. J. A. D. Appleby and D. W. Reynolds, On necessary and sufficient conditions for exponential stability in linear Volterra integro-differential equations, *J. Integral Equations Appl.*, **16** (2004), 221-240.

2. J. A. D. Appleby, I. Gyori and D. W. Reynolds, On exact rates of decay of solutions of linear systems of Volterra equations with delay, *J. Math. Anal. Appl.*, **320** (2006), 56-77.

3. W. Arendt, Charles J. K. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Monographs in Mathematics, 96, Basel-Birkhäuser, 2001.

4. M. Bologna, Asymptotic solution for first and second order linear Volterra integro-differential equations with convolution kernels, *J. Phys. A: Math. Theor.*, **43** (2010), 375203.

5. T. A. Burton, *Volterra Integral and Differential Equations*, Second edition, Mathematics in Science and Engineering, 202. Elsevier B. V., Amsterdam, 2005.

6. R. H. Cameron and W. T. Martin, An unsymmetric Fubini theorem, *Bull. Amer. Math. Soc.*, **47** (1941), 121-125.

7. C. Corduneanu and V. Lakshmikantham, Equations with unbounded delay: a survey, *Nonlinear Anal.*, **4** (1980), 831-877.

8. C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, 1991.

9. M. C. Delfour, Linear hereditary differential systems and their control, *Lecture Notes in Economics and Mathematical Systems*, **106** (1974), 92-154.

10. R. D. Driver, Existence and stability of solutions of a delay differential system, *Arch. Rational Mech. Anal.*, **10** (1962), 401-426.

11. G. Gripenberg, S. O. Londen and O. Staffans, *Volterra Integral and Functional Equations*, Cambridge Univeristy Press, 1990.

12. R. A. Horn and Ch. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1993.

13. S. I. Grossman and R. K. Miller, Nonlinear Volterra integrodifferential systems with L^1-kernels, *J. Differ. Equations*, **13** (1973), 551-566.

14. K. Gopalsamy, Stability and decay rates in a class of linear integro-differential systems, *Funkcialaj Ekvacioj*, **26** (1983), 251-261.

15. R. K. Miller, Asymptotic stability properties of linear Volterra integrodifferential equations, *J. Differ. Equations*, **10** (1971), 485-506.

16. A. Ilchmann and P. H. A. Ngoc, On positivity and stability of linear time-varying Volterra equations, *Positivity*, **13** (2009), 671-681.

17. S. Murakami, Exponential stability for fundemental solutions of some linear functional differential equations. in: *Proceedings of the International Symposium: Functional Differential Equations*, T. Yoshizwa and J. Kato, eds. Singapore, World Scientific, 1990, pp. 259-263,

18. S. Murakami, Exponential asymptotic stability for scalar linear Volterra equations, *Differential Integral Equations*, **4** (1991), 519-525.

19. S. Murakami and P. H. A. Ngoc, On stability and robust stability of positive linear Volterra equations in Banach lattices, *Central European Journal of Mathematics*, **8** (2010), 966-984.

20. T. Naito, J. S. Shin, S. Murakami and P. H. A. Ngoc, Characterizations of positive linear Volterra integro-differential systems, *Integral Equations and Operator Theory*, **58** (2007), 255-27.

21. P. H. A. Ngoc, Stability radii of positive linear Volterra-Stieltjes equations, *Journal of Differential Equations* **243** (2007), 101-122.

22. P. H. A. Ngoc, T. Naito, J. S. Shin and S. Murakami, On stability and robust stability of positive linear Volterra equations, *SIAM Journal on Control and Optimization*, **47** (2008), 975-996.

23. P. H. A. Ngoc, On positivity and stability and linear Volterra systems with delay, *SIAM Journal on Control and Optimization*, **48** (2009), 1939-1960.

24. P. H. A. Ngoc, S. Murakami, T. Naito, J. S. Shin and Y. Nagabuchi, On positive linear Volterra-Stieltjes differential equations, *Integral Equations and Operator Theory*, **64** (2009), 325-355.

25. P. H. A. Ngoc, Stability of linear Volterra-Stieltjes differential equations, *SIAM Journal on Control and Optimization*, **49** (2011), 205-226.

26. N. K. Son and D. Hinrichsen, Robust stability of positive continuous-time systems, *Numer. Funct. Anal. Optim.*, **17** (1996), 649-659.

27. M. Vidyasagar, *Nonlinear System Analysis*, Prentice Hall, New Jersey, 1978.

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